Mathematical
Inequalities
& Applications
Volume 10, Number 1 (2007), 29–43

DECISION FUNCTIONS AND CHARACTERIZATION OF THEIR PROPERTIES

BORBÁLA FAZEKAS

(*communicated by A. Prekopa*)

1. Introduction

In order to introduce the notion of a decision function, let us consider one of the possible examples. We assume a parliament whose members would like to make fair decisions. All the members have their own opinions and they are looking for a procedure that produces a final common decision from these opinions. A decision function accomplishes this procedure.

Naturally we would like our decisions to be fair in a certain sense, so a decision function has to possess some "good" properties. Namely,

(i) the final decision should not depend on the order of the opinions (symmetry);

(ii) if everybody has the same opinion, then the final decision should be equal to this joint opinion (reflexivity);

(iii) if there are two groups in the parliament (say left wing and right wing), and both groups have already made their decision, then the final common decision should be between these two decisions (internality);

(iv) the opinion of one member should not influence much the final decision, if "sufficiently many" members take part at the session, i.e., odd ball opinions are neglected (regularity);

(v) and, finally, a decision function has to be able to work with arbitrarily many opinions (for example, if due to an epidemy some of the members cannot take part at the session of the parliament).

Now let us define decision functions precisely. (This definition and the subsequent theorem are special (real) cases of the definition and theorem in [9].)

Let I be an open real interval throughout this paper. Further, let $\mathbb{R}_+$ denote the set of all positive real numbers and $\mathbb{R}_0^n$ the set $\{(r_1,\dots,r_n)\in\mathbb{R}^n \mid r_1,\dots,r_n \geqslant 0,\ r_1+\dots+r_n>0\}$ for $n\in\mathbb{N}$.

DEFINITION 1.1 A function $D\colon \bigcup_{i=1}^{\infty} I^i \to I$ is called a *decision function*, if it is

(i) *symmetric*, i.e.,

$$D(x_1,\dots,x_n)=D(x_{p_1},\dots,x_{p_n})$$

Mathematics subject classification (2000): 26B25, 90B50, 91B06, 93E24, 26E60, 26E25.

Key words and phrases: decision function, convexity, least squares method, monotonicity, continuity, power means, Gini means, quasi-arithmetic means.

Paper MIA-10-04

holds for every $n \in \mathbb{N}$, $x_1, \dots, x_n \in I$ and for every permutation $p\colon \{1,\dots,n\} \to \{1,\dots,n\}$;

(*ii*) *reflexive,* i.e.,

$$D(\underbrace{x,\dots,x}_{n-\text{times}}) = x$$

holds for every $n \in \mathbb{N}$ and $x \in I$;

(*iii*) *internal,* i.e.,

$$\begin{aligned}\min\{D(x_1,\dots,x_n), D(x_{n+1},\dots,x_{n+m})\} &\leqslant D(x_1,\dots,x_{n+m}) \\ &\leqslant \max\{D(x_1,\dots,x_n), D(x_{n+1},\dots,x_{n+m})\}\end{aligned}$$

holds for every $n, m \in \mathbb{N}$ and $x_1,\dots,x_{n+m} \in I$;

(*iv*) *regular*, i.e.,

$$\lim_{k\to\infty} D(x_0, \underbrace{x_1,\dots,x_1}_{k-\text{times}}, \dots, \underbrace{x_n,\dots,x_n}_{k-\text{times}}) = D(x_1,\dots,x_n)$$

holds for every $n \in \mathbb{N}$ and $x_0, x_1, \dots, x_n \in I$.

EXAMPLE 1.1. It is easy to see that the arithmetic mean is an example for a decision function. Moreover, this function gives the idea how one can generate other decision functions. Let $d\colon I \times I \to \mathbb{R}$ be defined by $d(x,y) = (x-y)^2$. We can obtain the arithmetic mean via the well-known least squares method as follows:

$$\frac{x_1 + \cdots + x_n}{n} = \operatorname*{argmin}_{y\in I} \big(d(x_1,y) + \cdots + d(x_n,y)\big),$$

where $\operatorname{argmin}_{x\in I} f(x)$ denotes the (unique) point of minimum of the function f over I. We can generalize this method to obtain further decision functions.

DEFINITION 1.2. A function $d\colon I \times I \to \mathbb{R}$ is called a *decision generating function*, if

(*i*) for every $n \in \mathbb{N}$ and $x_1,\dots,x_n \in I$ the function

$$y \mapsto d(x_1,y) + \cdots + d(x_n,y) \tag{1.1}$$

is strongly convex (i.e., convex and there is no proper subinterval of I, where it is constant);

(*ii*) for every $x, y \in I$ it holds that

$$0 = d(x,x) \leqslant d(x,y).$$

Let $\delta(I)$ denote the set of all decision generating functions over I.

For every decision generating function $d \in \delta(I)$ we can define a function D_d with the help of a generalized least squares method as follows:

$$D_d(x_1,\dots,x_n) = \operatorname*{argmin}_{y\in I} \big(d(x_1,y) + \cdots + d(x_n,y)\big), \quad (n \in \mathbb{N},\ x_1,\dots,x_n \in I).$$

(This definition is correct because of the strong convexity of the function in (1.1).)

