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DUALITY IN NONLINEAR COMPLEMENTARITY THEORY BY USING INVERSIONS AND SCALAR DERIVATIVES

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Abstract. The notion of infinitesimal exceptional family of elements will be introduced. By using a special inversion mapping a duality between the exceptional family of elements and the infinitesimal exceptional family of elements will be presented. By using this duality and the notion of scalar derivatives existence theorems for complementarity problems in Hilbert spaces will be presented. Remark (important!): The notion of duality will be introduced not for the sake of "playing" with a new notion, but in order to prove Theorem 8.8, which provides a powerful tool for solving complementarity problems.

1. Introduction

In 1995 G. Isac introduced a new topological method in Complementarity Theory. This method is based on the notion of exceptional family of elements (EFE), which is related to the topological degree and to the Leray-Schauder alternative.

The notion of (EFE) was presented in a talk given at the Institute of Applied Mathematics of Academia SINICA (Beijing, China). This new topological method was published in 1997 [14]. After 1997 many papers, based on this method, have been published [1]-[3], [6], [8]-[19], [22], [23], [25], [28]-[39]. The main result presented in [14] is the following theorem: "If $(H, \langle \cdot, \cdot \rangle)$ is a Hilbert space, $K \subset H$ a closed convex cone and $f : H \to H$ is a completely continuous field, then either the complementarity problem defined by K and f has a solution, or f is without (EFE)."

This theorem shows that it is very important to know when a given mapping is without (EFE). Several classes of mappings with this property were presented in [1]-[3], [6], [8]-[19], [22], [23], [25], [28]-[39].

We note that for a mapping the nonexistence of (EFE) is a kind of very general coercivity condition.

In this paper we present the notion of "infinitesimal exceptional family of element (IEFE)". This notion was defined here by S. Z. Németh.

If we consider the couple ((EFE), (IEFE)) we remark a kind of duality, through a special inversion function. By this duality we put in evidence new classes of mappings

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for which the complementarity problem has a solution. Some interesting relations between the solvability of the complementarity problem and the scalar derivative are also established. The scalar derivative was introduced in [26] and studied in several papers, as for example [26], [27] and [20] among others.

Remark (important!): The duality in this paper is introduced not for the sake of "playing" with a new notion, it provides a tool for finding new solutions of the complementarity problems by using conditions expressed with the scalar derivatives (see Theorem 8.8). By using the computational formulae of Theorem 4.5, in many cases these conditions can be verified explicitly. We also remark that we used the term of duality inspired by the duality related to inversive geometry, subject which first appears in the work of the famous ancient geometer Apollonius of Perga, whose works have had a very great influence on the development of mathematics, in particular his famous book Conics introduced terms which are familiar to us today such as parabola, ellipse and hyperbola. Therefore - although different from the familiar notion of duality in nonlinear programming - we consider that our term of duality has a well founded root in the history of mathematics and especially duality in geometry. However, if a more appropriate term is suggested by anybody we would be happy to replace our term.

This paper could open a challenging new research direction in Complementarity Theory.

2. Preliminaries

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $K \subset H$ a nonempty closed set. Consider the following axioms:

1. $K + K \subseteq K$;
2. $\lambda K \subseteq K$, for all $\lambda \in \mathbb{R}_+$.

If K satisfies 2., then it is called a *closed wedge*. If K satisfies 1. and 2., then it called a *closed convex cone*.

Suppose that K is a closed convex cone. The *dual cone* K^* of K is the closed convex cone defined by $K^* = \{y \in H \mid \langle x, y \rangle \geqslant 0 \text{ for all } x \in K\}$.

We say that the mapping $T : H \to H$ is *completely continuous*, if it is continuous and for any bounded set $B \subset H$, $T(B)$ is relatively compact. A completely continuous field on H is a mapping $f : H \to H$ such that $f = I - T$, where I is the identical mapping of H (i.e., $I(x) = x$, for all $x \in H$) and T is a completely continuous mapping. In the particular case $H = \mathbb{R}^n$, any continuous mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is a completely continuous field, since $f = I - (I - f)$.

