Mathematical
Inequalities
& Applications
Volume 9, Number 4 (2006), 725–738

ON A NONLINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATION IN BANACH SPACES

M. B. DHAKNE AND S. D. KENDRE

(*communicated by Ram N. Mohapatra*)

Abstract. The objective of the present paper is to study the local existence, global existence, uniqueness, continuous dependence, asymptotic stability and other properties of solutions of a nonlinear Volterra integrodifferential equation in Banach spaces of more general type. The technique employed in our analysis is based on treating the equation in the domain of the infinitesimal generator of semigroups of linear operators in a Banach space with graph norm and using results from linear semigroup theory.

1. Introduction

Let X be a general Banach space with norm $\|\ \|$. In this paper, we investigate an abstract nonlinear Volterra integrodifferential equation of the type

$$x'(t) = Ax(t) + \int_0^t \{a(t,s)f(s,\, x(s)) + g(t,s,\, x(s))\}ds + f_0(t), \quad t \geqslant 0; \tag{1}$$

$$x(0) = x_0 \in X; \tag{2}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$, $t \geqslant 0$ in X, nonlinear functions $f: R^+ \times X \to X$, $g: R^+ \times R^+ \times X \to X$, $f_0: R^+ \to X$ and the kernel $a: R^+ \times R^+ \to R$ are continuous and x_0 is a given element of X.

The equations of these types or their special forms arise naturally in many areas of applied mathematics as mathematical model of physical process such as heat flow in materials with memory, see [1, 2, 8, 10] and some of the references listed therein. Many recent papers and monographs have dealt with the existence, uniqueness, continuation and other properties of solutions of special forms of the equations $(1.1) - (1.2)$, see [6, 7, 11, 12] and references given therein. In an interesting paper [12] G.F.Webb has studied the special form of $(1.1) - (1.2)$ by using linear semigroup theory and the well-known Banach fixed-point theorem. Our general formulation of $(1.1) - (1.2)$ is

Mathematics subject classification (2000): 45N05, 45J05.

Key words and phrases: Volterra integrodifferential equation, semigroup, infinitesimal generator, graph norm.

This research was supported by University Grants Commission of India vide No. F 8-18/2001(SR-I) dated March 11, 2001.

an attempt to generalize the results of G.F.Webb [12] and is motivated by the model of integrodifferential equation studied by M.L.Heard [5] and Dhakne and Pachpatte [3, 4].

In the present paper, we study the basic problems such as the local existence, global existence, uniqueness, continuous dependence upon initial values and asymptotic stability of solutions of the equations $(1.1)-(1.2)$. The main tool employed in our analysis is based on treating the equation in the domain of A with graph norm and using results from linear semigroup theory.

The paper is organized as follows. In section 2, we present the preliminaries and statement of our main results. Section 3 deals with the proofs of Theorems. Finally in section 4 we give an example to illustrate the applications of main results.

2. Preliminaries and statements of results

Before proceeding to the statements of our main results, we setforth some preliminaries from [6, 7] and hypotheses that will be used in our further discussion.

DEFINITION 1. A function $x\colon [0,t_1] \to X$, for some $t_1 > 0$, is said to be a local solution of $(1.1)-(1.2)$ if

(i) $x\colon [0,t_1] \to X$ is continuous from $[0,t_1]$ to $D(A)$;

(ii) $x\colon [0,t_1] \to X$ is differentiable

and satisfies $(1.1)-(1.2)$. If the closed interval $[0,t_1]$ replaced by R^+ then the local solution of $(1.1)-(1.2)$ is called global solution.

DEFINITION 2. The solutions $x_1(t)$ and $x_2(t)$ of (1.1) with $x_1(0) = x_0^*$ and $x_2(0) = x_0^{**}$ respectively are called exponentially asymptotically stable in the graph norm if

$$\|x_1(t) - x_2(t)\|_A \leqslant \|x_0^* - x_0^{**}\|_A \, e^{\delta t}, \quad t \geqslant 0;$$

where $\delta \leqslant 0$.

Let A be the infinitesimal generator of a C_0 − *semigroup* of bounded linear operators $T(t), t \geqslant 0$ in Banach Space X satisfying $\|T(t)\| \leqslant e^{\omega t}, t \geqslant 0$, where ω is a real constant. Let $D(A)$ and $[D(A)]$ denote the domain of A and the Banach space with graph norm defined by $\|x\|_A = \|x\| + \|Ax\|$, $x \in D(A)$ respectively.

We need the following Lemma proved in [6, p. 486].

LEMMA 2.1. *Let* $k\colon [0,t_1] \to X$ *be continuously differentiable. If for* $0 \leqslant t \leqslant t_1$, $q(t)$ *is defined by*

$$q(t) = \int_0^t T(t-s)k(s)ds$$

then $q(t) \in D(A)$, q *is continuously differentiable and*

$$q'(t) = Aq(t) + k(t) = \int_0^t T(t-s)k'(s)ds + T(t)k(0).$$

For convenience, we list the following hypotheses used in our subsequent discussion.

(H_1) The function $f_0\colon R^+ \to X$ is continuously differentiable.

(H_2) The kernel function $a\colon R^+ \times R^+ \to R$ is continuous, and continuously differentiable in the first argument.

(H_3) The function $f: R^+ \times D \rightarrow X$ where D is an open subset of $[D(A)]$, is continuous and for each $x_0 \in D$ there exists a neighborhood Dx_0 about x_0 and continuous function $b: R^+ \rightarrow R^+$ such that

$$\|f(t,x_1) - f(t,x_2)\| \leqslant b(t)\, \|x_1 - x_2\|_A\,, \tag{3}$$

for all $t \in R^+, \quad x_1, x_2 \in Dx_0$.

