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THE HAMY SYMMETRIC FUNCTION AND ITS GENERALIZATION

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Abstract. In the paper we investigate and generalize the Hamy symmetric function: $F_n(x,r) = \sum_{1\leqslant i_1<i_2<\ldots<i_r\leqslant n}\left(\prod_{j=1}^{r} x_{i_j}\right)^{\frac{1}{r}}$. The Schur-convexity is discussed and some analytic inequalities are established by use of the theory of majorization.

1. Introduction

Let $R_+^n = \{x = (x_1, x_2, \ldots, x_n) \mid x_i > 0, i = 1, 2, \ldots, n\}$. The unweighted arithmetic and geometric means of x, denoted by $A_n(x)$, $G_n(x)$, respectively, are defined as follows

$$A_n(x) = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad G_n(x) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}. \tag{1.1}$$

The order relation among these means is the well-known "$A - G$" inequality, that is, $G_n(x) \leqslant A_n(x)$. The classical inequality has evoked the interest of many mathematicians, and numerous proofs, generalizations and refinements were published. See, for example, [1, 2, 4, 5, 11, 12] and the references cited therein.

The Hamy symmetric function ([4], [12, p. 67]) is defined as

$$F_n(x,r) = F_n(x_1, x_2, \ldots, x_n; r) = \sum_{1\leqslant i_1<i_2<\ldots<i_r\leqslant n}\left(\prod_{j=1}^{r} x_{i_j}\right)^{\frac{1}{r}}, \quad r = 1, 2, \ldots, n. \tag{1.2}$$

Corresponding to this is the $r-$ th order Hamy mean

$$\sigma_n(x,r) = \frac{1}{\binom{n}{r}}\sum_{1\leqslant i_1<i_2<\ldots<i_r\leqslant n}\left(\prod_{j=1}^{r} x_{i_j}\right)^{\frac{1}{r}}, \tag{1.3}$$

where $\binom{n}{r} = \frac{n!}{(n-r)!r!}$. T. Hara *et al.* [4] established the following refinement of the classical arithmetic and geometric means inequality:

$$G_n(x) = \sigma_n(x,n) \leqslant \sigma_n(x,n-1) \leqslant \ldots \leqslant \sigma_n(x,2) \leqslant \sigma_n(x,1) = A_n(x). \tag{1.4}$$

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The paper [5] by H. T. Ku, M. C. Ku and X. M. Zhang contains some interesting inequalities including the fact that $(\sigma_n(x,r))^r$ is log-concave. The more results can also be found in the book [2] by P. S. Bullen.

Now we define the following new symmetric function

$$F_n^*(x,r) = F_n^*(x_1,x_2,...,x_n;r) = \sum_{i_1+i_2+...+i_n=r} (x_1^{i_1}x_2^{i_2}...x_n^{i_n})^{\frac{1}{r}}, \tag{1.5}$$

where $i_1,i_2,...,i_n$ are non-negative integers, $r \in N = \{1,2,...\}$. It is obvious that

$$F_n(x,r) = \sum_{i_1+i_2+...+i_n=r}^{i_j=0 \ or \ 1} (x_1^{i_1}x_2^{i_2}...x_n^{i_n})^{\frac{1}{r}}.$$

Thus, $F_n^*(x,r)$ generalizes the function $F_n(x,r)$ and so may be called generalized Hamy symmetric function.

The main purpose of the paper is to investigate Schur-convexity of the Hamy symmetric function $F_n(x,r)$ and its generalization $F_n^*(x,r)$. Some analytic inequalities, including Ky Fan type inequalities, are established by use of the theory of majorization, for which, the interested reader can see the popular book [7] by Marshall and Olkin.

The Schur-convex function was introduced by I. Schur in 1923 [7]. It has many important applications in analytic inequalities. Hardy *et al.* were also interested in some inequalities that are related to Schur-convex functions [9]. The following definitions can be found in many references such as [7, 12].

