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MIXED QUASI REGULARIZED VARIATIONAL INEQUALITIES

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Abstract. In this paper, we introduce and study a new class of variational inequalities, known as mixed quasi regularized variational inequality in the setting of nonconvexity. We use the auxiliary principle technique to suggest and analyze some iterative schemes for regularized variational inequalities. We prove that the convergence of these iterative methods requires either pseudomonotonicity or partially relaxed strongly monotonicity. Our proofs of convergence are very simple. As special cases, we obtain earlier results for solving general variational inequalities involving the convex sets.

1. Introduction

Variational inequalities theory, which was introduced by Stampacchia [1], provides us with a simple, general and unified framework to study a wide class of problems arising in pure and applied sciences. During the last three decades, there has been considerable activity in the development of numerical techniques for solving variational inequalities. There are a substantial number of numerical methods including projection method and its variant forms, Wiener-Hopf equations, auxiliary principle, and descent framework for solving variational inequalities and complementarity problems; see [2-12]. It is worth mentioning that almost all the results regarding the existence and iterative schemes for variational inequalities , which have been investigated and considered so far, if the underlying set is a convex set. This is because all the techniques are based on the properties of the projection operator over convex sets, which may not hold in general, when the sets are nonconvex. To overcome these difficulties, one uses the concept of uniformly prox-regular (smooth sets), see [13, 14]. It is known that uniformly prox-regular sets are nonconvex sets and include convex sets as special case. In this paper, we introduce and consider a new class of variational inequalities, known as mixed quasi regularized variational inequality. These mixed quasi regularized variational inequalities are more general and include variational inequalities and related optimization problems as special case. Since the underlying set is a nonconvex set, it is not possible to extend the usual projection and resolvent techniques for solving regularized variational inequalities. Fortunately, these difficulties can be overcome by using the auxiliary principle, which is manily due to Glowinski, Lions and Tremolieres (Ref. 5). Noor

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(Refs. 6, 11, 15, 16) has used this technique to develop some iterative schemes for solving various classes of variational inequalities. We point out that this technique does not involve the projection or resolvent of the operator and is flexible. In this paper, we show that the auxiliary principle technique can be used to suggest and analyze a class of iterative methods for solving regularized (nonconvex) variational inequalities. We also prove that the convergence of these new methods either require pseudomonotonicity or partially relaxed strongly monotonicity. As special cases, one obtain several known and new results for variational inequalities and related optimization problems. Results obtained in this paper, represent an improvement and refinement of the known results for nonconvex variational inequalities.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot,\cdot\rangle$ and $\|.\|$ respectively. Let K be a nonempty closed convex set in H. First of all, we recall the following well-known concepts from nonlinear convex analysis, see [13, 14].

DEFINITION 2.1. The proximal normal cone of K at u is given by

$$N^P(K;u) := \{\xi \in H : u \in P_K[u+\alpha\xi]\},$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in K : d_K(u) = \|u-u^*\|\}.$$

Here $d_K(.)$ is an usual distance function to the subset K, that is

$$d_K(u) = \inf_{v\in K} \|v-u\|.$$

The proximal normal cone $N^P(K;u)$ has the following characterization.

LEMMA 2.1. *Let K be a closed subset in H. Then $\zeta \in N^P(K;u)$ if and only if there exists a constant $\alpha > 0$ such that*

$$\langle \zeta, v-u\rangle \leqslant \alpha\|v-u\|^2, \quad \forall v \in K.$$

DEFINITION 2.2. The Clarke normal cone, denoted by $N^C(K;u)$, is defined as

$$N^C(K;u) = \overline{co}[N^P(K;u)],$$

where $\overline{co}$ means the closure of the convex hull. Clearly $N^P(K;u) \subset N^C(K;u)$, but the converse is not true. Note that $N^C(K;u)$ is always closed and convex, whereas $N^P(K;u)$ is convex, but may not be closed, see [14]. Poliquin et al [14] and Clarke et al [13] have introduced and studied a new class of nonconvex sets, which are also called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions. In particular, we have:

DEFINITION 2.3. For a given $r \in (0,\infty]$, a subset K is said to be normalized uniformly r-prox-regular if and only if every nonzero proximal normal to K can be realized by an r-ball, that is, $\forall u \in K$ and $0 \neq \xi \in N^P(K;u)$ with $\|\xi\| = 1$, one has

$$\langle \xi, v-u\rangle \leqslant (1/2r)\|v-u\|^2, \quad \forall v \in K.$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see [13, 14]. It is clear that if $r = \infty$, then uniform r-prox-regularity of K is equivalent to the convexity of K. This fact plays an important part in this paper. It is known that if K is a uniformly r-prox-regular set, then the proximal normal cone $N^P(K;u)$ is closed as a set-valued mapping. Thus, we have $N^C(K;u) = N^P(K;u)$. For sake of simplicity, we denote $N(K;u) = N^C(K;u) = N^P(K;u)$. and take $\gamma = \frac{1}{2r}$. Clearly, if $r = \infty$, then $\gamma = 0$.