(H_4) The function $g: R^+ \times R^+ \times D \rightarrow X$ is continuously differentiable with respect to its first argument and for each $x_0 \in D$ there exists a neighborhood Dx_0 about x_0 and continuous functions $c: R^+ \times R^+ \rightarrow R^+$ and $d: R^+ \times R^+ \rightarrow R^+$ such that

$$\|g(t,s,x_1) - g(t,s,x_2)\| \leqslant c(t,s)\, \|x_1 - x_2\|_A\,; \tag{4}$$

$$\|g_1(t,s,x_1) - g_1(t,s,x_2)\| \leqslant d(t,s)\, \|x_1 - x_2\|_A\,, \tag{5}$$

for all $t, s \in R^+, \quad x_1, x_2 \in Dx_0$.

Our main results are established in the following theorems.

THEOREM 2.2. *Suppose that the hypotheses $(H_1) - (H_4)$ hold. For each $x_0 \in D$ there exist $t_1 > 0$ and a unique solution $x: [0,t_1] \rightarrow X$ of the equations $(1.1) - (1.2)$. Further, if $D = Dx_0 = [D(A)]$, then the solution x of $(1.1) - (1.2)$ exists on R^+.*

THEOREM 2.3. *Suppose that the hypotheses $(H_1) - (H_4)$ hold and $x_0 \in D$. Suppose that x_1 and x_2 satisfy the equation (1.1) for $0 \leqslant t \leqslant t_1$ with $x_1(0) = x_0^*$ and $x_2(0) = x_0^{**}$ respectively and $x_1(t),\ x_2(t) \in Dx_0$ then there exist continuous functions $\alpha, \beta, \gamma: R^+ \rightarrow R^+$ such that*

$$\|x_1(t) - x_2(t)\|_A \leqslant \|x_0^* - x_0^{**}\|_A exp\{(\beta(t) + \gamma(t) + M + \omega)t\}\,;\ 0 \leqslant t \leqslant t_1$$

where $M = \int_0^{t_1} \alpha(s)ds$.

COROLLARY 2.4. *Suppose the hypotheses of Theorem 2.3 hold with $D = Dx_0 = [D(A)]$. Assume there exist constants $\alpha_0, \beta_0, \gamma_0$ such that*

$$\int_0^t e^{-w(t-s)}\{|a(t,s)|\, b(s) + c(t,s) + |a_1(t,s)|\, b(s) + d(t,s)\}ds \leqslant \alpha_0;$$

$$e^{-w(t-s)}\,\{|a(t,s)|\, b(s) + c(t,s)\} \leqslant \beta_0;$$

$$|a(t,t)|\, b(t) + c(t,t) \leqslant \gamma_0;$$

for $t \geqslant 0$ and $\alpha_0 + \beta_0 + \gamma_0 + \omega = \delta \leqslant 0$. Then the solutions of (1.1) are exponentially asymptotically stable in the graph norm, in the sense : if $x_1(t),\ x_2(t)$ are the solutions of (1.1) with $x_1(0) = x_0^*,\ x_2(0) = x_0^{**}$ respectively, then

$$\|x_1(t) - x_2(t)\|_A \leqslant \|x_0^* - x_0^{**}\|_A\, e^{\delta t}, \quad t \geqslant 0.$$

THEOREM 2.5. *Suppose that the hypotheses $(H_1) - (H_4)$ hold. Let $f(t,x), g(t,s,x)$ and $g_1(t,s,x)$ be Lipschitz continuous on bounded sets of x_0 in D uniformly in finite intervals of t. If $x_0 \in D$ and x is a noncontinuable solution of $(1.1) - (1.2)$ on $[0,l)$, then either $l = +\infty$ or given any closed bounded set U in D there exist a sequence $t_k \rightarrow l^-$ such that $x(t_k) \notin U$.*

COROLLARY 2.6. *Suppose that the hypotheses of Theorem 2.5 hold. Let $D = [D(A)]$. If $x_0 \in D$ and x is a noncontinuable solution of $(1.1)-(1.2)$ on $[0,l)$ then either $l = +\infty$ or $\lim_{t \to l-} \|x(t)\|_A = +\infty$.*

3. Proofs of Theorems

First, we prove the existence of a solution of equations $(1.1)-(1.2)$. Let $x_0 \in D$ and let N be a neighborhood about x_0 in $[D(A)]$ such that $\overline{N} \subset Dx_0$. Let $t_1 > 0$ and $C = C([0,t_1];\overline{N})$. Define the mapping K on C by

$$\begin{aligned}(Kx)(t) = T(t)x_0 + \int_0^t T(t-s)\int_0^s \{a(s,\tau)f(\tau,x(\tau)) + g(s,\tau,x(\tau))\}\, d\tau ds \\ + \int_0^t T(t-s)f_0(s)ds, \quad x \in C, \quad 0 \leqslant \tau \leqslant s \leqslant t \leqslant t_1.\end{aligned} \tag{3.1}$$

Define

$$k(s) = \int_0^s \{a(s,\tau)f(\tau,x(\tau)) + g(s,\tau,x(\tau))\}\, d\tau, \quad 0 \leqslant \tau \leqslant s \leqslant t_1.$$

We show that $k(s)$ is continuously differentiable from $[0,t_1]$ to X. Now,

$$\begin{aligned}\frac{k(s+h)-k(s)}{h} =& h^{-1}\int_s^{s+h} \{a(s+h,\tau)f(\tau,x(\tau)) + g(s+h,\tau,x(\tau))\}\, d\tau \\ &+ \int_0^s \{\left[\frac{a(s+h,\tau)-a(s,\tau)}{h}\right] f(\tau,x(\tau)) \\ &+ \left[\frac{g(s+h,\tau,x(\tau))-g(s,\tau,x(\tau))}{h}\right]\} d\tau.\end{aligned} \tag{3.2}$$

