

## A REMARK ON AN INEQUALITY FOR THE PRIME COUNTING FUNCTION

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(communicated by J. Sándor)

*Abstract.* We note that the inequalities  $0.92 \frac{x}{\log(x)} < \pi(x) < 1.11 \frac{x}{\log(x)}$  do not hold for all  $x \geq 30$ , contrary to some references. These estimates on  $\pi(x)$  came up recently in papers on algebraic number theory.

### 1. Chebyshev's estimates for $\pi(x)$

Let  $\pi(x)$  denote the number of primes not greater than  $x$ , i.e.,

$$\pi(x) = \sum_{p \leq x} 1.$$

One of the first works on the function  $\pi(x)$  is due to Chebyshev. He proved (see [2]) in 1852 the following explicit inequalities for  $\pi(x)$ , holding for all  $x \geq x_0$  with some  $x_0$  sufficiently large:

$$\begin{aligned}
 c_1 \frac{x}{\log(x)} &< \pi(x) < c_2 \frac{x}{\log(x)}, \\
 c_1 &= \log(2^{1/2} 3^{1/3} 5^{1/5} 30^{-1/30}) \approx 0.921292022934, \\
 c_2 &= \frac{6}{5} c_1 \approx 1.10555042752.
 \end{aligned}$$

This can be found in many books on analytic number theory (see for example [1], [3], [11] and [14]). But it seems that this result is sometimes cited incorrectly: it is claimed that the estimates are valid for all  $x \geq 30$ . For example, in [6], page 21 we read that

$$c_1 \frac{x}{\log(x)} < \pi(x) < c_2 \frac{x}{\log(x)}, \quad \forall x \geq 30.$$

But a quick numerical computation shows that this is wrong. To give an example, take  $x = 100$ . Then we have  $\pi(x) = 25$  and

$$c_2 \frac{x}{\log(x)} \approx 24.00672250690558538515780234 < 25.$$

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Actually, the inequality is far from true for small  $x$ . We have the following result:

**THEOREM 1.1.** *Let  $c_2 \approx 1.10555042752$  be Chebyshev's constant. Then the inequality*

$$\pi(x) < c_2 \frac{x}{\log(x)}$$

*is true for all  $x \geq 96098$ . For  $x = 96097$  it is false.*

*Proof.* In [10] it is shown that

$$\pi(x) < \frac{x}{\log(x) - 1.11}, \quad x \geq 4.$$

The RHS is less or equal to  $c_2 x / \log(x)$  if and only if

$$x \geq \exp\left(\frac{1.11 \cdot c_2}{c_2 - 1}\right) \approx 112005.18.$$

This shows the claim for  $x \geq 112006$ . Since  $x / \log(x)$  is a monotonously increasing function it is enough to check the claimed estimate for intergers  $x$  in the intervall [96098, 112006] by computer. For  $x = 96097$  we have  $\pi(96097) = 9260$  and  $c_2 x / \log(x) \approx 9259.92$ .

The incorrect inequality was also used in a former version of Khare's proof of Serre's modularity conjecture for the level one case, see [8], [9]. Let  $\mathbb{F}$  be a finite field of characteristic  $p$ . The conjecture stated that an odd, irreducible Galois representation  $\rho: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F})$  which is unramified outside  $p$  is associated to a modular form on  $SL_2(\mathbb{Z})$ . Khare's proof is an elaborate induction on  $p$ . Starting with a  $p$  for which the conjecture is known one wants to prove the conjecture for a larger prime  $P$ . Kahre's arguments do only work if  $P$  and  $p$  are not Fermat primes, and if

$$\frac{P}{p} \leq a$$

for certain values  $a > 1$ , close to 1. At this point Khare used the incorrect estimate on  $\pi(x)$ , as explained above. Fortunately the proof easily could be repaired by using better estimates on  $\pi(x)$  provided by Rosser and Schoenfeld [12], and Dusart [4].

Indeed, P. Dusart proved inequalities for  $\pi(x)$  which are much better than Chebyshev's estimates. He verifies this for smaller  $x$  numerically. Nevertheless he claims in his thesis [5], that Chebyshev gave the following inequality

$$0.92 \frac{x}{\log(x)} < \pi(x) < 1.11 \frac{x}{\log(x)}, \quad x \geq 30,$$

which is equally wrong.