From now onward, the set K is uniformly prox-regular set, unless otherwise specified.

For given nonlinear continuous operators $T, g :\longrightarrow H$, and a continuous bifunction $\varphi(.,.) : H \times H \longrightarrow R \cup \{+\infty\}$, we consider the problem of finding $u \in H : g(u) \in K$ such that

$$\langle Tu, g(v)-g(u)\rangle + \varphi(g(v),g(u)) - \varphi(g(u),g(u)) + \gamma\|g(v)-g(u)|^2 \geqslant 0, \ \forall g(v) \in K, \tag{2.1}$$

which is called the *general mixed quasi regularized variational inequality.*

Note that, if $\gamma = 0$, then uniformly prox-regular set K becomes the convex set K and consequently problem (2.1) reduces to finding $u \in H : g(u) \in K$ such that

$$\langle Tu, g(v)-g(u)\rangle + \varphi(g(v),g(u)) - \varphi(g(u),g(u)) \geqslant 0, \ \forall g(v) \in K, \tag{2.2}$$

which is known as general mixed quasi variational inequality and has been studied extensively in recent years.

If $\varphi(v,u) = \varphi(v)$, $\forall v \in H$, then problem (2.1) is equivalent to finding $u \in H$: $g(u) \in K$ such that

$$\langle Tu, g(v)-g(u)\rangle + \varphi(g(v)) - \varphi(g(u)) + \gamma\|g(v)-g(u)\|^2 \geqslant 0, \ \forall g(v) \in K, \tag{2.3}$$

which is called the regularized mixed variational inequality, introduced and studied by Noor [16, 18] using the auxiliary principle technique and resolvent operator method.

If $\varphi(.)$ is the indicator function of the set K, then problem (2.3) reduces to finding $u \in H : g(u) \in K$ such that

$$\langle Tu, g(v)-g(u)\rangle + \gamma\|g(v)-g(u)\|^2 \geqslant 0, \ \forall g(v) \in K, \tag{2.4}$$

is called the regularized variational inequality, introduced and studied by Noor [16, 18, 19] using the auxiliary principle technique respectively.

In particular, if $\gamma = 0$, then problem (2.4) is eaxactly the general variational inequality problem introduced and studied by Noor [6] in 1988. For the recently applications, numerical methods, sensitivity analysis and local uniqueness of solutions of variational inequalities, see [1-20] and the references therein. For $g = I$, the indentity operator, we obtain the various classes of classical variational inequalities, which have been studied extensively.

3. Main results

We use the auxiliary principle technique, which is mainly due to Glowinski, Lions and Tremolieres [5] as developed by Noor [6, 11, 16], to suggest and analyze some iterative methods for mixed quasi regularized variational inequalities (2.1).

For a given $u \in K$, where K is a prox-regular set in H, consider the problem of finding a solution $w \in H : g(w) \in K$ such that

$$\begin{aligned} \langle \rho Tw + g(w) - g(u), g(v) - g(w) \rangle &\geqslant -\gamma \|g(v) - g(w)\|^2 \\ &+ \rho\{\varphi(g(w), g(w)) - \varphi(g(v), g(w))\}, \quad \forall g(v) \in K, \end{aligned} \tag{3.1}$$

where $\rho > 0$ is a constant. Inequality of type (3.1) is called the auxiliary mixed quasi regularized variational inequality. Note that if $w = u$, then w is a solution of (2.1). This simple observation enables us to suggest the following iterative method for solving (2.1).

Algorithm 3.1. For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned} \langle \rho Tu_{n+1} + g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1}) \rangle &\geqslant -\gamma \|g(u_{n+1}) - g(v)\|^2 \\ &+ \rho\{\varphi(g(u_{n+1}), g(u_{n+1}))) - \varphi(g(v), g(u_{n+1}))\} \ \forall g(v) \in K. \end{aligned} \tag{3.2}$$

Algorithm 3.1 is called the proximal point algorithm for solving mixed quasi regularized variational inequalities (2.1). In particular, if $\gamma = 0$, then the prox-regular set K becomes the standard convex set K, and consequently Algorithm 3.1 reduces to:

Algorithm 3.2. For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes

$$\begin{aligned} &\langle \rho Tu_{n+1} + g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1}) \rangle \\ &\qquad \geqslant \rho\{\varphi(g(u_{n+1}), g(u_{n+1})) - \varphi(g(v), g(u_{n+1}))\}, \ \forall g(v) \in K. \end{aligned}$$

In brief, for suitable and appropriate choice of the operators and spaces, one can obtain several new and previously known methods for solving different classes of variational inequalities and related optimization problems.