For fixed $n \geqslant 2$, let

$$x = (x_1,x_2,...,x_n), \qquad y = (y_1,y_2,...,y_n)$$

be two $n-$ tuples of real numbers. Let

$$x_{[1]} \geqslant x_{[2]} \geqslant ... \geqslant x_{[n]}, \qquad y_{[1]} \geqslant y_{[2]} \geqslant ... \geqslant y_{[n]},$$

be their ordered components.

DEFINITION 1.1. ([7, p. 55]) The $n-$ tuple x is said to be majorized by y (in symbols $x \prec y$), if

$$\sum_{i=1}^{m} x_{[i]} \leqslant \sum_{i=1}^{m} y_{[i]}, \quad m = 1,2,...,n-1; \tag{1.6}$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}. \tag{1.7}$$

DEFINITION 1.2. ([7, p. 54]) A real-valued function ϕ defined on a set $\Omega \subset R^n$ is said to be Schur-convex function on Ω if

$$x \prec y \quad on \quad \Omega \Longrightarrow \phi(x) \leqslant \phi(y).$$

If, in addition, $\phi(x) < \phi(y)$ whenever $x \prec y$ but x is not a permutation of y, then ϕ is said to be strictly Schur-convex on Ω. ϕ is Schur-concave function on Ω if and only if $-\phi$ is Schur-convex function; ϕ is a strictly Schur-concave function on Ω if and only if $-\phi$ is strictly Schur-convex function on Ω.

2. Lemmas

In order to verify our results, the following lemmas are necessary.

LEMMA 2.1. ([6, *p.* 259; 7, *p.* 57]) *Let* $f(x) = f(x_1, x_2, ..., x_n)$ *be symmetric and have continuous partial derivatives on* $I^n = I \times I \times ... \times I$ $(n$ *copies*), *where* I *is an open interval. Then* $f : I^n \to R$ *is Schur-convex if and only if*

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geqslant 0 \tag{2.1}$$

on I^n. *It is strictly Schur-convex if* (2.1) *is a strict inequality for* $x_i \neq x_j$, $1 \leqslant i, j \leqslant n$.

Since $f(x)$ is symmetric, the Schur's condition, i.e. (2.1), can be reduced as [7, p. 57]

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \geqslant 0, \tag{2.2}$$

and f is strictly Schur-convex if (2.2) is a strict inequality for $x_1 \neq x_2$. The Schur's condition that guarantees a symmetric function being Schur-concave is the same as (2.1) or (2.2) except for the direction of the inequality.

In Schur's condition, the domain of $f(x)$ does not have to be a Cartesian product I^n. Lemma 2.1 remains true if we replace I^n by a set $A \subseteq R^n$ with the following properties (see [7, p. 57]):

(i) A is convex and has a nonempty interior;

(ii) A is symmetric in the sense that $x \in A$ implies $Px \in A$ for any $n \times n$ permutation matrix P.

LEMMA 2.2. ([8]) *Suppose that* $x_i > 0$, $i = 1, 2, ..., n$, $\sum_{i=1}^{n} x_i = s$, *and that* $c \geqslant s$. *Then*

(i) $\frac{c-x}{nc/s-1} = \left(\frac{c-x_1}{nc/s-1}, ..., \frac{c-x_n}{nc/s-1}\right) \prec (x_1, x_2, ..., x_n) = x$;

(ii) $\frac{c+x}{s+nc} = \left(\frac{c+x_1}{s+nc}, \frac{c+x_2}{s+nc}, ..., \frac{c+x_n}{s+nc}\right) \prec \left(\frac{x_1}{s}, \frac{x_2}{s}, ..., \frac{x_n}{s}\right) = \frac{x}{s}$.