The question is: where lies the origin for this error? Chebyshev himself proved inequalities in [2] with his constants  $c_1$  and  $c_2 = \frac{6}{5}c_1$  indeed for all  $x \geq 30$ , but for inequalities involving  $\psi(x) = \sum_{n \leq x} \Lambda(n)$  instead of  $\pi(x)$ . His estimates concerning

$\psi(x)$  seem to be correct for all  $x \geq 30$ . For example, he shows by elementary means that, for all  $x \geq 30$ ,

$$\begin{aligned}\psi(x) &< \frac{6}{5}c_1x + \frac{5}{4\log(6)}\log^2(x) + \frac{5}{4}\log(x) + 1, \\ \psi(x) &> c_1x - \frac{5}{2}\log(x) - 1.\end{aligned}$$

To derive from this inequalities on  $\pi(x)$  for  $x \geq 30$ , we have to estimate

$$\psi(x) = \sum_{p \leq x} \left[ \frac{\log(x)}{\log(p)} \right] \log(p).$$

Using the estimates  $[y] \leq y < [y] + 1 \leq 2[y]$  for  $y \geq 1$  we obtain

$$\psi(x) \leq \pi(x) \log(x) \leq 2\psi(x), \quad x \geq 2.$$

On the RHS we cannot do easily much better than  $2\psi(x)$ . Hence we obtain

$$c_1 \frac{x}{\log(x)} < \pi(x) < 2c_2 \frac{x}{\log(x)}, \quad x \geq 30.$$

On the other hand we know that

$$\pi(x) = \frac{\psi(x)}{\log(x)} + O\left(\frac{x}{\log^2(x)}\right), \quad x \geq 2,$$

so that we obtain, as  $x$  tends to infinity,

$$(c_1 + o(1)) \frac{x}{\log(x)} \leq \pi(x) \leq (c_2 + o(1)) \frac{x}{\log(x)}.$$

Chebyshev used these estimates to prove Bertrand's postulate: each interval  $(n, 2n]$  for  $n \geq 1$  contains at least one prime. Moreover his results were a first step towards the proof of the prime number theorem.

## 2. Other estimates for $\pi(x)$

There are many interesting inequalities on the function  $\pi(x)$ . Let us first consider inequalities of the form

$$A \frac{x}{\log(x)} < \pi(x) < B \frac{x}{\log(x)}$$

for all  $x \geq x_0$ , where  $x_0$  depends on the constant  $A \leq 1$  and respectively on  $B > 1$ . On the LHS we can choose  $A$  equal to 1, if  $x \geq 17$ . In fact, we have [5]

$$\frac{x}{\log(x)} < \pi(x), \quad \forall x \geq 17.$$

Note that for  $x = 16.999$  we have  $x/\log(x) \approx 6.0000257$ , but  $\pi(x) = 6$ . Consider the RHS of the above inequalities: if we want to hold such inequalities on  $\pi(x)$  for all  $x \geq x_0$  with a smaller  $x_0$ , we need to enlarge the constant  $B$ . Conversely, if we

need this inequality for smaller  $B$ , we have to enlarge  $x_0$ . The prime number theorem ensures that we can choose  $B$  as close to 1 as we want, provided  $x_0$  is sufficiently large. The following result of Dusart [4] enables us to derive adjusted versions for the above inequalities:

**THEOREM 2.1.** (Dusart) *For real  $x$  we have the following sharp bounds:*

$$\begin{aligned}\pi(x) &\geq \frac{x}{\log(x)} \left( 1 + \frac{1}{\log(x)} + \frac{1.8}{\log^2(x)} \right), & x \geq 32299, \\ \pi(x) &\leq \frac{x}{\log(x)} \left( 1 + \frac{1}{\log(x)} + \frac{2.51}{\log^2(x)} \right), & x \geq 355991.\end{aligned}$$

One can derive, for example, the following inequalities.

$$\begin{aligned}\pi(x) &< 1.095 \cdot \frac{x}{\log(x)}, & x \geq 284860, \\ \pi(x) &< 1.25506 \cdot \frac{x}{\log(x)}, & x \geq 17.\end{aligned}$$

Among other inequalities on  $\pi(x)$  we mention the following ones:

$$\frac{x}{\log(x) - m} < \pi(x) < \frac{x}{\log(x) - M}$$

for all  $x \geq x_0$  with real constants  $m$  and  $M$ . They have been studied by various authors. A good reference is the article [10]. There it is shown, for example, that

$$\begin{aligned}\pi(x) &> \frac{x}{\log(x) - \frac{28}{29}}, & x \geq 3299, \\ \pi(x) &< \frac{x}{\log(x) - 1.11}, & x \geq 4.\end{aligned}$$

The second inequality can also be used to obtain results on our estimate  $\pi(x) < B \frac{x}{\log(x)}$ , in particular for smaller  $x$ , where the second inequality of Theorem 2.1 is not valid. However we have

$$\frac{x}{\log(x)} \left( 1 + \frac{1}{\log(x)} + \frac{2.51}{\log^2(x)} \right) < \frac{x}{\log(x) - 1.11}, \quad x \geq 28516.$$

For  $x > 10^6$  and  $a = 1.08366$  we can use [10]

$$\pi(x) < \frac{x}{\log(x) - a}.$$

Here the upper bound of Dusart is better only as long as  $x \geq 2846396$ . Finally we mention the book [13], providing many references on inequalities on  $\pi(x)$ , and the recent article [7], where lower and upper bounds for  $\pi(x)$  of the form  $\frac{n}{H_n - c}$  are discussed, where  $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$ .

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