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BOUNDS FOR THE RATIO AND DIFFERENCE BETWEEN PARALLEL SUM AND SERIES VIA MOND–PEČARIĆ METHOD

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(*communicated by T. Furuta*)

Abstract. Upper bounds for the ratio and the difference between parallel sum and series of operator connections in the sense of Anderson-Duffin-Trapp are obtained, in which the Mond-Pečarić method for convex functions is applied: Let A and B be positive operators on a Hilbert space such that $0 < mI \leqslant A, B \leqslant MI$ for some scalars $m < M$. Then we show an upper bound of the difference of parallel sum and series :

$$(A+B)-(A:B) \leqslant 2\left(M+m-\sqrt{Mm}\right)I.$$

As an application, we show a noncommutative Kantorovich inequality: For positive operators A and B such that $0 < mI \leqslant A, B \leqslant MI$,

$$\frac{A+B}{2} \leqslant \frac{(M+m)^2}{4Mm}\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}$$

and moreover we show the following refinement:

$$\frac{2\sqrt{Mm}}{M+m}\,\frac{A+B}{2} \leqslant A \sharp B \leqslant \frac{M+m}{2\sqrt{Mm}}\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1},$$

where $A \sharp B$ is the geometric mean.

1. Introduction

Motivated by a study of electrical network connection, Anderson and Duffin [1] introduced the concept of parallel sum of two positive semidefinite matrices and sequently Anderson and Trapp [2] have extended this notion to positive operators on a Hilbert space H. If A and B are impedance matrices of two resistive n-port networks, then their parallel sum $A : B$ defined by

$$A : B = (A^{-1}+B^{-1})^{-1}$$

is the impedance matrix of parallel connection and their series

$$A+B$$

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is the impedance matrix of series connection. Some properties of parallel sum of two positive semidefinite matrices are discussed. For example, Anderson and Duffin [1] showed the following estimate of two impedance above: If $A_1, \cdots, A_n$ are positive semidefinite, then

$$\sum_{i=1}^{n} A_i \geqslant n^2 \prod_{i=1}^{n} : A_i, \tag{1}$$

where

$$\prod_{i=1}^{n} : A_i = A_1 : A_2 : \cdots : A_n.$$

In fact, the inequality (1) is a generalization of the classical inequality between the arithmetic mean and the harmonic mean.

Thus we consider upper bounds for the ratio and the difference between two impedance matrices above. We attempt to determine an upper estimate α such that

$$\sum_{i=1}^{n} A_i \leqslant \alpha \prod\nolimits_{i=1}^{n} : A_i$$

and an upper estimate β such that

$$\sum_{i=1}^{n} A_i - \prod\nolimits_{i=1}^{n} : A_i \leqslant \beta I.$$

We regard these constants as two types of energy loss of two impedance matrices.

Throughout this paper, we discuss parallel sum and series in the framework of operator theory on a Hilbert space.

Our purpose in this paper is to give upper bounds for two types of energy loss of two impedances in terms of the spectra for given positive operators on a Hilbert space, in which the Mond-Pečarić method for convex functions [4] is applied. As an application, we show a noncommuatative Kantorovich inequality.

2. Mond-Pečarić method

A capital letter means a bounded linear operator on a Hilbert space H. An operator A is said to be positive ($A \geqslant 0$) if $(Ax, x) \geqslant 0$ for all $x \in H$. We denote by $B(H)$ the algebra of all bounded linear operators on H.

In this section, we prove a few lemmas on positive linear maps to obtain upper bounds for the ratio and the difference between parallel sum and series of operator connections in the sense of Anderson-Duffin-Trapp [1, 2].

Let Φ be a normalized positive linear map on $B(H)$. Then it follows from [3, Corollary 4.2] that Jensen's operator inequality implies Kadison's Schwarz inequality as follows:

$$\Phi(A^{-1})^{-1} \leqslant \Phi(A) \tag{2}$$

for every positive invertible operator A.

By using the Mond-Pečarić method [4], we have the following reverse inequality of (2) without the assumption of the normalization of Φ.