Next, recall that the complete symmetric function [7, p. 81] is defined by

$$c_r = c_r(x) = \sum_{i_1+i_2+...+i_n=r} x_1^{i_1} x_2^{i_2} ... x_n^{i_n}, \tag{2.3}$$

where $i_1, i_2, ..., i_n$ are non-negative integers, $r \in N = \{1, 2, ...\}$, and define $c_0(x) = 1$. K. Z. Guan [3] established its property as follows

LEMMA 2.3. *Assume that* $x_i > 0$, $i = 1, 2, ..., n$. *Let*

$$\overline{x}_i = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n).$$

Then

$$c_r(x) = x_i c_{r-1}(x) + c_r(\overline{x}_i).$$

3. Main results

In this section we investigate the Schur-convexity of $F_n(x,r)$ and $F_n^*(x,r)$. Some analytic inequalities are established by use of the theory of majorization.

THEOREM 3.1. *The Hamy symmetric function* $F_n(x,r)$, $r=1,2,...,n$, *is Schur-concave in* R_+^n, *and is increasing for all* x_i, $i=1,2,...,n$.

Proof. It is clear that $F_n(x,r)$ is increasing with respect to x_i, $i=1,2,...,n$, and so is omitted. Below, we prove that $F_n(x,r)$ is a Schur-concave function in R_+^n. By Lemma 2.1, noting that $F_n(x,r)$ is symmetric and has continuous partial derivatives in R_+^n, we only need to prove

$$(x_1-x_2)\left(\frac{\partial F_n(x,r)}{\partial x_1}-\frac{\partial F_n(x,r)}{\partial x_2}\right)\leqslant 0.$$

To this end, we consider the following possible cases for r.

(i) For $r=1$ we are done due to $F_n(x,1)=\sum_{i=1}^n x_i$.

(ii) For $2\leqslant r\leqslant n$. The r-th order symmetric function ([7, p. 78]) is defined as

$$E_n(x,r)=E_n(x_1,x_2,...,x_n;r)=\sum_{1\leqslant i_1<i_2<...<i_r\leqslant n}\prod_{j=1}^r x_{i_j}.$$

The following property can be found in [2, p. 324]:

$$\begin{aligned}E_n(x_1,x_2,...,x_n;r)=x_1x_2&E_{n-2}(x_3,x_4,...,x_n;r-2)\\&+(x_1+x_2)E_{n-2}(x_3,x_4,...,x_n;r-1)\\&+E_{n-2}(x_3,x_4,...,x_n;r).\end{aligned}\tag{3.1}$$

Fix r and let $u=(u_1,u_2,...,u_n)$ and $u_i=\sqrt[r]{x_i}$, $i=1,2,...,n$, we have

$$F_n(x_1,x_2,...,x_n;r)=E_n(u_1,u_2,...,u_n;r).$$

Differentiating $F_n(x,r)$ with respect to x_1 and using (3.1), we obtain

$$\begin{aligned}\frac{\partial F_n(x,r)}{\partial x_1}&=\sum_{k=1}^n\frac{\partial E_n(u,r)}{\partial u_k}\cdot\frac{\partial u_k}{\partial x_1}=\frac{\partial E_n(u,r)}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_1}\\&=\frac{1}{rx_1}\cdot\sqrt[r]{x_1x_2}\cdot E_{n-2}(u_3,u_4,...,u_n;r-2)\\&\quad+\frac{\sqrt[r]{x_1}}{rx_1}\cdot E_{n-2}(u_3,u_4,...,u_n;r-1).\end{aligned}$$

Similarly, we can also get

$$\begin{aligned}\frac{\partial F_n(x,r)}{\partial x_2}&=\sum_{k=1}^n\frac{\partial E_n(u,r)}{\partial u_k}\cdot\frac{\partial u_k}{\partial x_2}=\frac{\partial E_n(u,r)}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_2}\\&=\frac{1}{rx_2}\cdot\sqrt[r]{x_1x_2}\cdot E_{n-2}(u_3,u_4,...,u_n;r-2)\\&\quad+\frac{\sqrt[r]{x_2}}{rx_2}\cdot E_{n-2}(u_3,u_4,...,u_n;r-1).\end{aligned}$$