Taking limit as $h \to 0$ on both sides of (3.2), we obtain

$$\begin{aligned}k'(s) =& \{a(s,s)f(s,x(s)) + g(s,s,x(s))\} \\ &+ \int_0^s \{a_1(s,\tau)f(\tau,x(\tau)) + g_1(s,\tau,x(\tau))\}\, d\tau.\end{aligned} \tag{8}$$

From hypotheses $(H_2)-(H_4)$, it follows that $k'(s)$ is continuous and therefore $k(s)$ is continuously differentiable. For $x \in C, \ \ 0 \leqslant s \leqslant t \leqslant t_1$ from the Lemma 2.1, we have

$$\int_0^t T(t-s)k(s)ds, \quad \int_0^t T(t-s)f_0(s)ds \in D(A)$$

and by Theorem $(2.2.1)(b)$ in [7] it follows that

$$(Kx)(t) = T(t)x_0 + \int_0^t T(t-s)k(s)ds + \int_0^t T(t-s)f_0(s)ds \in D(A).$$

Now, from equations (3.1), (3.3) and Lemma 2.1, we get

$$\begin{aligned}(AKx)(t) =& AT(t)x_0 + \int_0^t T(t-s)\left\{a(s,s)f(s,x(s)) + g(s,s,x(s))\right\}ds \\ &+ \int_0^t T(t-s)\int_0^s \left\{a_1(s,\tau)f(\tau,x(\tau)) + g_1(s,\tau,x(\tau))\right\}d\tau ds \\ &- \int_0^t \left\{a(t,s)f(s,x(s)) + g(t,s,x(s))\right\}ds \\ &+ \int_0^t T(t-s)f_0'(s)ds + T(t)f_0(0) - f_0(t).\end{aligned} \tag{3.4}$$

For $x \in C,$ from (3.1), (3.4) and the hypotheses $(H_1)-(H_4),$ it is clear that (Kx) and (AKx) are both continuous from $[0,t_1]$ to $X.$ If t_1 is chosen sufficiently small then K maps C into $C.$

Using (3.1), hypotheses $(H_1)-(H_3)$ and (2.2), we get

$$\|(Kx)(t)-(Ky)(t)\| \leqslant \int_0^t e^{\omega(t-s)}\int_0^s \left\{|a(s,\tau)|\,b(\tau)+c(s,\tau)\right\}\|x(\tau)-y(\tau)\|_A\,d\tau ds. \tag{10}$$

Similarly by using (3.4), hypotheses $(H_1)-(H_4),$ we obtain

$$\begin{aligned}\|(AKx)(t)-(AKy)(t)\| \leqslant& \int_0^t e^{\omega(t-s)}\left\{|a(s,s)|\,b(s)+c(s,s)\right\}\|x(s)-y(s)\|_A\,ds \\ &+ \int_0^t e^{\omega(t-s)}\int_0^s \left\{|a_1(s,\tau)|\,b(\tau)+d(s,\tau)\right\}\|x(\tau)-y(\tau)\|_A\,d\tau ds \\ &+ \int_0^t \left\{|a(t,s)|\,b(s)+c(t,s)\right\}\|x(s)-y(s)\|_A\,ds\end{aligned} \tag{3.6}$$

Using the definition of graph norm and equations (3.5) and (3.6), we obtain

$$\begin{aligned}\|(Kx)(t)-(Ky)(t)\|_A \leqslant& \int_0^t e^{\omega(t-s)}\left\{|a(s,s)|\,b(s)+c(s,s)\right\}\|x(s)-y(s)\|_A\,ds \\ &+ \int_0^t e^{\omega(t-s)}\int_0^s \left\{|a(s,\tau)|\,b(\tau)+c(s,\tau)\right\}\|x(\tau)-y(\tau)\|_A\,d\tau ds \\ &+ \int_0^t e^{\omega(t-s)}\int_0^s \left\{|a_1(s,\tau)|\,b(\tau)+d(s,\tau)\right\}\|x(\tau)-y(\tau)\|_A\,d\tau ds \\ &+ \int_0^t \left\{|a(t,s)|\,b(s)+c(t,s)\right\}\|x(s)-y(s)\|_A\,ds.\end{aligned} \tag{3.7}$$

Since $b\colon[0,t_1] \to R^+,$ $c\colon[0,t_1]\times[0,t_1] \to R^+$ and $d\colon[0,t_1]\times[0,t_1] \to R^+$ are continuous functions on compact domain, the functions $b,$ c and d are bounded i.e. there exists constants M_1 , M_2 and M_3 such that

$$b(t) \leqslant M_1,\;\; c(t,s) \leqslant M_2 \;\; and \;\; d(t,s) \leqslant M_3,\;\; for\, 0 \leqslant s \leqslant t \leqslant t_1. \tag{13}$$

Also $a(t,s)$ is continuous and continuously differentiable for $0 \leqslant s \leqslant t \leqslant t_1$, there exists positive constants N_1 and N_2 such that

$$|a(t,s)| \leqslant N_1 \quad \text{and} \quad |a_1(t,s)| \leqslant N_2, \quad \text{for} \quad 0 \leqslant s \leqslant t \leqslant t_1. \tag{14}$$

It is clear that

$$e^{\omega(t-s)} \leqslant e^{|\omega|t_1}, \quad \text{for} \quad 0 \leqslant s \leqslant t \leqslant t_1. \tag{15}$$