3. Complementarity problem

DEFINITION 3.1. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $f : K \to H$ a mapping. The *Nonlinear Complementarity Problem* defined by f and the cone K is

$$NCP(f, K) : \begin{cases} \text{find } x_* \in K \text{ such that} \\ f(x_*) \in K^* \text{ and } \langle x_*, f(x_*) \rangle = 0. \end{cases}$$

4. Scalar derivatives

Denote by $(H, \langle \cdot, \cdot \rangle)$ a Hilbert space and $\| \cdot \|$ the norm generated by $\langle \cdot, \cdot \rangle$. Let $C_1, C_2 \subseteq H$ such that 0 is a non-isolated point of C_1 and x_0 a non-isolated point of C_2. Let $F, G: C_2 \to H$.

The following definition is an extension of Definition 2.2 [26]:

DEFINITION 4.1. The limit

$$\underline{F}^{\#}(x_0, C_1) = \liminf_{\substack{x \to x_0 \\ x - x_0 \in C_1}} \frac{\langle F(x) - F(x_0), x - x_0 \rangle}{\|x - x_0\|^2}$$

is called the *lower scalar derivative of f in* x_0 *in the direction of* C_1. Taking $\limsup$ in place of $\liminf$, we can define the upper scalar derivative of F at x_0 in the direction of C_1 similarly. It will be denoted by $\overline{F}^{\#}(x_0, C_1)$. If $C_1 = H$ or $C_1 = C_2$ and $x_0 = 0$, then without confusion, we can omit the phrase “in the direction of C_1” from the definitions. In this case, we omit C_1 from the corresponding notations.

Definition 4.1 can be extended for the unordered pair of mappings (F, G). The idea was inspired by the notion of derivative of a function with respect to another function [5].

DEFINITION 4.2. The limit

$$\underline{(F, G)}^{\#}(x_0, C_1) = \liminf_{\substack{x \to x_0 \\ x - x_0 \in C_1}} \frac{\langle F(x) - F(x_0), G(x) - G(x_0) \rangle}{\|x - x_0\|^2}$$

is called the *lower scalar derivative of the (unordered) pair of mappings* (F, G) *in* x_0 *in the direction of* C_1. Taking $\limsup$ in place of $\liminf$, we can define the upper scalar derivative of rank p of the (unordered) pair of mappings (F, G) at x_0 in the direction of C_1 similarly. It will be denoted by $\overline{(F, G)}^{\#}(x_0, C_1)$. If $C_1 = H$ or $C_1 = C_2$ and $x_0 = 0$, then without confusion, we can omit the phrase "in the direction of C_1" from the definitions. In this case, we omit C_1 from the corresponding notations.

REMARK 4.1. If $G = I$, we obtain Definition 4.1.

THEOREM 4.1. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space,* $x \in H$ *and* $K \subseteq H$ *a closed wedge. If* $F, G: H \to H$ *are Frechét differentiable in* x, *with the differentials* $dF(x), dG(x)$, *respectively, then*

$$\underline{(F, G)}^{\#}(x, K) = \underline{(dF(x), dG(x))}^{\#}(0, K),$$
$$\overline{(F, G)}^{\#}(x, K) = \overline{(dF(x), dG(x))}^{\#}(0, K),$$

Proof. For the expressions

$$\frac{F(x + v) - F(x) - dF(x)(v)}{\|v\|} = \omega(F)(x, v)$$

and

$$\frac{G(x+v)-G(x)-dG(x)(v)}{\|v\|}=\omega(G)(x,v)$$

we have $\lim_{v\to 0}\omega(F)(x,v)=0$ and $\lim_{v\to 0}\omega(G)(x,v)=0$, hence

$$\begin{aligned}
\underline{(F,G)}^{\#}(x,K) &= \liminf_{\substack{v\to 0\\ v\in K}}\frac{\langle F(x+v)-F(x),G(x+v)-G(x)\rangle}{\|v\|^2}\\
&= \liminf_{\substack{v\to 0\\ v\in K}}\left\langle \omega(F)(x,v)+dF(x)\left(\frac{v}{\|v\|}\right),\omega(G)(x,v)+dG(x)\left(\frac{v}{\|v\|}\right)\right\rangle\\
&= \lim_{\substack{v\to 0\\ v\in K}}\langle\,\omega(F)(x,v),\omega(G)(x,v)\rangle+\lim_{\substack{v\to 0\\ v\in K}}\left\langle \omega(F)(x,v),dG(x)\left(\frac{v}{\|v\|}\right)\right\rangle\\
&\quad+\lim_{\substack{v\to 0\\ v\in K}}\left\langle dF(x)\left(\frac{v}{\|v\|}\right),\omega(G)(x,v)\right\rangle\\
&\quad+\liminf_{\substack{v\to 0\\ v\in K}}\left\langle dF(x)\left(\frac{v}{\|v\|}\right),dG(x)\left(\frac{v}{\|v\|}\right)\right\rangle.
\end{aligned}$$