Thus

$$(x_1-x_2)\left(\frac{\partial F_n(x,r)}{\partial x_1}-\frac{\partial F_n(x,r)}{\partial x_2}\right)$$
$$=-\frac{\sqrt[r]{x_1x_2}}{rx_1x_2}(x_1-x_2)^2\cdot E_{n-2}(u_3,u_4,...,u_n;r-2)$$
$$+\frac{1}{r}(x_1-x_2)(x_1^{\frac{1}{r}-1}-x_2^{\frac{1}{r}-1})\cdot E_{n-2}(u_3,u_4,...,u_n;r-1).$$

From the fact that the function $x^{\frac{1}{r}-1}$ is decreasing in $(0,+\infty)$, it follows that

$$(x_1-x_2)\left(\frac{\partial F_n(x,r)}{\partial x_1}-\frac{\partial F_n(x,r)}{\partial x_2}\right)\leqslant 0.$$

Therefore, the proof is complete.

Using Theorem 3.1 and Lemma 2.2, we can easily establish the following consequences.

COROLLARY 3.2. *Suppose that* $x_i>0, i=1,2,...,n$, $\sum_{i=1}^n x_i=s$, *and that* $c\geqslant s$. *Then*

$$\frac{F_n(c-x;r)}{F_n(x;r)}\geqslant\left(\frac{nc}{s}-1\right).$$

REMARK 3.1. By Corollary 3.2, let $c=s=1$, the following statements are true.
(*i*) $\frac{F_n(1-x;r)}{F_n(x;r)}\geqslant(n-1)$;

(*ii*) $\prod_{i=1}^n(x_i^{-1}-1)\geqslant(n-1)^n$ (Weierstrass inequality [11, p. 260]).

COROLLARY 3.3. *Assume that* $x_i>0, i=1,2,...,n$, $\sum_{i=1}^n x_i=s$, *and that* $c\geqslant s$. *Then*

$$\frac{F_n(c+x;r)}{F_n(x;r)}\geqslant\left(\frac{nc}{s}+1\right).$$

In particular, let $c=s=1$, we can get the Weierstrass inequality (see [11, p. 260])

$$\prod_{i=1}^n(x_i^{-1}+1)\geqslant(n+1)^n.$$

Now we investigate the generalized Hamy symmetric function $F_n^*(x,r)$.

THEOREM 3.4. *The generalized Hamy symmetric function* $F_n^*(x,r)$, $r=1,2,...,n$, *is Schur-concave in* R_+^n.

Proof. By Lemma 2.1, we only need to prove that

$$(x_1-x_2)\left(\frac{\partial F_n^*(x,r)}{\partial x_1}-\frac{\partial F_n^*(x,r)}{\partial x_2}\right)\leqslant 0. \quad (3.2)$$

To this end, we consider the two possible cases for r.

(i) When $r=1$, it is clear that $F_n^*(x,1)=\sum_{i=1}^n x_i$. One can easily find that (3.2) holds.

(ii) When $2\leqslant r\leqslant n$. Fix r and let $u=(u_1,u_2,...,u_n)$, $u_i=\sqrt[r]{x_i}$, $i=1,2,...,n$, we have $F_n^*(x,r)=c_r(u)$. From Lemma 2.3, it follows that

$$\frac{\partial c_r(u)}{\partial u_k}=c_{r-1}(u)+u_k\frac{\partial c_{r-1}(u)}{\partial u_k},\quad k=1,2,...,n. \tag{3.3}$$

Using Lemma 2.3 and (3.3) repeatedly yields

$$\frac{\partial c_r(u)}{\partial u_k}=c_{r-1}(u)+u_kc_{r-2}(u)+u_k^2c_{r-3}(u)+...+u_k^{r-2}c_1(u)+u_k^{r-1}. \tag{3.4}$$