Using the results $(3.8)-(3.10)$ in (3.7), we obtain

$$\|(Kx)(t)-(Ky)(t)\|_A \leqslant \lambda \, \|x(t)-y(t)\|_A\, ; \tag{16}$$

where

$$\lambda = e^{|\omega|t_1}\left\{N_1M_1+M_2+N_2M_1+M_3\right\}\frac{t_1^2}{2}+\left\{(e^{|\omega|t_1}+1)(N_1M_1+M_2)\right\}t_1.$$

We observe that if t_1 is chosen sufficiently small then λ satisfies $0<\lambda<1$. Therefore K is a contraction mapping on C. By contraction mapping theorem there exists a unique $x \in C$ such that $Kx = x$. From (3.1) and Lemma 2.1 we have $x\colon [0,t_1] \to [D(A)]$ is continuous, $x\colon [0,t_1] \to X$ is differentiable and x satisfies $(1.1)-(1.2)$.

If $D = Dx_0 = [D(A)]$, then N can be chosen as $[D(A)]$. In this case t_1 does not depend on x_0 nor on $f_0(t)$ and the solution x can be continued to $+\infty$ i.e. we observe that a solution x of $(1.1)-(1.2)$ defined on a closed interval $[0,t_1]$ can be extended to a larger interval $[0,t_1+\eta]$, $\eta>0$ by defining $x(t+t_1)=u(t)$ where $u(t)$ is a solution of

$$u'(t) = Au(t)+\int_0^{t+t_1}\{a(t+t_1,s)f(s,u(s))+g(t+t_1,s,u(s))\}ds+f_0(t+t_1); \tag{17}$$

$$u(0)=x(t_1). \tag{18}$$

Above discussion guarantees the existence of the solution of equations $(3.12)-(3.13)$ on an interval of positive length $\eta>0$. Repeating this process, we observe that x exists on R^+ and the proof the of Theorem 2.2 is complete.

Define continuous functions α, β, $\gamma\colon R^+ \to R^+$ as follows

$$\alpha(t) = \max_{0\leqslant s\leqslant t} e^{-\omega(t-s)}\{|a(t,s)|\,b(s)+c(t,s)+|a_1(t,s)|\,b(s)+d(t,s)\}, \tag{19}$$

$$\beta(t) = \max_{0\leqslant s\leqslant t} e^{-\omega(t-s)}\left\{|a(t,s)|\,b(s)+c(t,s)\right\} \tag{20}$$

and

$$\gamma(t) = \max_{0\leqslant s\leqslant t}\left\{|a(s,s)|\,b(s)+c(s,s)\right\}. \tag{21}$$

Let

$$p_1(t) = e^{-\omega t}\|x_1(t)-x_2(t)\|, \tag{22}$$

$$p_2(t) = e^{-\omega t}\|A(x_1(t)-x_2(t))\| \tag{23}$$

and

$$p(t) = p_1(t)+p_2(t) = e^{-\omega t}\|x_1(t)-x_2(t)\|_A. \tag{24}$$

The functions $x_1(t)$ and $x_2(t)$ satisfy the equation (1.1) for $0 \leqslant t \leqslant t_1$ with $x_1(0) = x_0^*$ and $x_2(0) = x_0^{**}$ respectively. Then by using Theorem 2.2, equations (3.1) and (3.4); we obtain

$$\begin{aligned} x_1(t) = T(t)x_0^* &+ \int_0^t T(t-s) \int_0^s \{a(s,\tau)f(\tau, x_1(\tau)) + g(s,\tau,x_1(\tau))\} \, d\tau ds \\ &+ \int_0^t T(t-s)f_0(s)ds, \end{aligned} \tag{3.20}$$

$$\begin{aligned} x_2(t) = T(t)x_0^{**} &+ \int_0^t T(t-s) \int_0^s \{a(s,\tau)f(\tau, x_2(\tau)) + g(s,\tau,x_2(\tau))\} \, d\tau ds \\ &+ \int_0^t T(t-s)f_0(s)ds, \end{aligned} \tag{3.21}$$

$$\begin{aligned} Ax_1(t) = AT(t)x_0^* &+ \int_0^t T(t-s)\{a(s,s)f(s,x_1(s)) + g(s,s,x_1(s))\}ds \\ &+ \int_0^t T(t-s) \int_0^s \{a_1(s,\tau)f(\tau,x_1(\tau)) + g_1(s,\tau,x_1(\tau))\} \, d\tau ds \\ &- \int_0^t \{a(t,s)f(s,x_1(s)) + g(t,s,x_1(s))\} \, ds \\ &+ \int_0^t T(t-s)f_0'(s)ds + T(t)f_0(0) - f_0(t) \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} Ax_2(t) = AT(t)x_0^{**} &+ \int_0^t T(t-s) \{a(s,s)f(s,x_2(s)) + g(s,s,x_2(s))\} \, ds \\ &+ \int_0^t T(t-s) \int_0^s \{a_1(s,\tau)f(\tau,x_2(\tau)) + g_1(s,\tau,x_2(\tau))\} \, d\tau ds \\ &- \int_0^t \{a(t,s)f(s,x_2(s)) + g(t,s,x_2(s))\} \, ds \\ &+ \int_0^t T(t-s)f_0'(s)ds + T(t)f_0(0) - f_0(t). \end{aligned} \tag{3.23}$$

Using (3.20) and (3.21) in (3.17), and making use of hypotheses $(H_1) - (H_3)$, (2.2) and (3.19), we get

$$p_1(t) \leqslant \|x_0^* - x_0^{**}\| + \int_0^t \int_0^s e^{-\omega(s-\tau)} \{|a(s,\tau)| \, b(\tau) + c(s,\tau)\} \, p(\tau) d\tau ds. \tag{29}$$