In the last relation we used the fact that the first three limits exist since

$$|\langle\,\omega(F)(x,v),\omega(G)(x,v)\rangle\,|\leqslant\|\omega(F)(x,v)\|\cdot\|\omega(G)(x,v)\|\to 0 \text{ for } v\to 0, v\in K,$$
$$\left|\left\langle\omega(F)(x,v),dG(x)\left(\frac{v}{\|v\|}\right)\right\rangle\right|\leqslant\|\omega(F)(x,v)\|\cdot\|d(G)(x)\|\to 0 \text{ for } v\to 0, v\in K,$$

and

$$\left|\left\langle dF(x)\left(\frac{v}{\|v\|}\right),\omega(G)(x,v)\right\rangle\right|\leqslant\|d(F)(x)\|\cdot\|\omega(G)(x,v)\|\to 0 \text{ for } v\to 0, v\in K.$$

Hence,

$$\underline{(F,G)}^{\#}(x)=\liminf_{\substack{v\to 0\\ v\in K}}\frac{\langle\,dF(x)(v),dG(x)(v)\rangle}{\|v\|^2}=\underline{(dF(x),dG(x))}^{\#}(0,K).$$

A similar way yields the proof of the second relation of the theorem.

Particularly we have as follows:

THEOREM 4.2. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $x\in H$ and $K\subseteq H$ a closed wedge. If $f:H\to H$ is Frechét differentiable in x, with the differential $dF(x)$, then*

$$\underline{f}^{\#}(x,K)=\underline{dF(x)}^{\#}(0,K),$$

$$\overline{f}^{\#}(x,K)=\overline{dF(x)}^{\#}(0,K),$$

THEOREM 4.3. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space and* $K \subseteq H$ *a closed wedge. If* $\Phi_1, \Phi_2 : H \to H$ *are positively homogeneous, then*

$$\underline{(\Phi_1, \Phi_2)}^{\#}(0, K) = \inf_{\substack{\|u\|=1 \\ u \in K}} \langle \Phi_1(u), \Phi_2(u) \rangle ,$$

$$\overline{(\Phi_1, \Phi_2)}^{\#}(0, K) = \sup_{\substack{\|u\|=1 \\ u \in K}} \langle \Phi_1(u), \Phi_2(u) \rangle ,$$

Proof. We prove only the first equality. The second equality can be proved similarly. We have

$$\begin{aligned} \underline{(\Phi_1, \Phi_2)}^{\#}(0, K) &= \liminf_{\substack{v \to 0 \\ v \in K}} \frac{\langle \Phi_1(v), \Phi_2(v) \rangle}{\|v\|^2} \\ &\leqslant \liminf_{t \to 0} \frac{\langle \Phi_1(tu), \Phi_2(tu) \rangle}{\|tu\|^2} \\ &= \langle \Phi_1(u), \Phi_2(u) \rangle , \end{aligned}$$

for all $u \in K$ with $\|u\| = 1$. Therefore,

$$\underline{\Phi}^{\#}(0, K) \leqslant \inf_{\substack{\|u\|=1 \\ u \in K}} \langle \Phi_1(u), \Phi_2(u) \rangle .$$

Conversely,

$$\frac{\langle \Phi_1(v), \Phi_2(v) \rangle}{\|v\|^2} = \left\langle \Phi_1 \left(\frac{v}{\|v\|} \right), \Phi_2 \left(\frac{v}{\|v\|} \right) \right\rangle \geqslant \inf_{\substack{\|u\|=1 \\ u \in K}} \langle \Phi_1(u), \Phi_2(u) \rangle ,$$

for all $v \in K \setminus \{0\}$, which implies

$$\underline{\Phi}^{\#}(0, K) = \liminf_{\substack{v \to 0 \\ v \in K}} \frac{\langle \Phi_1(v), \Phi_2(v) \rangle}{\|v\|^2} \geqslant \inf_{\substack{\|u\|=1 \\ u \in K}} \langle \Phi_1(u), \Phi_2(u) \rangle .$$

Hence,

$$\underline{\Phi}^{\#}(0, K) = \inf_{\substack{\|u\|=1 \\ u \in K}} \langle \Phi_1(u), \Phi_2(u) \rangle .$$