LEMMA 1. *Let Φ be a positive linear map on $B(H)$ such that $\Phi(I) = kI$ for some positive scalar k. If A is a positive operator on a Hilbert space H such that $0 < mI \leqslant A \leqslant MI$ for some scalars $m < M$, then for each $\alpha > 0$*

$$\Phi(A) \leqslant \alpha\Phi(A^{-1})^{-1} + \beta(m,M,\alpha,k)I, \tag{3}$$

where

$$\beta(m,M,\alpha,k) = \begin{cases} k(m+M) - 2\sqrt{\alpha mM} & \text{if} \quad m \leqslant \frac{\sqrt{\alpha Mm}}{k} \leqslant M, \\ (k - \frac{\alpha}{k})M & \text{if} \quad \frac{\sqrt{\alpha Mm}}{k} \leqslant m, \\ (k - \frac{\alpha}{k})m & \text{if} \quad M \leqslant \frac{\sqrt{\alpha Mm}}{k}. \end{cases} \tag{4}$$

Proof. By the convexity of $f(t) = t^{-1}$, we have

$$A^{-1} \leqslant \frac{\frac{1}{M} - \frac{1}{m}}{M - m}(A - mI) + \frac{1}{m}I = -\frac{1}{Mm}A + \frac{M+m}{Mm}I$$

and hence

$$\Phi(A^{-1}) \leqslant -\frac{1}{Mm}\Phi(A) + \frac{M+m}{Mm}\Phi(I) = -\frac{1}{Mm}\Phi(A) + \frac{k(M+m)}{Mm}I$$

The last equality holds by the assumption of $\Phi(I) = kI$.

Therefore it follows that

$$\Phi(A) - \alpha\Phi(A^{-1})^{-1} \leqslant \Phi(A) - \alpha\left(-\frac{1}{Mm}\Phi(A) + \frac{k(M+m)}{Mm}I\right)^{-1}.$$

Since $kmI \leqslant \Phi(A) \leqslant kMI$, put

$$h(t) = t - \alpha\left(\frac{k(M+m) - t}{Mm}\right)^{-1} \quad \text{on} \quad [km, kM].$$

Then we have

$$h'(t) = \frac{(t - k(M+m))^2 - \alpha Mm}{(t - k(M+m))^2}.$$

It follows that the equation $h'(t) = 0$ has exactly one solution $t_0 = k(M+m) - \sqrt{\alpha Mm}$. If $km \leqslant t_0 \leqslant kM$, then we have $\max_{km \leqslant t \leqslant kM} h(t) = h(t_0)$ and the condition $km \leqslant t_0 \leqslant kM$ is equaivalent to the condition

$$m \leqslant \frac{\sqrt{\alpha Mm}}{k} \leqslant M.$$

If $kM \leqslant t_0$, then $h(t)$ is increasing on $[km, kM]$ and hence we have $\beta = \max_{km \leqslant t \leqslant kM} h(t) = h(t_0) = (k - \frac{\alpha}{k})M$ for $t_0 = kM$. Similarly, we have $\beta = \max_{km \leqslant t \leqslant kM} h(t) = h(t_0) = (k - \frac{\alpha}{k})m$ for $t_0 = km$. Since

$$\Phi(A) - \alpha\Phi(A^{-1})^{-1} \leqslant \max_{km \leqslant t \leqslant kM} h(t)I,$$

we have the desired inequality (3). □

REMARK 2. By the construction of $\beta(m,M,\alpha,k)$ in Lemma 1, it follows that for each $\alpha>0$

$$\left(k-\frac{\alpha}{k}\right)M\,,\ \left(k-\frac{\alpha}{k}\right)m\ \leqslant\ k(m+M)-2\sqrt{\alpha mM}$$

and hence

$$\beta(m,M,\alpha,k)\leqslant k(m+M)-2\sqrt{\alpha mM}.$$

By Lemma 1, we have the following upper bounds for the ratio and the difference in the inequality (2):

LEMMA 3. *Let Φ be a positive linear map on $B(H)$ such that $\Phi(I)=kI$ for some positive scalar k. If A is a positive operator on a Hilbert space H such that $0<mI\leqslant A\leqslant MI$ for some scalars $m<M$, then*

$$\Phi(A)\leqslant\frac{k^2(M+m)^2}{4Mm}\Phi(A^{-1})^{-1} \tag{5}$$

and

$$\Phi(A)-\Phi(A^{-1})^{-1}\leqslant(k(m+M)-2\sqrt{Mm})I. \tag{6}$$

Proof. If we choose α such that $k(m+M)-2\sqrt{\alpha mM}=0$ in (4) of Lemma 1, then it follows that $\alpha=\frac{k^2(M+m)^2}{4Mm}$ and α satisfies the condition $m\leqslant\frac{\sqrt{\alpha mM}}{k}\leqslant M$. Thus we have (5). Also, if we put $\alpha=1$ in (3) of Lemma 1, then it follows from Remark 2 that $\beta(m,M,1,k)\leqslant k(m+M)-2\sqrt{Mm}$ and hence we have (6). □