Differentiating $F_n^*(x,r)$ with respect to x_1 and using (3.4), we obtain

$$\begin{aligned}\frac{\partial F_n^*(x,r)}{\partial x_1}&=\sum_{k=1}^n\frac{\partial c_r(u)}{\partial u_k}\cdot\frac{\partial u_k}{\partial x_1}=\frac{\partial c_r(u)}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_1}\\&=\left(c_{r-1}(u)+u_1\frac{\partial c_{r-1}(u)}{\partial u_1}\right)\frac{\sqrt[r]{x_1}}{rx_1}\\&=\left(c_{r-1}(u)+u_1c_{r-2}(u)+u_1^2c_{r-3}(u)+...+u_1^{r-2}c_1(u)+u_1^{r-1}\right)\frac{\sqrt[r]{x_1}}{rx_1}\\&=\frac{1}{r}\sum_{j=1}^r c_{r-j}(u)x_1^{\frac{j-r}{r}}.\end{aligned}$$

Similarly, we also get

$$\begin{aligned}\frac{\partial F_n^*(x,r)}{\partial x_2}&=\sum_{k=1}^n\frac{\partial c_r(u)}{\partial u_k}\cdot\frac{\partial u_k}{\partial x_2}=\frac{\partial c_r(u)}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_2}\\&=\frac{1}{r}\sum_{j=1}^r c_{r-j}(u)x_2^{\frac{j-r}{r}}.\end{aligned}$$

Because the function $x^{\frac{j-r}{r}}$ $(j=1,2,...,r-1)$ is decreasing in $(0,+\infty)$, one can easily find that

$$(x_1-x_2)\left(\frac{\partial F_n^*(x,r)}{\partial x_1}-\frac{\partial F_n^*(x,r)}{\partial x_2}\right)=\frac{(x_1-x_2)}{r}\sum_{j=1}^{r-1}c_{r-j}(u)(x_1^{\frac{j-r}{r}}-x_2^{\frac{j-r}{r}})\leqslant 0.$$

Combining these argument, we have completed the proof of the theorem.

COROLLARY 3.5. *The function* $\frac{F_n^*(x,r)}{F_n^*(x,1)}$, $2\leqslant r\leqslant n$, *is Schur-concave in* R_+^n.

Proof. Let $\Phi=\frac{F_n^*(x,r)}{F_n^*(x,1)}$. Differentiating Φ with respect to x_i shows that

$$\frac{\partial\Phi}{\partial x_i}=\frac{1}{(F_n^*(x,1))^2}\left(F_n^*(x,1)\cdot\frac{\partial F_n^*(x,r)}{\partial x_i}-F_n^*(x,r)\right),\quad i=1,2. \tag{3.5}$$

From (3.5) and Theorem 3.4, it follows that

$$(x_1 - x_2)\left(\frac{\partial \Phi}{\partial x_1} - \frac{\partial \Phi}{\partial x_2}\right) = \frac{(x_1 - x_2)}{F_n^*(x,1)} \cdot \left(\frac{\partial F_n^*(x,r)}{\partial x_1} - \frac{\partial F_n^*(x,r)}{\partial x_2}\right) \leqslant 0.$$

By Lemma 2.1, we have proven $\frac{F_n^*(x,r)}{F_n^*(x,1)}$ to be Schur-concave in R_+^n.

Let $0 < x_i \leqslant 1/2,\ i = 1,2,...,n.$ The following inequality

$$\frac{G_n(x)}{G_n(1-x)} \leqslant \frac{A_n(x)}{A_n(1-x)},$$

where $1-x = (1-x_1, 1-x_2, ..., 1-x_n)$, commonly referred to as the Ky Fan inequality ([10, p. 5]) has stimulated an interest of many researchers. New proofs, improvements and generalizations of it were published (see for instance [3] and [13-15]). Using Corollary 3.6 and Lemma 2.2 and noting $F_n^*(x,n) = G_n(x)$, we get the following inequality which generalizes the Ky Fan inequality.