Particularly we have as follows:

THEOREM 4.4. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space and* $K \subseteq H$ *a closed wedge. If* $\Phi : H \to H$ *is positively homogeneous, then*

$$\underline{\Phi}^{\#}(0, K) = \inf_{\substack{\|u\|=1 \\ u \in K}} \langle \Phi(u), u \rangle ,$$

$$\overline{\Phi}^{\#}(0, K) = \sup_{\substack{\|u\|=1 \\ u \in K}} \langle \Phi(u), u \rangle .$$

Theorems 4.1 and 4.3 imply:

THEOREM 4.5. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $x\in H$ and $K\subseteq H$ a closed wedge. If $F,G:H\to H$ are Frechét differentiable in x, with differentials $dF(x)$ and $dG(x)$, respectively, then*

$$\underline{(F,G)}^{\#}(x,K)=\inf_{\substack{\|u\|=1\\ u\in K}}\langle dF(x)(u),dG(x)(u)\rangle,$$

$$\overline{(F,G)}^{\#}(x,K)=\sup_{\substack{\|u\|=1\\ u\in K}}\langle dF(x)(u),dG(x)(u)\rangle,$$

Theorems 4.2 and 4.4 imply:

THEOREM 4.6. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space, $x\in H$ and $K\subseteq H$ a closed wedge. If $F:H\to H$ is Frechét differentiable in x, with the differential $dF(x)$, then*

$$\underline{F}^{\#}(x,K)=\inf_{\substack{\|u\|=1\\ u\in K}}\langle dF(x)(u),u\rangle,$$

$$\overline{F}^{\#}(x,K)=\sup_{\substack{\|u\|=1\\ u\in K}}\langle dF(x)(u),u\rangle.$$

Scalar derivatives were studied in [26], [27] and succesfully applied to fixed point theorems in [20], [21].

5. Inversions

Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space and $\|\cdot\|$ the norm generated by $\langle\cdot,\cdot\rangle$. The following definition is an extension of Example 5.1 p.169 [4]:

DEFINITION 5.1. The operator

$$i:H\setminus\{0\}\to H\setminus\{0\};\ \ i(x)=\frac{x}{\|x\|^2}$$

is called *inversion* (of pole 0).

It is easy to see that i is one to one and $i^{-1}=i$.

Let $K\subseteq H$ be a closed convex cone and $f:K\to H$. Since $K\setminus\{0\}$ is an invariant set of i the following definition makes sense.

DEFINITION 5.2. The *inversion* (of pole 0) of the mapping f is the mapping $\mathscr{I}(f):K\to H$ defined by:

$$\mathscr{I}(f)(x)=\begin{cases}\|x\|^2(f\circ i)(x) & \text{if } x\neq 0,\\ 0 & \text{if } x=0.\end{cases}$$

It is easy to see that the inversion of mappings $\mathscr{I}$ is a one to one operator on the set of mappings $\{f\,|\,f:K\to H; f(0)=0\}$ and $\mathscr{I}^{-1}=\mathscr{I}$, i.e., $\mathscr{I}(\mathscr{I}(f))=f$.

The properties of inversions were studied in detail in [20].

6. Exceptional family of elements

The next definition can be found in [14] and [16].

DEFINITION 6.1. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $f : H \to H$ a mapping. We say that a family of elements $\{x_r\}_{r>0} \subset K$ is an *exceptional family of elements* for f with respect to K, if for every real number $r > 0$, there exists a real number $\mu_r > 0$ such that the vector $u_r = \mu_r x_r + f(x_r)$ satisfies the following conditions:

1. $u_r \in K^*$,
2. $\langle u_r, x_r \rangle = 0$,
3. $\|x_r\| \to +\infty$ as $r \to +\infty$.

The next theorem is Theorem 9 of [12].

THEOREM 6.1. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $f : H \to H$ a completely continuous field. If f is without exceptional family of elements with respect to K, then the problem $NCP(f, K)$ has a solution.*

The next definition can be found in [9] and [16].

DEFINITION 6.2. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $f : H \to H$ a mapping. We say that the mapping f satisfies *condition* Θ with respect to K if

$$\begin{cases} \text{there exists } \rho > 0 \text{ such that for each } x \in K \text{ with } \|x\| > \rho, \\ \text{there exists } p \in K \text{ with } \|p\| < \|x\| \text{ such that } \langle x - p, f(x) \rangle \geqslant 0. \end{cases} \tag{1}$$

The next definition is a particular case of condition Θ_g of [25] with $g = I$.