Similarly, using (3.22) and (3.23) in (3.18), and also using hypotheses $(H_1) - (H_4)$ and equation (3.19), we obtain

$$p_2(t) \leqslant \|Ax_0^* - Ax_0^*\| + \int_0^t \{|a(s,s)|\, b(s) + c(s,s)\}\, p(s)ds$$
$$+ \int_0^t \int_0^s e^{-\omega(s-\tau)} \{|a_1(s,\tau)|\, b(\tau) + d(s,\tau)\}\, p(\tau)d\tau ds \tag{3.25}$$
$$+ \int_0^t e^{-\omega(t-s)} \{|a(t,s)|\, b(s) + c(t,s)\}\, p(s)ds.$$

Adding (3.24), (3.25) and using the definitions of $\alpha(t)$, $\beta(t)$ and $\gamma(t)$, we obtain

$$p(t) \leqslant \|x_0^* - x_0^{**}\| + \|Ax_0^\star - Ax_0^{\star\star}\| + \int_0^t \{|a(s,s)|\, b(s) + c(s,s)\}\, p(s)ds$$
$$+ \int_0^t \int_\tau^t e^{-\omega(s-\tau)} [\{|a(s,\tau)|\, b(\tau) + c(s,\tau)\} + \{|a_1(s,\tau)|\, b(\tau) + d(s,\tau)\}] p(\tau) ds d\tau$$
$$+ \int_0^t e^{-\omega(t-s)} \{|a(t,s)|\, b(s) + c(t,s)\}\, p(s)ds$$
$$\leqslant \|x_0^* - x_0^{**}\|_A + \{\beta(t) + \gamma(t) + M\} \int_0^t p(s)ds. \tag{3.26}$$

Thanks to Gronwall's inequality and applying it to the equation (3.26), we get

$$p(t) \leqslant \|x_0^\star - x_0^{\star\star}\|_A\, exp(\{\beta(t) + \gamma(t) + M\}\, t),$$

which yields

$$\|x_1(t) - x_2(t)\|_A \leqslant \|x_0^* - x_0^{**}\|_A\, exp(\{\beta(t) + \gamma(t) + M + \omega\}\, t). \tag{32}$$

Thus, the proof of the Theorem 2.3 is complete.

Using definitions of continuous functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and hypotheses of Corollary 2.4, it is clear that

$$\int_0^t \alpha(s)ds \leqslant \alpha_0,\ \beta(t) \leqslant \beta_0 \quad \text{and} \quad \gamma(t) \leqslant \gamma_0 \quad \text{for} \quad t \geqslant 0.$$

Then by using (3.27), we have

$$\|x_1(t) - x_2(t)\|_A \leqslant \|x_0^* - x_0^{**}\|_A\, exp(\{\alpha_0 + \beta_0 + \gamma_0 + \omega\}\, t)$$
$$\leqslant \|x_0^* - x_0^{**}\|_A\, exp(\delta t), \tag{3.28}$$

and the proof of the Corollary 2.4 is complete.

We prove the Theorem 2.5 by method of contradiction. Suppose that $l < \infty$ and conclusion of the theorem is false. Then there exists a closed bounded set U in D such that $x(t) \in U$ for $0 \leqslant t < l$. For $0 < t+h < l$ and using Theorem 2.2, we have

$$\begin{aligned}\|x(t+h)-x(t)\| \leqslant &\|T(t+h)x_0 - T(t)x_0\| \\ &+ \|\int_0^{t+h} T(t+h-s)\int_0^s \{a(s,\tau)f(\tau,x(\tau))+g(s,\tau,x(\tau))\}\,d\tau ds \\ &- \int_0^t T(t-s)\int_0^s \{a(s,\tau)f(\tau,x(\tau))+g(s,\tau,x(\tau))\}\,d\tau ds\| \\ &+ \left\|\int_0^{t+h} T(t+h-s)f_0(s)ds - \int_0^t T(t-s)f_0(s)ds\right\|.\end{aligned} \tag{3.29}$$

Now, by substitution, we get

$$\begin{aligned}&\int_0^{t+h} T(t+h-s)\int_0^s \{a(s,\tau)f(\tau,x(\tau))+g(s,\tau,x(\tau))\}\,d\tau ds \\ &\quad= \int_{-h}^0 T(t-s)\int_0^{s+h} \{a(s+h,\tau)f(\tau,x(\tau))+g(s+h,\tau,x(\tau))\}d\tau ds \\ &\quad+ \int_0^t T(t-s)\int_0^{s+h} \{a(s+h,\tau)f(\tau,x(\tau))+g(s+h,\tau,x(\tau))\}d\tau ds\end{aligned} \tag{3.30}$$

Similarly, we have

$$\begin{aligned}&\int_0^{s+h} \{a(s+h,\tau)f(\tau,x(\tau))+g(s+h,\tau,x(\tau))\}\,d\tau \\ &\quad= \int_{-h}^0 \{a(s+h,\tau+h)f(\tau+h,x(\tau+h))+g(s+h,\tau+h,x(\tau+h))\}d\tau \\ &\quad+ \int_0^s \{a(s+h,\tau+h)f(\tau+h,x(\tau+h))+g(s+h,\tau+h,x(\tau+h))\}d\tau\end{aligned} \tag{3.31}$$