COROLLARY 3.6. *Assume that $x_i > 0,\ i = 1,2,...,n,$ and that $\sum_{i=1}^n x_i \leqslant 1$, then*

$$\frac{A_n(x)}{A_n(1-x)} \geqslant \frac{F_n^*(x,r)}{F_n^*(1-x,r)}, \quad r = 2,3,...,n. \tag{3.6}$$

Having in mind Theorem 3.4, it is naturally to ask whether the function $F_n^*(x,r)$ is Schur-concave in R_+^n for $r > n$. We point it out as an open problem which is interesting to be investigated. However, for $n = 2$, we can prove the following

THEOREM 3.7. *The function $F_2^*(x,r) = F_2^*(x_1,x_2;r)$ is strictly Schur-concave in R_+^2 for $r \in N$.*

Proof. Differentiating $F_2^*(x,r)$ with respect to x_i, $i = 1,2$, we have

$$\frac{\partial F_2^*(x,r)}{\partial x_1} = 1 + \sum_{i=1}^{r-1}\left(1 - \frac{i}{r}\right)\left(\frac{x_2}{x_1}\right)^{\frac{i}{r}},$$

and

$$\frac{\partial F_2^*(x,r)}{\partial x_2} = 1 + \sum_{i=1}^{r-1}\left(1 - \frac{i}{r}\right)\left(\frac{x_1}{x_2}\right)^{\frac{i}{r}}.$$

Thus, we get

$$(x_1 - x_2)\left(\frac{\partial F_2^*(x,r)}{\partial x_1} - \frac{\partial F_2^*(x,r)}{\partial x_2}\right) = \sum_{i=1}^{r-1}\left(1 - \frac{i}{r}\right)(x_1 - x_2)\left(\left(\frac{x_2}{x_1}\right)^{\frac{i}{r}} - \left(\frac{x_1}{x_2}\right)^{\frac{i}{r}}\right).$$

On the other hand, for $x_1 \neq x_2$, one can easily find that

$$(x_1 - x_2)\left(\left(\frac{x_2}{x_1}\right)^{\frac{i}{r}} - \left(\frac{x_1}{x_2}\right)^{\frac{i}{r}}\right) < 0, \quad i = 1,2,...,r-1.$$

From this there follows that

$$(x_1 - x_2)\left(\frac{\partial F_2^*(x,r)}{\partial x_1} - \frac{\partial F_2^*(x,r)}{\partial x_2}\right) < 0.$$

By Lemma 2.1, the proof is complete.

For $F_2^*(x,r) = F_2^*(x_1,x_2;r)$, $r \in N$, we can also establish the following ratio inequality.

THEOREM 3.8. *Assume that* $b_1 > b_2 > 0$, $\frac{a_1}{b_1} \geqslant \frac{a_2}{b_2} > 0, r \in N$. *Then*

$$\frac{F_2^*(a_1,a_2;r)}{F_2^*(b_1,b_2;r)} \geqslant \frac{M_r(a_1,a_2)}{M_r(b_1,b_2)}, \tag{3.7}$$

where as usual $M_r(a,b) = (\frac{a^{\frac{1}{r}}+b^{\frac{1}{r}}}{2})^r$ *denotes the* $r-th$ *power-mean of* a *and* b.