REMARK 4. If Φ is normalized, that is, $\Phi(I)=I$, then by Lemma 3 we have the following results due to Mond-Pečarić [8], cf. [4, Theorem 1.32]:

$$\Phi(A)\leqslant\frac{(M+m)^2}{4Mm}\Phi(A^{-1})^{-1} \tag{7}$$

and

$$\Phi(A)-\Phi(A^{-1})^{-1}\leqslant(\sqrt{M}-\sqrt{m})^2I. \tag{8}$$

REMARK 5. Let A be a positive operator such that $0<mI\leqslant A\leqslant MI$ and $\Phi(I)=kI$ for some positive scalar k. Then

$$kmI\leqslant\Phi(A)\leqslant kMI$$

and

$$\frac{m}{k}I\leqslant\Phi(A^{-1})^{-1}\leqslant\frac{M}{k}I.$$

Therefore it follows that

$$\Phi(A)-\alpha\Phi(A^{-1})^{-1}\leqslant(kM-\frac{\alpha m}{k})I.$$

However, it follows from easy calculation that for each $\alpha>0$

$$\beta(m,M,\alpha,k)\leqslant kM-\frac{\alpha m}{k},$$

where $\beta(m,M,\alpha,k)$ is defined as (4) of Lemma 1.

3. Main result

We state our main theorem, in which upper bounds for the ratio and the difference between parallel sum and series of operator connections are given.

THEOREM 6. *If A and B are positive operators on H such that $0 < mI \leqslant A$, $B \leqslant MI$ for some scalars $m < M$, then for each $\alpha > 0$*

$$A + B \leqslant \alpha(A : B) + \beta(m, M, \alpha, k = 2)I, \tag{9}$$

where

$$\beta(m, M, \alpha, k = 2) = \begin{cases} 2(m+M) - 2\sqrt{\alpha mM} & \text{if} \quad m \leqslant \frac{\sqrt{\alpha Mm}}{2} \leqslant M, \\ (2 - \frac{\alpha}{2})M & \text{if} \quad \frac{\sqrt{\alpha Mm}}{2} \leqslant m, \\ (2 - \frac{\alpha}{2})m & \text{if} \quad M \leqslant \frac{\sqrt{\alpha Mm}}{2}. \end{cases} \tag{10}$$

In particular,

$$A + B \leqslant \frac{(M+m)^2}{Mm}(A : B) \tag{11}$$

and

$$A + B - (A : B) \leqslant 2\left(M + m - \sqrt{Mm}\right) I. \tag{12}$$

Proof. Let a map $\Psi : B(H) \oplus B(H) \mapsto B(H) \oplus B(H)$ be defined by

$$\Psi \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} A+B & 0 \\ 0 & A+B \end{pmatrix}.$$

Then Ψ is a positive linear map such that $\Psi(I) = 2I$. Since

$$m \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leqslant \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \leqslant M \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

it follows from Lemma 1 that for each $\alpha > 0$

$$\Psi \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} - \alpha \Psi \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix}^{-1} \leqslant \beta(m, M, \alpha, k = 2) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

We have the desired inequality (9) by rearranging the expression above.

If we choose α such that $2((M+m) - \sqrt{\alpha Mm}) = 0$ in (9), then it follows that $\alpha = \frac{(M+m)^2}{Mm}$ and α satisfies the condition $m \leqslant \frac{\sqrt{\alpha Mm}}{2} \leqslant M$. Thus we have (11). Also, if we put $\alpha = 1$ in (9), then it follows that

$$\beta(m, M, \alpha = 1, k = 2) \leqslant 2\left(M + m - \sqrt{Mm}\right)$$

and hence we have (12). $\square$

Similarly, we have the following n-variable version of Theorem 6.

THEOREM 7. *If A_i are positive operators on H such that $0 < mI \leqslant A_i \leqslant MI$ for some scalars $m < M$ for $i = 1, 2, \cdots, n$, then for each $\alpha > 0$*

$$\sum_{i=1}^{n} A_i \leqslant \alpha \prod_{i=1}^{n} : A_i + \beta(m, M, \alpha, k = n)I, \tag{13}$$

where $\beta(m, M, \alpha, k = n)$ is defined as (4).