Using (3.30) and (3.31), we obtain

$$\begin{aligned}&\|\int_0^{t+h} T(t+h-s)\int_0^s \{a(s,\tau)f(\tau,x(\tau))+g(s,\tau,x(\tau))\}\,d\tau ds \\ &-\int_0^t T(t-s)\int_0^s \{a(s,\tau)f(\tau,x(\tau))+g(s,\tau,x(\tau))\}\,d\tau ds\| \\ &\leqslant \int_{-h}^0 \|T(t-s)\|\int_{-h}^0 \{|a(s+h,\tau+h)|\,\|f(\tau+h,x(\tau+h))\| \\ &+\|g(s+h,\tau+h,x(\tau+h))\|\}d\tau ds \\ &+\int_{-h}^0 \|T(t-s)\|\int_0^s \{|a(s+h,\tau+h)|\,\|f(\tau+h,x(\tau+h))\| \\ &+\|g(s+h,\tau+h,x(\tau+h))\|\}d\tau ds \\ &+\int_0^t \|T(t-s)\|\int_{-h}^0 \{|a(s+h,\tau+h)|\,\|f(\tau+h,x(\tau+h))\| \\ &+\|g(s+h,\tau+h,x(\tau+h))\|\}d\tau ds \\ &+\int_0^t \|T(t-s)\|\int_0^s \{\|a(s+h,\tau+h)f(\tau+h,x(\tau+h))-a(s,\tau)f(\tau,x(\tau))\| \\ &+\|g(s+h,\tau+h,x(\tau+h))-g(s,\tau,x(\tau))\|\}d\tau ds\end{aligned} \tag{3.32}$$

Using the same technique as above, we get

$$\begin{aligned}&\left\|\int_0^{t+h} T(t+h-s)f_0(s)ds-\int_0^t T(t-s)f_0(s)ds\right\|\\&\leqslant\int_{-h}^0\|T(t-s)\|\,\|f_0(s+h)\|\,ds+\int_0^t\|T(t-s)\|\,\|f_0(s+h)-f_0(s)\|\,ds.\end{aligned}\tag{3.33}$$

Since $T(t)$ and $f_0(t)$ are continuous for $0\leqslant t<t+h<l$

$$\|T(t+h)-T(t)\|\to 0 \quad \text{as} \quad h\to 0 \tag{39}$$

$$\|f_0(t+h)-f_0(t)\|\to 0 \quad \text{as} \quad h\to 0 \tag{40}$$

From $(3.32)-(3.35)$ and (3.29), we obtain

$$\begin{aligned}&\|x(t+h)-x(t)\|\\&\leqslant\int_{-h}^0\|T(t-s)\|\int_{-h}^0\{|a(s+h,\tau+h)|\,\|f(\tau+h,x(\tau+h))\|\\&+\|g(s+h,\tau+h,x(\tau+h))\|\}d\tau ds\\&+\int_{-h}^0\|T(t-s)\|\int_0^s\{|a(s+h,\tau+h)|\,\|f(\tau+h,x(\tau+h))\|\\&+\|g(s+h,\tau+h,x(\tau+h))\|\}d\tau ds\\&+\int_0^t\|T(t-s)\|\int_{-h}^0\{|a(s+h,\tau+h)|\,\|f(\tau+h,x(\tau+h))\|\\&+\|g(s+h,\tau+h,x(\tau+h))\|\}d\tau ds\\&+\int_0^t\|T(t-s)\|\int_0^s\{|a(s+h,\tau+h)|\,\|f(\tau+h,x(\tau+h))-f(\tau+h,x(\tau))\|\\&+\|a(s+h,\tau+h)f(\tau+h,x(\tau))-a(s,\tau)f(\tau,x(\tau))\|\\&+\|g(s+h,\tau+h,x(\tau+h))-g(s+h,\tau+h,x(\tau))\|\\&+\|g(s+h,\tau+h,x(\tau))-g(s,\tau,x(\tau))\|\}d\tau ds\\&+\int_{-h}^0\|T(t-s)\|\,\|f_0(s+h)\|\,ds\end{aligned}\tag{3.36}$$

For each t, $T(t)$ is bounded operator, therefore there exists a constant M_0 such that $\|T(t)\|\leqslant M_0$. The functions $f(t,.), g(t,.,), g_1(t,.,.), f_0(t)$ and $f_0'(t)$ are continuous for $0\leqslant s\leqslant t\leqslant l$, there exists constants L_1, L_2, L_3, L_4 and L_5 such that

$$\|f(t,.)\|\leqslant L_1, \|g(t,.,.)\|\leqslant L_2, \|g_1(t,.,.)\|\leqslant L_3, \|f_0(t)\|\leqslant L_4, \text{ and } \|f_0'(t)\|\leqslant L_5. \tag{42}$$

Since $f(t,x)$, $g(t,s,x)$ and $g_1(t,s,x)$ are Lipschitz continuous on bounded sets of x_0 uniformly in finite intervals of t, there exists constants M_1, M_2 and M_3 such that

$$\|f(t,x_1)-f(t,x_2)\|\leqslant M_1\|x_1-x_2\|_A; \tag{43}$$

$$\|g(t,s,x_1)-g(t,s,x_2)\| \leqslant M_2\,\|x_1-x_2\|_A \tag{44}$$

and

$$\|g_1(t,s,x_1)-g_1(t,s,x_2)\| \leqslant M_3\,\|x_1-x_2\|_A\,. \tag{45}$$

Using $(3.37)-(3.40)$ in equation (3.36), we get

$$\|x(t+h)-x(t)\| \leqslant L(h)+M\int_0^t \|x(s+h)-x(s)\|_A\,ds \tag{46}$$

where

$$M=M_0\left\{N_1M_1+M_2\right\}l \quad \text{and} \quad L(h)=M_0\left\{N_1L_1+L_2\right\}\left\{\frac{3h^2}{2}+lh\right\}+M_0L_4h.$$