For the convergence analysis of Algorithm 3.1, we recall the following concepts and results.

DEFINITION 3.1. For all $u, v, z \in H$, an operator $T : H \to H$ is said to be:
(i) g*-monotone*, if

$$\langle Tu - Tv, g(u) - g(v) \rangle \geqslant 0.$$

(ii) *g-pseudomonotone with respect to the bifunction* $\varphi(.,.)$, if

$$\langle Tu, g(v)-g(u)\rangle + \varphi(g(v),g(u)) - \varphi(g(u),g(u)) \geqslant 0$$
$$\Longrightarrow$$
$$\langle Tv, g(v)-g(u)\rangle + \varphi(g(v),g(u)) - \varphi(g(u),g(u)) \leqslant 0.$$

(iii) *partially relaxed strongly g-monotone,* if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, g(z)-g(v)\rangle \geqslant -\alpha\|g(z)-g(u)\|^2.$$

Note that for $z = u$, partially relaxed strongly g-monotonicity reduces to g-monotonicity. It is known [11] that cocoercivity implies partially relaxed strongly monotonicity, but the converse is not true. It is known [3] that monotonicity implies pseudomonotonicity; but the converse is not true. Consequently, the class of pseudomonotone operators is bigger than the one of monotone operators.

DEFINITION 3.2. The bifunction $\varphi(.,.) : H \times H \longrightarrow R \cup \{+\infty\}$ is called *skew-symmetric,* if and only if,

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) - \varphi(v,v) \geqslant 0, \ \forall u,v \in H.$$

Clearly if the skew-symmetric bifunction $\varphi(.,.)$ is bilinear,

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) = \varphi(u-v,u-v) \geqslant 0, \ \forall u,v \in H.$$

LEMMA 3.1. $\forall u,v \in H,$

$$2\langle u,v\rangle = \|u+v\|^2 - \|u\|^2 - \|v\|^2. \tag{3.3}$$

We now consider the convergence criteria of Algorithm 3.1. The analysis is in the spirit of Noor [11, 16, 19].

THEOREM 3.1. *Let $u \in H : g(u) \in K$ be a solution of (2.1) and let u_{n+1} be the approximate solution obtained from Algorithm 3.1. If the operator T is g-pseudomonotone with respect to the bifunction $\varphi(.,.)$ and the bifunction $\varphi(.,.)$ is skew-symmetric, then*

$$\{1-\gamma\}\|g(u_{n+1})-g(u)\|^2 \leqslant \|g(u_n)-g(u)\|^2 - \{1-\gamma\}\|g(u_{n+1})-g(u_n)\|^2. \tag{3.4}$$

Proof. Let $u \in H : g(u) \in K$ be a solution of (2.1). Then

$$\langle Tu, g(v)-g(u)\rangle + \gamma\|g(v)-g(u)\|^2 \geqslant \varphi(g(u),g(u)) - \varphi(g(v),g(u)), \ \forall g(v) \in K. \tag{3.5}$$

Now taking $v = u_{n+1}$ in (3.5), we have

$$\langle Tu, g(u_{n+1})-g(u)\rangle + \gamma\|g(u_{n+1})-g(u)\|^2 \geqslant \varphi(g(u),g(u)) - \varphi(g(u_{n+1}),g(u)),$$

which implies that

$$\langle Tu_{n+1}, g(u_{n+1})-g(u)\rangle + \gamma\|g(u_{n+1})-g(u)\|^2 \geqslant \varphi(g(u),g(u)) - \varphi(g(u_{n+1}),g(u)), \tag{3.6}$$

since T is g-pseudomonotone operator with respect to the bifunction $\varphi(.,.)$.