Proof. Simply calculating shows that

$$F_2^*(a_1,a_2;r) = a_2\left(\frac{a_1}{a_2} + \left(\frac{a_1}{a_2}\right)^{\frac{r-1}{r}} + \left(\frac{a_1}{a_2}\right)^{\frac{r-2}{r}} + \dots + \left(\frac{a_1}{a_2}\right)^{\frac{1}{r}} + 1\right),$$

and

$$M_r(a_1,a_2) = a_2\left(\frac{(\frac{a_1}{a_2})^{\frac{1}{r}}+1}{2}\right)^r.$$

Thus

$$\frac{F_2^*(a,r)}{F_2^*(b,r)} = \frac{a_2}{b_2}\cdot\frac{\frac{a_1}{a_2} + \left(\frac{a_1}{a_2}\right)^{\frac{r-1}{r}} + \left(\frac{a_1}{a_2}\right)^{\frac{r-2}{r}} + \dots + \left(\frac{a_1}{a_2}\right)^{\frac{1}{r}} + 1}{\frac{b_1}{b_2} + \left(\frac{b_1}{b_2}\right)^{\frac{r-1}{r}} + \left(\frac{b_1}{b_2}\right)^{\frac{r-2}{r}} + \dots + \left(\frac{b_1}{b_2}\right)^{\frac{1}{r}} + 1},$$

and

$$\frac{M_r(a_1,a_2)}{M_r(b_1,b_2)} = \frac{a_2}{b_2}\cdot\left(\frac{(\frac{a_1}{a_2})^{\frac{1}{r}}+1}{(\frac{b_1}{b_2})^{\frac{1}{r}}+1}\right)^r.$$

Let $x = \frac{a_1}{a_2}$, $y = \frac{b_2}{b_2}$ ($x \geqslant y > 1$) and $\phi(x) = \frac{x+x^{\frac{r-1}{r}}+x^{\frac{r-2}{r}}+\dots+x^{\frac{1}{r}}+1}{(x^{\frac{1}{r}}+1)^r}$. In order to prove (3.7), we only have to verify the function $\phi(x)$ be non-decreasing for $x \in (1,+\infty)$. We consider the following two possible cases for r.

Case 1. For $r = 1$, we are done due to $\phi(x) = 1$.

Case 2. For $r \geqslant 2$. Calculating $\phi(x)$, we obtain

$$\phi(x) = \frac{x^{\frac{r+1}{r}}-1}{(x^{\frac{1}{r}}-1)(x^{\frac{1}{r}}+1)^r} \quad (x > 1).$$

Let $u = x^{\frac{1}{r}}$, then $u > 1$ and $\phi(x) = f(u) = \frac{u^{r+1}-1}{(u-1)(u+1)^r}$. Taking logarithm on $f(u)$ and differentiating it with respect to u, we have

$$\frac{f'(u)}{f(u)} = \frac{(r+1)u^r}{u^{r+1}-1} - \frac{1}{u-1} - \frac{r}{u+1}$$
$$= \frac{1}{(u^2-1)(u^{r+1}-1)}\cdot\varphi(u),$$

where $\varphi(u) = (r-1)u^{r+1} - (r+1)u^r + (r+1)u - r + 1 \ (u > 1)$.

Now

$$\varphi'(u) = (r-1)(r+1)u^r - (r+1)ru^{r-1} + (r+1),$$
$$\varphi''(u) = (r-1)r(r+1)u^{r-2}(u-1) \geqslant 0 \quad (u > 1).$$

From $\varphi'(1) = \varphi(1) = 0$, it follows that $\varphi(u) \geqslant 0$ or $f'(u) \geqslant 0 \quad (u > 1)$. Therefore $f(u)$ is non-decreasing for $u \in (1, +\infty)$, which shows that the function $\phi(x)$ is no-decreasing for $x \in (1, +\infty)$.

Summarizing the above discussion, we have proven the theorem.

Using Theorem 3.8, we can get the following Ky Fan type inequality.

COROLLARY 3.9. *If* $0 < a_1, a_2 \leqslant \frac{1}{2}, r \in N$, *then*

$$\frac{F_2^*(a_1, a_2; r)}{F_2^*(1-a_1, 1-a_2; r)} \geqslant \frac{M_r(a_1, a_2)}{M_r(1-a_1, 1-a_2)}.$$

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