DEFINITION 6.3. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $f : H \to H$ a mapping. We say that the mapping f satisfies *condition* $\tilde{\Theta}$ with respect to K if

$$\begin{cases} \text{there exists } \rho > 0 \text{ such that for each } x \in K \text{ with } \|x\| > \rho, \\ \text{there exists } p \in K \text{ with } \langle p, x \rangle < \|x\|^2 \text{ such that } \langle x - p, f(x) \rangle \geqslant 0. \end{cases} \tag{2}$$

The next lemma shows that condition $\tilde{\Theta}$ is an extension of condition Θ.

LEMMA 6.1. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $f : H \to H$ a mapping. If f satisfies condition Θ with respect to K, then it satisfies condition $\tilde{\Theta}$ with respect to K.*

Proof. Since f satisfies condition Θ with respect to K, there exists $\rho > 0$ such that for each $x \in K$ with $\|x\| > \rho$, there exists $p \in K$ with $\|p\| < \|x\|$ such that $\langle x - p, f(x) \rangle \geqslant 0$. By the Cauchy inequality

$$\langle p, x \rangle \leqslant \|p\| \|x\| < \|x\|^2.$$

Hence, f satisfies condition $\tilde{\Theta}$ with respect to K.

The next theorem is proved in [9]. It also follows from Lemma 6.1 and Theorem 6.3.

THEOREM 6.2. *Let H be a Hilbert space, $K \subset H$ a closed convex cone and $f : H \to H$ a mapping. If f satisfies condition Θ with respect to K, then it is without exceptional family of elements wih respect to K.*

The next result follows from the proof of Theorem 4 [9].

THEOREM 6.3. *Let H be a Hilbert space, $K \subset H$ a closed convex cone and $f : H \to H$ a mapping. If f satisfies condition $\tilde{\Theta}$ with respect to K, then it is without exceptional family of elements wih respect to K.*

7. Infinitesimal exceptional family of elements

DEFINITION 7.1. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space $K \subset H$ a closed convex cone and $g : K \to H$ a mapping. We say that $\{y_r\}_{r>0} \subset K$ is an *infinitesimal exceptional family of elements* for g with respect to K, if for every real number $r > 0$, there exists a real number $\mu_r > 0$ such that the vector $v_r = \mu_r y_r + g(y_r)$ satisfies the following conditions:

1. $v_r \in K^*$,
2. $\langle v_r, y_r \rangle = 0$,
3. $y_r \to 0$ as $r \to +\infty$.

DEFINITION 7.2. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $g : H \to H$ a mapping. We say that the mapping g satisfies *condition* ${}^i\Theta$ with respect to K if

$$\begin{cases} \text{there exists } \lambda > 0 \text{ such that for each } y \in K \setminus \{0\} \text{ with } \|y\| < \lambda, \\ \text{there exists } q \in K \text{ with } \|q\| < \|y\| \text{ such that } \langle y - q, g(y) \rangle \geqslant 0. \end{cases} \tag{3}$$

DEFINITION 7.3. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $g : H \to H$ a mapping. We say that the mapping g satisfies *condition* ${}^i\tilde{\Theta}$ with respect to K if

$$\begin{cases} \text{there exists } \lambda > 0 \text{ such that for each } y \in K \setminus \{0\} \text{ with } \|y\| < \lambda, \\ \text{there exists } q \in K \text{ with } \langle q, y \rangle < \|y\|^2 \text{ such that} \langle y - q, g(y) \rangle \geqslant 0. \end{cases} \tag{4}$$

The next lemma shows that condition ${}^i\tilde{\Theta}$ is an extension of condition ${}^i\Theta$ and it can be proved similarly to Lemma 6.1.