Taking $v = u$ in (3.2), we get

$$\langle \rho Tu_{n+1} + g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle \\ \geqslant -\gamma\|g(u) - g(u_{n+1})\|^2\rho\{\varphi(g(u_{n+1}), g(u_{n+1})) - \varphi(g(u0, g(u_{n+1}))\},$$

which can be written as

$$\begin{aligned} &\langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle \\ &\quad\geqslant \rho\langle Tu_{n+1}, g(u_{n+1}) - g(u)\rangle + \rho\{\varphi(g(u_{n+1}), g(u_{n+1})) \\ &\qquad - \varphi(g(u), g(u_{n+1}))\} - \gamma\|g(u) - g(u_{n+1})\|^2 \\ &\quad\geqslant -\gamma\|g(u) - g(u_{n+1})\|^2 + \rho\{\varphi(g(u), g(u)) - \varphi(g(u), g(u_{n+1})) \\ &\qquad - \varphi(g(u_{n+1}), g(u)) + \varphi(g(u_{n+1}), g(u_{n+1}))\} - \gamma\|\|g(u_{n+1}) - g(u)\|^2, \end{aligned} \tag{3.7}$$

where we have used (3.6) and the fact thatthe bifunction $\varphi(.,.)$ is skew-symmetric.

Setting $u = u - u_{n+1}$ and $v = u_{n+1} - u_n$ in (3.3), we obtain

$$2\langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle = \|g(u) - g(u_n)\|^2 - \|g(u) - g(u_{n+1})\|^2 - \|u_{n+1} - u_n\|^2. \tag{3.8}$$

Combining (3.7) and (3.8), we have

$$\{1-\gamma\}\|g(u_{n+1}) - g(u)\|^2 \leqslant \|g(u_n) - g(u)\|^2 - \{1-\gamma\}\|g(u_{n+1}) - g(u_n)\|^2,$$

the required result, (3.4). □

THEOREM 3.2. *Let H be a finite dimensional space. If $\gamma \leqslant 1$, then the sequence $\{u_n\}_1^\infty$ given by Algorithm 3.1 converges to a solution u of (2.1), provided the operator g is bijective.*

Proof. Let $u \in K : g(u) \in K$ be a solution of (2.1). From (3.4), it follows that the sequence $\{||g(u) - g(u_n)||\}$ is nonincreasing and consequently $\{u_n\}$ is bounded under the assumptions of the Theorem. Furthermore, we have

$$\sum_{n=0}^{\infty}\{1-\gamma)\}\|g(u_{n+1}) - g(u_n)\|^2 \leqslant \|g(u_0) - g(u)\|^2,$$

which implies that

$$\lim_{n\to\infty}\|g(u_{n+1}) - g(u_n)\| = 0. \tag{3.9}$$

Let $\hat{u}$ be the limlit point of $\{u_n\}_1^\infty$; a subsequence $\{u_{n_j}\}_1^\infty$ of $\{u_n\}_1^\infty$ converges to $\hat{u} \in H$. Replacing w_n by u_{n_j} in (3.2), taking the limit $n_j \longrightarrow \infty$ and using (3.9), we have

$$\langle T\hat{u}, g(v) - g(\hat{u})\rangle + \gamma\|g(v) - g(\hat{u})\| \geqslant \varphi(g(\hat{u}), g(\hat{u})) - \varphi(g(v), g(\hat{u})), \quad \forall g(v) \in K,$$

which implies that $\hat{u}$ solves the regularized mixed quasi variational inequality (2.1) and

$$\|g(u_{n+1}) - g(u)\|^2 \leqslant \|g(u_n) - g(u)\|^2.$$

Thus, it follows from the above inequality that $\{u_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty} g((u_n)) = g(\hat{u}).$$

which implies that $\lim_{n\longrightarrow\infty} u_n = \hat{u},$ since g is bijective. □

It is well-known that to implement the proximal point methods, one has to calculate the approximate solution implicitly, which is in itself a difficult problem. To overcome this drawback, we suggest another iterative method, the convergence of which requires only the partially relaxed strongly monotonicity, which is a weaker condition that cocoercivity.

For a given $u \in H : g(u) \in K,$ consider the problem of finding $w \in H : g(w) \in K$ such that

$$\begin{aligned}\langle \rho Tu + g(w) - g(u), g(v) - g(w)\rangle &+ \gamma\|g(v) - g(w)\|^2 \\ &\geqslant \rho\{\varphi(g(w), g(w)) - \varphi(g(v), g(w)),\ \forall g(v) \in K,\end{aligned} \tag{3.10}$$

which is also called the auxiliary mixed quasi regularized variational inequality. Note that problems (3.1) and (3.10) are quite different. If $w = u,$ then clearly w is a solution of the regularized mixed quasi variational inequality (2.1). This fact enables us to suggest and analyze the following iterative method for solving (2.1).