From equation (3.4), we obtain

$$\begin{aligned}
\|Ax(t+h)-Ax(t)\| \leqslant{}& \|T(t+h)Ax_0-T(t)Ax_0\| \\
&+\|\int_0^{t+h} T(t+h-s)[\{a(s,s)f(s,x(s))+g(s,s,x(s))\} \\
&+\int_0^s \{a_1(s,\tau)f(\tau,x(\tau))+g_1(s,\tau,x(\tau))\}\,d\tau]ds \\
&-\int_0^t T(t-s)[\{a(s,s)f(s,x(s))+g(s,s,x(s))\} \\
&+\int_0^s \{a_1(s,\tau)f(\tau,x(\tau))+g_1(s,\tau,x(\tau))\}\,d\tau]ds\| \qquad (3.42)\\
&+\|\int_0^{t+h} \{a(t+h,s)f(s,x(s))+g(t+h,s,x(s))\}\,ds \\
&-\int_0^t \{a(t,s)f(s,x(s))+g(t,s,x(s))\}\,ds\| \\
&+\left\|\int_0^{t+h} T(t+h-s)f_0'(s)ds-\int_0^t T(t-s)f_0'(s)ds\right\| \\
&+\|T(t+h)f_0(0)-T(t)f_0(0)\|+\|f_0(t+h)-f_0(t)\|\,.
\end{aligned}$$

By using substitution, we get

$$\begin{aligned}
\|Ax(t+h)-Ax(t)\| \leqslant{}& \|T(t+h)Ax_0-T(t)Ax_0\| \\
&+\int_{-h}^0 \|T(t-s)\|\,\{|a(s+h,s+h)|\,\|f(s+h,x(s+h))\| \\
&+\|g(s+h,s+h,x(s+h))\|\}ds \\
&+\int_{-h}^0 \|T(t-s)\|\int_{-h}^0 \{|a_1(s+h,\tau+h)|\,\|f(\tau+h,x(\tau+h))\| \\
&+\|g_1(s+h,\tau+h,x(\tau+h))\|\}d\tau ds \\
&+\int_{-h}^0 \|T(t-s)\|\int_0^s \{|a_1(s+h,\tau+h)|\,\|f(\tau+h,x(\tau+h))\| \\
&+\|g_1(s+h,\tau+h,x(\tau+h))\|\}d\tau ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \|T(t-s)\| \int_{-h}^0 \{|a_1(s+h,\tau+h)|\,\|f(\tau+h,x(\tau+h))\| \\
&+\|g_1(s+h,\tau+h,x(\tau+h))\|\}d\tau ds \\
&+\int_{-h}^0 \{|a(t+h,s+h)|\,\|f(s+h,x(s+h))\| \\
&+\|g(t+h,s+h,x(s+h))\|\}ds \\
&+\int_0^t \|T(t-s)\|\,\{|a(s+h,s+h)|\,\|f(s+h,x(s+h))-f(s+h,x(s))\| \\
&+\|a(s+h,s+h)f(s+h,x(s))-a(s,s)f(s,x(s))\| \\
&+\|g(s+h,s+h,x(s+h))-g(s+h,s+h,x(s))\| \\
&+\|g(s+h,s+h,x(s))-g(s,s,x(s))\|\}ds \\
&+\int_0^t \|T(t-s)\| \int_0^s \{|a_1(s+h,\tau+h)|\,\|f(\tau+h,x(\tau+h))-f(\tau+h,x(\tau))\| \qquad (3.43)\\
&+\|a_1(s+h,\tau+h)f(\tau+h,x(\tau))-a_1(s,\tau)f(\tau,x(\tau))\| \\
&+\|g_1(s+h,\tau+h,x(\tau+h))-g_1(s+h,\tau+h,x(\tau))\| \\
&+\|g_1(s+h,\tau+h,x(\tau))-g_1(s,\tau,x(\tau))\|\}d\tau ds \\
&+\int_0^t \{|a(t+h,s+h)|\,\|f(s+h,x(s+h))-f(s+h,x(s))\| \\
&+\|a(t+h,s+h)f(s+h,x(s))-a(t,s)f(s,x(s))\| \\
&+\|g(t+h,s+h,x(s+h))-g(t+h,s+h,x(s))\| \\
&+\|g(t+h,s+h,x(s))-g(t,s,x(s))\|\}ds \\
&+\int_0^t \|T(t-s)\|\,\{\|f_0'(s+h)-f_0'(s)\|\}\,ds+\int_{-h}^0 \|T(t-s)\|\|f_0(s+h)\|\,ds \\
&+\|T(t+h)f_0(0)-T(t)f_0(0)\|+\|f_0(t+h)-f_0(t)\|
\end{aligned}
$$

Using (3.37) – (3.40) in (3.43), we get

$$\|Ax(t+h)-Ax(t)\| \leqslant \overline{L(h)}+\overline{M}\int_0^t \|x(s+h)-x(s)\|_A\,ds \qquad (49)$$

where

$$
\begin{aligned}
\overline{L(h)} &= (M_0h+h)(N_1L_1+L_2)+(N_2L_1+L_3)M_0\left(\frac{3h^2}{2}+lh\right)+M_0L_5h \\
\overline{M} &= M_0(N_1M_1+M_2)+M_0(N_2M_1+M_3)l+N_1M_1+M_2
\end{aligned}
$$

Now, by using equations (3.41) and (3.44), we obtain

$$
\begin{aligned}
\|x(t+h)-x(t)\|_A &= \|x(t+h)-x(t)\|+\|Ax(t+h)-Ax(t)\| \\
&\leqslant \left[L(h)+\overline{L(h)}\right]+\left[M+\overline{M}\right]\int_0^t \|x(s+h)-x(s)\|_A\,ds
\end{aligned}
$$

Again, thanks to Gronwall's inequality, and applying to the above result, we get

$$\|x(t+h)-x(t)\|_A \leqslant \left[L(h)+\overline{L(h)}\right] exp\left[(M+\overline{M})t\right], \text{ for } 0<t<t+h<l. \quad (50)$$

Thus $\lim_{t\to l-} x(t)$ exists in $[D(A)]$ and is in D. By Theorem 2.2, $x(t)$ can be continued past l, contradicting the noncontinuability hypothesis and the proof of the Theorem 2.5 is complete.