LEMMA 7.1. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $g : H \to H$ a mapping. If g satisfies condition ${}^i\Theta$ with respect to K, then it satisfies condition ${}^i\tilde{\Theta}$ with respect to K.*

THEOREM 7.1. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $g : H \to H$ a mapping. If g satisfies condition ${}^i\tilde{\Theta}$ with respect to K, then it is without infinitesimal exceptional family of elements wih respect to K.*

Proof. Suppose to the contrary, that g has infinitesimal family of elements $\{y_r\}_{r>0} \subset K$ with respect to K. For any $r > 0$ such that $\|y_r\| < \rho$ there is an element $q_r \in K$ with $\langle q_r, y_r \rangle < \|y_r\|^2$ satisfying relation (3), i.e.,

$$\langle y_r - q_r, g(y_r) \rangle \geqslant 0.$$

Since, according to Definition 7.1, $\langle v_r, y_r \rangle = 0$ and $v_r \in K^*$, we have

$$\begin{aligned} 0 &\leqslant \langle y_r - q_r, g(y_r) \rangle \\ &= \langle y_r - q_r, v_r - \mu_r y_r \rangle \\ &= -\mu_r \|y_r\|^2 - \langle q_r, v_r \rangle + \mu_r \langle q_r, y_r \rangle \\ &\leqslant -\mu_r (\|y_r\|^2 - \langle q_r, y_r \rangle) < 0, \end{aligned}$$

which is a contradiction.

REMARK 7.1. At first sight Theorem 7.1 seems to be a direct consequence of Theorem 8.2 and Theorem 8.4, proved in the next section. However, note that there might be an infinitesimal family of elements of g which contains zero.

COROLLARY 7.1. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $g : H \to H$ a mapping. If g satisfies condition ${}^i\Theta$ with respect to K, then it is without infinitesimal exceptional family of elements wih respect to K.*

Proof. By Lemma 7.1 g satisfies condition ${}^i\tilde{\Theta}$ with respect to K. Hence, by Theorem 7.1 g is without infinitesimal exceptional family of elements wih respect to K.

REMARK 7.2. At first sight Corollary 7.1 seems to be a direct consequence of Theorem 8.2 and Theorem 8.5, proved in the next section.. However, note that there might be an infinitesimal family of elements of g which contains zero.

8. A duality and main results

THEOREM 8.1. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space $K \subset H$ a closed convex cone and $f : K \to H$ a mapping. Then $x_* \neq 0$ is a solution of $NCP(f, K)$ if and only if y_* is a solution of $NCP(g, K)$, where $y_* = i(x_*)$ is the inversion of x_* and $g = \mathscr{I}(f)$ is the inversion of f.*

Proof.

$$\langle y_*, \mathscr{I}(f)(y_*) \rangle = \langle y_*, \|y_*\|^2 f(i(y_*)) \rangle .$$

Hence,

$$\langle y_*, \mathscr{I}(f)(y_*) \rangle = \|y_*\|^4 \langle i(y_*), f(i(y_*)) \rangle .$$

Since $i^{-1} = i$, we have

$$\langle y_*, g(y_*) \rangle = \frac{1}{\|x_*\|^4} \langle x_*, f(x_*) \rangle . \tag{5}$$

It can be similarly proved that

$$\langle g(y_*), z\rangle = \frac{1}{\|x_*\|^2}\langle f(x_*), z\rangle, \tag{6}$$

for every $z \in K$. By using (5),

$$\langle x_*, f(x_*)\rangle = 0$$

if and only if

$$\langle y_*, g(y_*)\rangle = 0.$$

By using (6), $f(x_*) \in K^*$ if and only if $g(y_*) \in K^*$.

THEOREM 8.2. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $f : K \to H$ a mapping. $\{x_r\}_{r>0} \subset K \setminus \{0\}$ is an exceptional family of elements for f with respect to K if and only if $\{y_r\}_{r>0} \subset K \setminus \{0\}$ is an infinitesimal exceptional family of elements for g with respect to K, where $y_r = i(x_r)$ and $g = \mathscr{I}(f)$.*

Proof. Bearing in mind the notations of Definition 7.1, we have

$$v_r = \mu_r y_r + \|y_r\|^2 f(i(y_r)).$$

Hence,

$$v_r = \|y_r\|^2(\mu_r i(y_r) + f(i(y_r))).$$

Since $i^{-1} = i$, we have

$$v_r = \frac{1}{\|x_r\|^2}(\mu_r x_r + f(x_r)).$$

Hence,

$$v_r = \frac{1}{\|x_r\|^2}u_r.$$

Therefore,

$$\langle v_r, y_r\rangle = \frac{1}{\|x_r\|^4}\langle u_r, x_r\rangle \tag{7}$$

and

$$\langle v_r, z\rangle = \frac{1}{\|x_r\|^2}\langle u_r, z\rangle, \tag{8}$$

for every $z \in K$. Since $\|x_r\| \cdot \|y_r\| = 1$, $\|x_r\| \to +\infty$ if and only if $y_r \to 0$. By using (7),

$$\langle u_r, x_r\rangle = 0$$

if and only if

$$\langle v_r, y_r\rangle = 0.$$

By using (8), $u_r \in K^*$ if and only if $v_r \in K^*$.