In particular,

$$\sum_{i=1}^{n} A_i \leqslant n^2 \frac{(M+m)^2}{4Mm} \prod_{i=1}^{n} : A_i \tag{14}$$

and

$$\sum_{i=1}^{n} A_i - \prod_{i=1}^{n} : A_i \leqslant \left(n(M+m) - 2\sqrt{Mm} \right) I. \tag{15}$$

Proof. Let a map $\Psi : B(H) \oplus \cdots \oplus B(H) \mapsto B(H) \oplus \cdots \oplus B(H)$ be defined by

$$\Psi \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{pmatrix} = \begin{pmatrix} A_1 + \cdots + A_n & & 0 \\ & \ddots & \\ 0 & & A_1 + \cdots + A_n \end{pmatrix}.$$

Then we can prove (13) by the same way as Theorem 6. □

4. Noncommutative Kantorovich inequality

Motivated by a study of parallel sum due to Anderson and Duffin [1], and Anderson and Trapp [2], Kubo and Ando [7] introduced the notion of operator mean. A map $(A, B) \to A\ \sigma\ B$ in the cone of positive invertible operators is called an operator mean if the following conditions are satisfied:

monotonicity $A \leqslant C$ and $B \leqslant D$ imply $A\ \sigma\ B \leqslant C\ \sigma\ D$,

upper continuity $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\ \sigma B_n \downarrow A\ \sigma\ B$,

transformer inequality $T^*(A\ \sigma\ B)T \leqslant (T^*AT)\ \sigma\ (T^*BT)$ for every operator T,

normalized condition $A\ \sigma\ A = A$.

A key for the theory is that there is a one-to-one correspondence between an operator mean σ and a nonnegative operator monotone function $f(x)$ on $[0, \infty)$ through the formula

$$f(x) = 1\ \sigma\ x \qquad \text{for all } x > 0,$$

or

$$A\ \sigma\ B = A^{\frac{1}{2}}(1\ \sigma\ A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = A^{\frac{1}{2}}f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} \quad \text{for all } A, B > 0.$$

We say that f is the representing function for σ. In this case, notice that $f(t)$ is operator monotone if and only if it is operator concave. The operator mean with representing function $tf(t^{-1})$ is called the transpose of σ and denoted by σ°:

$$A\ \sigma^\circ\ B = B\ \sigma\ A \qquad \text{for every positive } A \text{ and } B.$$

An operator mean is called symmetric if $\sigma = \sigma^\circ$. The operator mean with representing function $f(t^{-1})^{-1}$ is called the adjoint of σ and denoted by σ^*:

$$A\ \sigma^*\ B = (A^{-1}\ \sigma\ B^{-1})^{-1} \qquad \text{for every positive invertible } A \text{ and } B.$$

Simple examples of operator means are the arithmetic mean, in symbol ∇,

$$A \nabla B = \frac{A+B}{2}.$$

The normalized parallel sum is called the harmonic mean, in symbol $!$,

$$A\ !\ B = 2(A\ :\ B).$$

For invertible A, B, the geometric mean $A \sharp B$ is

$$A \sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}.$$

Then the following harmonic-geometric-arithmeric mean inequality holds

$$A\ !\ B \leqslant A \sharp B \leqslant A \nabla B.$$

Furthermore, the arithmetic mean is the maximum of all symmetric operator means while the harmonic mean is the minimum.

On the other hand, Kantorovich [6] proved the following inequality. If the sequence $\{a_i\}$ ($i = 1, 2, \cdots, n$) of positive numbers has the property

$$0 < m \leqslant a_i \leqslant M$$

and $\{x_i\}$ $(i = 1, 2, \cdots, n)$ denotes another sequence with $\sum_{i=1}^n x_i^2 = 1$, then the inequality

$$\sum_{i=1}^n a_i x_i^2 \ \sum_{i=1}^n \frac{1}{a_i} x_i^2 \ \leqslant \ \frac{(M+m)^2}{4Mm} \tag{16}$$

holds. In fact, the inequality (16) is considered as a ratio type reverse inequality of harmonic - arithmetic mean inequality

$$\left(\sum_{i=1}^n \frac{1}{a_i} x_i^2 \right)^{-1} \leqslant \sum_{i=1}^n a_i x_i^2.$$

Prof. S. Izumino suggested that Theorem 6 implies the following noncommutative Kantorovich inequality:

THEOREM 8. *If A and B are positive operators on H such that $0 < mI \leqslant A$, $B \leqslant MI$ for some scalars $m < M$, then*

$$A \nabla B \leqslant \frac{(M+m)^2}{4Mm} A\ !\ B \tag{17}$$

and

$$A \nabla B - A\ !\ B \leqslant (\sqrt{M} - \sqrt{m})^2 I. \tag{18}$$

Proof. It follows from the inequality (11) in Theorem 6 that

$$A+B \leqslant 4\frac{(M+m)^2}{4Mm}(A:B),$$

and hence

$$A \nabla B \leqslant \frac{(M+m)^2}{4Mm} A \,!\, B.$$

If we put $\alpha = 4$ in Theorem 6, then the condition $m \leqslant \sqrt{Mm} \leqslant M$ satisfies and

$$A+B \leqslant 4(A:B) + 2(M+m-2\sqrt{Mm})I.$$

Therefore, we have the desired inequality (18). □

As an application of Theorem 7, we have the following n-variable version of a noncommutative Kantorovich inequality. We use the notation

$$\prod_{i=1}^{n} ! A_i = A_1 \,!\, A_2 \,!\, \cdots \,!\, A_n = \left(\frac{A_1^{-1}+\cdots+A_n^{-1}}{n}\right)^{-1}.$$

THEOREM 9. *If A_i are positive operators on H such that $0 < mI \leqslant A_i \leqslant MI$ for some scalars $m < M$ for $i = 1, 2, \cdots, n$, then*

$$\frac{1}{n}\sum_{i=1}^{n} A_i \leqslant \frac{(M+m)^2}{4Mm} \prod_{i=1}^{n} ! A_i \tag{19}$$

and

$$\frac{1}{n}\sum_{i=1}^{n} A_i - \prod_{i=1}^{n} ! A_i \leqslant (\sqrt{M}-\sqrt{m})^2 I. \tag{20}$$

Proof. The inequality (19) follows from (14) in Theorem 7. If we put $\alpha = n^2$ in (13) of Theorem 7, then the condition $m \leqslant \frac{\sqrt{\alpha Mm}}{k} \leqslant M$ satisfies and $\beta(m, M, \alpha = n^2, k = n) = n(m+M-2\sqrt{mM})$. Therefore we have the desired inequality (20). □

REMARK 10. Prof. T. Furuta kindly pointed out that Theorem 9 is the special case where $r = -1$ and $s = 1$ in [9, Theorem 1] due to Pečarić and Mićić, also where $p = -1$ in [5, Theorem E] due to Furuta and Pečarić, which is one of typical examples applying the Mond-Pečarić method.

Furthermore we show a generalization of Theorem 8 by means of symmetric operator means.

THEOREM 11. *Let σ be a symmetric operator mean with the representing function f. If A and B are positive operators on H such that $0 < mI \leqslant A,\ B \leqslant MI$ for some scalars $m < M$, then*

$$\frac{m \,\sigma\, M}{m \nabla M} A \nabla B \leqslant A \,\sigma\, B \tag{21}$$

and

$$A \,\sigma^*\, B \leqslant \frac{m \nabla M}{m \,\sigma\, M} A \,!\, B. \tag{22}$$

Also,

$$A \nabla B - A \sigma B \leqslant M \left(\frac{m \nabla M}{m \sigma M} - 1 \right) I \tag{23}$$

and

$$A \sigma^* B - A \,!\, B \leqslant M \left(\frac{m \nabla M}{m \sigma M} - 1 \right) I. \tag{24}$$

To prove it, we need the following lemma.

LEMMA 12. *Let m and M be positive scalars. Then*

$$\frac{m \sigma^* M}{m \,!\, M} = \frac{m \nabla M}{m \sigma M}$$

for every symmetric operator mean σ.

Proof. Let f be the representing function for σ. Then it follows that

$$\begin{aligned} \frac{m \sigma^* M}{m \,!\, M} &= \frac{(m^{-1} \sigma M^{-1})^{-1}}{(m^{-1} \nabla M^{-1})^{-1}} = \frac{m^{-1} \nabla M^{-1}}{m^{-1} \sigma M^{-1}} \\ &= \frac{m+M}{2mM} \frac{m}{f\left(\frac{m}{M}\right)} = \frac{m \nabla M}{M \sigma m} \\ &= \frac{m \nabla M}{m \sigma M}. \end{aligned}$$

The last equality holds since σ is symmetric. □

Proof of Theorem 11. Since the representing function f is concave on $(0, \infty)$, it follows that

$$f(t) \geqslant \frac{f\left(\frac{M}{m}\right) - f\left(\frac{m}{M}\right)}{\frac{M}{m} - \frac{m}{M}} \left(t - \frac{m}{M} \right) + f\left(\frac{m}{M}\right) \qquad \text{for all } t \in \left[\tfrac{m}{M}, \tfrac{M}{m}\right].$$