Suppose $\overline{\lim}_{t\to l-} \|x(t)\|_A < \infty$. Assume that U is the closure of $\{x(t)\colon 0\leqslant t<l\}$ in $[D(A)]$. Then, it is clear that $U\subset D$, U is closed and bounded. By Theorem 2.5, it follows that $l=+\infty$ and proof of the corollary 2.6 is complete.

4. Application

In order to illustrate the application of our main result, consider the following nonlinear partial integrodifferential equation of the form

$$w_t(u,t) = w_{uu}(u,t)+\int_0^t \{a(t,s)F(s,w_{uu}(u,s))+G(t,s,w_{uu}(u,s))\}ds+H(u,t), \quad (51)$$

$$w(0,t)=w(1,t)=0,\ t>0, \quad (52)$$

$$w(u,0)=w_0(u),\ 0<u<1. \quad (53)$$

where the kernel function $a(t,s)\colon R^+\times R^+\to R$ is continuous, and continuously differentiable in the first argument, the functions $F(t,x)\colon R^+\times R\to R$ and $G(t,s,x)\colon R^+\times R^+\times R\to R$ are continuous and continuously differentiable, $F(t,x)$, $G(t,s,x)$ and $G_1(t,s,x)$ are Lipschitz continuous in x uniformly in t and the function $H(u,t)\colon [0,1]\times R^+\to R$ is continuously differentiable.

Let $X=L^2(0,1;R)$. We define an operator $A\colon X\to X$ by $Az=z''$ with domain $D(A)=\{z\in X\colon z''\in X \text{ and } z(0)=z(1)=0\}$. Define the functions $f\colon R^+\times D\to X$, $g\colon R^+\times R^+\times D\to X$ and $f_0\colon R^+\to X$ as follows

$$f(t,z)(u) = F(t,z''(u)),$$
$$g(t,s,z)(u) = G(t,s,z''(u))$$

and

$$f_0(t)(u) = H(u,t)$$

for $t>0$; $z\in X$ and $0<u<1$. From above choices of the functions, the problem $(4.1)-(4.3)$ can be formulated abstractly as

$$x'(t) = Ax(t)+\int_0^t \{a(t,s)f(s,\, x(s))+g(t,s,\, x(s))\}ds+f_0(t), \quad t\geqslant 0; \quad (54)$$

$$x(0)=x_0\in X. \quad (55)$$

Since all the hypotheses of Theorem 2.2 are satisfied and therefore a unique solution of the equations $(4.1)-(4.3)$ exists.

REFERENCES

[1] A. BELLENI MORANTE, *An integrodifferential equation arising from the theory of heat conduction in rigid material with memory*, Boll. Un. Mat. Ital., **15**, (1978), 470–482.

[2] A. BELLENI MORANTE, G. F. ROACH, *A mathematical model for Gamma ray transport in the cardiac region*, J. Math. Anal. Appl., **244**, (2000), 498–514.

[3] M. B. DHAKNE, B. G. PACHPATTE, *On perturbed abstract functional integrodifferential equation*, Acta Mathematical Scienta., **8**, (1988), 263–282.

[4] M. B. DHAKNE, B. G. PACHPATTE, *On some abstract functional integrodifferential equations*, Indian J. Pure and Appl. Math., **22**, (1991), 109–134.

[5] M. L. HEARD, *An abstract parabolic Volterra integrodifferential equation*, SIAM J. Maths. Anal., 1**3**, (1982), 81–105.

[6] T. KATO, *Perturbation theory for linear operators*, 2nd ed. Grundlehrender, Math. Wissenschaften Band 132 Springer Verlag, New York, 1980.

[7] G. E. LADAS, V. LAKSHMIKANTHAM, *Differential equations in abstract space*, Academic press, 1972.

[8] M. HIEBER, J. PRUSS, *Heat kernels and maximal Lp–Lq estimates for parabolic evolution equations*, Commun. In Partial Differential Equations., **22**, (9 and 10)(1997), 1647–1669.

[9] R. K. MILLER, *Volterra integral equations in a Banach Spaces*, Funkcial Ekvac., **18**, (1975), 163–194.

[10] J. A. NOHEL, *Nonlinear Volterra equations for heat flow in materials with memory*, in “Integral and functional differential equations" ed. T. Herdman et al, Lecture Notes in Pure And Appl. Math. **67**, Dekker, New York, (1981), 3–82.

[11] A. PAZY, *Semigroup of linear operators and applications to partial differential equations*, Springer Verlag, New York, (1983).

[12] G. F. WEBB, *An abstract semilinear Volterra integrodifferential equation*, Proc. Amer. Math. Soc., **69**, (1978), 255–260.

(Received June 17, 2005)

M. B. Dhakne
Department of Mathematics
Dr. Babasaheb Ambedkar Marathwada University
Aurangabad-431004
India (M. S.)
e-mail: mbdhakne@yahoo.com

S. D. Kendre
Department of Mathematics
University of Pune
Pune-411007
India (M. S.)
e-mail: subhashkendre@rediffmail.com