THEOREM 8.3. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $f : K \to H$ a completely continuous field with $f(0) \notin K^*$. If every infinitesimal exceptional family of elements for $g = \mathscr{I}(f)$ with respect to K contains 0, then the nonlinear complementarity problem $NCP(f, K)$ has a nonzero solution.*

Proof. Since $f(0) \notin K^*$, if $NCP(f,K)$ has a solution, then this solution is nonzero. By Theorem 6.1, it is enough to prove that f is without exceptional family of elements with respect to K. Suppose to the contrary that $\{x_r\}_{r>0}$ is an exceptional family of elements for f with respect to K. Since $f(0) \notin K^*$, by the definition of an exceptional family of elements $\{x_r\}_{r>0} \subset K \setminus \{0\}$. Hence, by Theorem 8.2, $g = \mathscr{I}(f)$ has an infinitesimal exceptional family of elements with respect to K which does not contain 0, which is a contradiction.

THEOREM 8.4. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $f : H \to H$ a mapping and $g = \mathscr{I}(f)$. Then, f satisfies condition $\tilde{\Theta}$ with respect to K if and only if g satisfies condition ${}^i\tilde{\Theta}$ with respect to K.*

Proof. Suppose that g satisfies condition ${}^i\tilde{\Theta}$ with respect to K and prove that f satisfies condition $\tilde{\Theta}$ with respect to K. Consider the constant λ of condition ${}^i\tilde{\Theta}$ and let

$$\rho = \frac{1}{\lambda}.$$

Let $x \in K$ with

$$\|x\| > \rho \tag{9}$$

and $y = i(x)$. Since

$$\|y\| = \frac{1}{\|x\|},$$

it follows that $\|y\| < \lambda$. Hence, by condition ${}^i\tilde{\Theta}$, there exists $q \in K$ with $\langle q, y \rangle < \|y\|^2$ such that $\langle y - q, g(y) \rangle \geqslant 0$. Let

$$p = \frac{q}{\|y\|^2}. \tag{10}$$

Since $\langle q, y \rangle < \|y\|^2$ and $i^{-1} = i$, relation (10) implies that

$$\langle p, x \rangle = \frac{\langle q, y \rangle}{\|y\|^4} < \frac{1}{\|y\|^2} = \|x\|^2. \tag{11}$$

On the other hand $\mathscr{I}^{-1} = \mathscr{I}$ implies that

$$\begin{aligned} \langle x - p, f(x) \rangle &= \langle x - p, \mathscr{I}(g)(x) \rangle \\ &= \langle x - p, \|x\|^2 g(i(x)) \rangle \\ &= \|x\|^4 \langle y - q, g(y) \rangle \\ &\geqslant 0 \end{aligned} \tag{12}$$

By (9), (11) and (12) f satisfies condition $\tilde{\Theta}$ with respect to K. Now, suppose that f satisfies condition $\tilde{\Theta}$ with respect to K and prove that g satisfies condition ${}^i\tilde{\Theta}$ with respect to K. Consider the constant $\rho > 0$ of condition $\tilde{\Theta}$ and let

$$\lambda = \frac{1}{\rho}.$$

Let $y \in K \setminus \{0\}$ with $\|y\| < \lambda$. We have to prove that there exists $q \in K$ with $\langle q, y \rangle < \|y\|^2$ such that $\langle y - q, g(y) \rangle \geqslant 0$. Since $f = \mathscr{I}(g)$, we can proceed as above.

The next theorem can be proved similarly to Theorem 8.4.

THEOREM 8.5. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone, $f : H \to H$ a mapping and $g = \mathscr{I}(f)$. Then, f satisfies condition Θ with respect to K if and only if g satisfies condition ${}^{i}\Theta$ with respect to K.*

THEOREM 8.6. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $f : K \to H$ a completely continuous field. If $g = \mathscr{I}(f)$ satisfies condition ${}^{i}\Theta$ with respect to K, then the nonlinear complementarity problem $NCP(f, K)$ has a solution.*

Proof. By Theorem 8.5, f satisfies condition Θ with respect to K. Hence, Theorem 6.2 and Theorem 6.1 implies that the nonlinear complementarity problem $NCP(f, K)$ has a solution.