Algorithm 3.3. For a given $u_0 \in K,$ compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned}\langle \rho Tu_n + g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle &\geqslant -\gamma\|g(v) - g(u_{n+1})\|^2 \\ + \rho\{\varphi(g(u_{n+1}), g(u_{n+1})) &- \varphi(g(v), g(u_{n+1}))\}, \forall g(v) \in K.\end{aligned} \tag{3.11}$$

Note that for $\gamma = 0,$ the prox-regular set K becomes a convex set K and Algorithm 3.3 reduces to:

Algorithm 3.4. For a given $u_0 \in K,$ calculate the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned}&\langle \rho Tu_n + g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle \\ &\qquad \geqslant \rho\{\varphi(g(u_{n+1}), g(u_{n+1})) - \varphi(g(v), g(u_{n+1}))\}, \forall g(v) \in K.\end{aligned}$$

In a similar way, for suitable and appropriate choice of the operators and spaces, one can obtain a number of new and known algorithms for solving various classes of (regularized) variational inequalities and related optimization problems.

We now study the convergence of Algorithm 3.3. The analysis is in the spirit of Theorem 3.1. For the sake of completeness and to convey an idea of the technique involved, we sketch the main points.

THEOREM 3.3. *Let the operator T be partially relaxed strongly monotone with constant $\alpha > 0$ and the bifunction $\varphi(.,.)$ be skew-symmetric. If u_{n+1} is the approximate solution obtained from Algorithm* 3.3 *and $u \in H : g(u) \in K$ is a solution of* (2.1)*, then*

$$\{1-\gamma\}\|g(u)-g(u_{n+1})\|^2 \leqslant \|g(u)-g(u_n)\|^2 - \{1-2\rho\alpha-\gamma\}\|g(u_n)-g(u_{n+1})\|^2. \tag{3.12}$$

Proof. Let $u \in H : g(u) \in K$ be a solution of (2.1). Then

$$\langle Tu, g(v)-g(u)\rangle + \gamma\|g(v)-g(u)\|^2 \geqslant \varphi g(u), g(u)) - \varphi(g(v), g(u)), \ \forall g(v) \in K. \tag{3.13}$$

Taking $v = u_{n+1}$ in (3.13), we have

$$\langle Tu, g(u_{n+1})-g(u)\rangle + \gamma\|g(u_{n+1})-g(u)\|^2 \geqslant \varphi(g(u), g(u)) - \varphi(g(u_{n+1}), g(u)). \tag{3.14}$$

Letting $v = u$ in (3.11), we obtain

$$\begin{aligned}&\langle \rho Tu_n + g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle \\ &\qquad \geqslant -\gamma\|g(u) - g(u_{n+1})\|^2 + \rho\{\varphi(g(u_{n+1}), g(u_{n+1})) - \varphi(g(u), g(u_{n+1}))\},\end{aligned}$$

which implies that

$$\begin{aligned}&\langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle \\ &\qquad \geqslant \langle \rho Tu_n, u_{n+1} - u\rangle - \gamma\|g(u) - g(u_{n+1})\|^2 \\ &\qquad\quad + \rho\{\varphi(g(u_{n+1}), g(u_{n+1})) - \varphi(g(u), g(u_{n+1}))\}, \\ &\qquad \geqslant \rho\langle Tu_n - Tu, g(u_{n+1}) - g(u_n)\rangle - \gamma\|g(u) - g(u_{n+1})\|^2 \\ &\qquad\quad + \rho\{\varphi(g(u), g(u)) - \varphi(g(u), g(u_{n+1})) - \varphi(g(u_{n+1}), g(u)) \\ &\qquad\quad + \varphi(g(u_{n+1}), g(u_{n+1}))\} - \gamma\|g(u_n) - g(u_{n+1})\|^2 \\ &\qquad \geqslant -\alpha\rho\|g(u_n) - g(u_{n+1})\|^2 \\ &\qquad\quad - \gamma\|g(u) - g(u_{n+1})\|^2 - \gamma\|g(u_n) - g(u_{n+1})\|^2,\end{aligned} \tag{3.15}$$

where we have used the fact that T is partially relaxed strongly g-monotone with constant $\alpha > 0$ and the bifunction $\varphi(.,.)$ is skew-symmetric.

Combining (3.8) and (3.15), we obtain the required result (3.12). □

Using essentially the technique of Theorem 3.2, one can study the convergence analysis of Algorithm 3.3.

REMARK 3.1. In this paper, we have shown that the auxiliary principle technique can be extended for solving mixed quasi regularized variational inequalities with suitable modifications. We note that this technique is independent of the projection and the resolvent of the operator. Moreover, we have studied the convergence analysis of these new methods under some mild conditions. It is worth mentioning that the ideas and techniques developed in this paper can be extended for mixed reularized equilibrium problems, which is the subject of future research.

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