Since $\frac{m}{M} I \leqslant A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \leqslant \frac{M}{m} I$, we have

$$f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \geqslant \frac{f\left(\frac{M}{m}\right) - f\left(\frac{m}{M}\right)}{\frac{M}{m} - \frac{m}{M}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} - \frac{m}{M} I \right) + f\left(\frac{m}{M}\right) I$$

and hence

$$\begin{aligned} A \sigma B = A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}} &\geqslant \frac{f\left(\frac{M}{m}\right) - f\left(\frac{m}{M}\right)}{\frac{M}{m} - \frac{m}{M}} \left(B - \frac{m}{M} A \right) + f\left(\frac{m}{M}\right) A \\ &= \frac{f\left(\frac{M}{m}\right) - f\left(\frac{m}{M}\right)}{\frac{M}{m} - \frac{m}{M}} B + \frac{\frac{M}{m} f\left(\frac{m}{M}\right) - \frac{m}{M} f\left(\frac{M}{m}\right)}{\frac{M}{m} - \frac{m}{M}} A \\ &= \frac{2\left(f\left(\frac{M}{m}\right) - f\left(\frac{m}{M}\right)\right)}{\frac{M}{m} - \frac{m}{M}} A \nabla B. \end{aligned}$$

The last equality holds since σ is symmetric, that is, $f(t) = tf(t^{-1})$. This relation also implies

$$\frac{2(f(\frac{M}{m}) - f(\frac{m}{M}))}{\frac{M}{m} - \frac{m}{M}} = \frac{2Mm}{M^2 - m^2}\left(1 - \frac{m}{M}\right) f\left(\frac{M}{m}\right) = \frac{2}{M+m}\, mf\left(\frac{M}{m}\right)$$
$$= \frac{m\ \sigma\ M}{m\ \nabla\ M},$$

and hence we have the desired inequality (21).

Replacing A by A^{-1} and B by B^{-1} in (21), it follows from $\frac{1}{M}I \leqslant A^{-1}, B^{-1} \leqslant \frac{1}{m}I$ that

$$\frac{m^{-1}\ \sigma\ M^{-1}}{m^{-1}\ \nabla\ M^{-1}}\, A^{-1}\ \nabla\ B^{-1} \leqslant A^{-1}\ \sigma\ B^{-1}.$$

Taking inverse of both sides, we have

$$\left(\frac{m^{-1}\ \sigma\ M^{-1}}{m^{-1}\ \nabla\ M^{-1}}\right)^{-1} (A^{-1}\ \nabla\ B^{-1})^{-1} \geqslant (A^{-1}\ \sigma\ B^{-1})^{-1}$$

and it follows from Lemma 12 that

$$A\ \sigma^*\ B \leqslant \frac{m\ \nabla\ M}{m\ \sigma\ M}\, A\ !\ B.$$

as desired.

It follows from the inequality (21) that

$$A\ \nabla\ B - A\ \sigma\ B \leqslant \left(\frac{m\ \nabla\ M}{m\ \sigma\ M} - 1\right) A\ \sigma\ B$$
$$\leqslant M\left(\frac{m\ \nabla\ M}{m\ \sigma\ M} - 1\right) I.$$

Similarly we have (24). □

As a special case of Theorem 11, we have the following refinement of Theorem 8.

THEOREM 13. *If A and B are positive operators on H such that $0 < mI \leqslant A$, $B \leqslant MI$ for some scalars $m < M$, then*

$$\frac{2\sqrt{Mm}}{M+m} A\ \nabla\ B \leqslant A\ \sharp\ B \leqslant \frac{M+m}{2\sqrt{Mm}} A\ !\ B \tag{25}$$

and

$$A\ \nabla\ B\ -\ \frac{(\sqrt{M}-\sqrt{m})^2}{2}\sqrt{\frac{M}{m}}I\ \leqslant\ A\ \sharp\ B\ \leqslant\ A\ !\ B\ +\ \frac{(\sqrt{M}-\sqrt{m})^2}{2}\sqrt{\frac{M}{m}}I \tag{26}$$

Proof. Since the geometric mean $\sharp$ is symmetric and selfadjoint, that is, $(\sharp)^* = \sharp = (\sharp)^\circ$, it follows from Theorem 11 if we put the representing function $f(t) = \sqrt{t}$. □

REMARK 14. The inequality (25) in Theorem 13 is a refinement of Corollary 5.39 in [4] if Φ is the identity map.

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