THEOREM 8.7. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $f : K \to H$ a completely continuous field. If $g = \mathscr{I}(f)$ satisfies condition ${}^{i}\tilde{\Theta}$ with respect to K, then the nonlinear complementarity problem $NCP(f, K)$ has a solution.*

Proof. By Theorem 8.4, f satisfies condition $\tilde{\Theta}$ with respect to K. Hence, Theorem 4 [25] implies that the nonlinear complementarity problem $NCP(f, K)$ has a solution.

THEOREM 8.8. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $f : K \to H$ a completely continuous field. If there is a $\delta > 0$ and a $h : B(0, \delta) \cap K \to K$ with $h(0) = 0$ and*

$$\begin{cases} \overline{h}^{\#}(0) < 1, \\ \underline{(I - h, \mathscr{I}(f))}^{\#}(0) > 0, \end{cases}$$

where $B(0, \delta) = \{z \in H : \|z\| < \delta\}$, then the nonlinear complementarity problem $NCP(f, K)$ has a solution.

Proof. Let $g = \mathscr{I}(f)$. Since $\overline{h}^{\#}(0) < 1$, there is a λ_1 with $0 < \lambda_1 < \delta$ such that for every $y \in K$ with $\|y\| < \lambda_1$ we have

$$\langle h(y), y \rangle < \|y\|^2. \tag{13}$$

Since

$$\underline{(I - h, g)}^{\#}(0) > 0,$$

there is a λ_2 with $0 < \lambda_2 < \delta$ such that for every $y \in K$ with $\|y\| < \lambda_2$ we have

$$\langle y - h(y), g(y) \rangle > 0. \tag{14}$$

Let $\lambda = \min\{\lambda_1, \lambda_2\}$. Obviously,

$$\lambda > 0. \tag{15}$$

For

$$\|y\| < \lambda \tag{16}$$

let $q = h(y)$. Then, relations (13) and (14) imply

$$\langle q, y \rangle < \|y\|^2. \tag{17}$$

and

$$\langle y - q, g(y) \rangle \geqslant 0, \tag{18}$$

respectively. Hence, relations (15), (16), (17) and (18) imply that g satisfies condition ${}^i\tilde{\Theta}$. Hence, Theorem 8.7 implies that the problem $NCP(f, K)$ has a solution.

REMARK 8.1. If h and $I(f)$ have extensions to H which are differentiable in 0, then the scalar derivatives in Theorem 8.8 can be calculated explicitly by using the formulae of Theorem 4.5.

In the particular case $h = 0$ we have as follows:

COROLLARY 8.1. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $f : K \to H$ a completely continuous field. If $\underline{\mathscr{I}(f)}^{\#}(0) > 0$, then the nonlinear complementarity problem $NCP(f, K)$ has a solution.*

REMARK 8.2. If $I(f)$ has an extension to H which is differentiable in 0, then the scalar derivatives in Corollary 8.1 can be calculated explicitly by using the first formula of Theorem 4.6.

REMARK 5. It is easy to see that the differentiability condition of the extension of $\mathscr{I}(f)$ in 0 can be satisfied if f has an extension to H denoted by the same letter, such that

$$f(x) = A(x) + \frac{1}{\|x\|}\mathscr{I}(\omega)(x),$$

where A is a linear bounded operator and $\omega : H \to H$ is a mapping such that $lim_{x \to 0}\omega(x) = 0$. In this case the differential in 0 of the extension of $\mathscr{I}(f)$ is A. In this way it is likely that one can generate a large class of complementarity problems which are solvable. The only problem could be the verification of the complet continuity of f in the infinite dimensional case. This should be the issue of future investigations. In the finite dimensional case the complet continuity is equivalent to continuity, which can be easily verified.

9. Comments

In this paper we introduced the notion of (IEFE). By using a special inversion mapping we presented a duality between the notion of (IEFE) and the notion of (EFE) (used in several papers [1]-[3], [6], [8]-[19], [22], [23], [25], [28]-[39]). By this duality and by also using the scalar derivative we obtained some existence theorems for complementarity problems in Hilbert spaces. In the view of the results presented in this paper, finding new classes of mappings without (IEFE) seems a very challenging subject which could be the topic of further papers.

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