The connection between the functions of form D_d and the decision functions is described by the following result due to Páles [9].

THEOREM 1.1. *Let $d \in \delta(I)$ be a decision generating function. Then the function D_d generated by d is a decision function.*

Conversely, if $D\colon \bigcup_{i=1}^{\infty} I^i \to I$ is a decision function, then there exists a decision generating function $d \in \delta(I)$ such that $D = D_d$.

EXAMPLE 1.2. Power means (or Hölder means, see [6], [5], [1], [7], [2] for the definition) are also decision functions. To see this, for $p \in \mathbb{R}$, define $d_p\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ by

$$d_p(x,y) = \begin{cases} \dfrac{px^{p+1} - (p+1)x^p y + y^{p+1}}{p(p+1)}, & \text{if} \quad p(p+1) \neq 0, \\ y(\ln y - \ln x) + x - y, & \text{if} \quad p = 0, \\ \ln x - \ln y + \dfrac{y}{x} - 1, & \text{if} \quad p+1 = 0. \end{cases}$$

Then it holds that $d_p(x,x) = 0$, $\partial_2 d_p(x,x) = 0$ and $\partial_2\partial_2 d_p(x,y) = y^{p-1} > 0$. Therefore the function $y \mapsto d_p(x,y)$ is strictly convex (and thus the mapping $y \mapsto d_p(x_1,y) + \cdots + d_p(x_n,y)$ as well) and $0 = d_p(x,x) \leqslant d_p(x,y)$ for $x,y > 0$. Thus d_p is a decision generating function.

For fixed $x_1,\ldots,x_n > 0$, the function

$$y \mapsto d_p(x_1,y) + \cdots + d_p(x_n,y)$$

is minimal at $y = y_0$ if and only if

$$\partial_2 d_p(x_1,y_0) + \cdots + \partial_2 d_p(x_n,y_0) = 0,$$

i.e., if

$$0 = \sum_{i=1}^{n} \frac{1}{p}(y_0^p - x_i^p), \qquad \text{if} \quad p \neq 0,$$

$$0 = \sum_{i=1}^{n} (\ln y_0 - \ln x_i), \quad \text{if} \quad p = 0.$$

This yields

$$y_0 = \begin{cases} \left(\dfrac{x_1^p + \cdots + x_n^p}{n}\right)^{\frac{1}{p}}, & \text{if} \quad p \neq 0, \\ \sqrt[n]{x_1 \cdots x_n}, & \text{if} \quad p = 0, \end{cases}$$

i.e., in view of Theorem 1.1, the decision function generated by d_p is the p^{th} power mean.

EXAMPLE 1.3. Let $\varphi\colon I \to \mathbb{R}$ be a continuous and strictly monotone increasing function. Define $d_\varphi\colon I \times I \to \mathbb{R}$ by

$$d_\varphi(x,y) = (x-y)\varphi(x) + \int_x^y \varphi(t)\,dt.$$

One can see, as above, that d_φ is a decision generating function and it generates the decision function $D_{d_\varphi}(x_1,\ldots,x_n) = \varphi^{-1}\left(\dfrac{\varphi(x_1) + \cdots + \varphi(x_n)}{n}\right)$. Thus, quasi-arithmetic means (see [5], [1], [7], [2]) turn out to be decision functions, too.

EXAMPLE 1.4. The Gini mean $G_{p,q}$ (cf. [4]) of the positive real numbers $x_1,\dots,x_n$ is defined by

$$G_{p,q}(x_1,\dots,x_n)=\begin{cases}\left(\dfrac{x_1^p+\cdots+x_n^p}{x_1^q+\cdots+x_n^q}\right)^{\frac{1}{p-q}}, & \text{if } \quad p\neq q,\\[2ex] \exp\left(\dfrac{x_1^p\ln x_1+\cdots+x_n^p\ln x_n}{x_1^p+\cdots+x_n^p}\right), & \text{if } \quad p=q,\end{cases}$$

where $p,q\in\mathbb{R}$ are fixed parameters. One can easily verify, by adapting the above arguments, that it is also a decision function generated by (e.g.) the decision generating function

$$d_{p,q}(x,y)=\begin{cases}\dfrac{(p-q)x^{p+1}-(p-q+1)x^py+x^qy^{p-q+1}}{(p-q)(p-q+1)}, & \text{if } (p-q)(p-q+1)\neq 0,\\ yx^p(\ln y-\ln x)+x^p(x-y), & \text{if } p-q=0,\\ x^{p+1}(\ln x-\ln y)+x^p(y-x), & \text{if } p-q+1=0.\end{cases}$$

Observe that, if $q=0$ then $d_{p,q}$ and $G_{p,q}$ reduce to d_p and to the p^{th} power mean, respectively.

EXAMPLE 1.5. Without going into the details, we mention that deviation means (cf. [3]) and also quasi-deviation means (cf. [8], [10]) are decision functions as well.

EXAMPLE 1.6. Not every "mean" is a decision function. The empirical median (defined for $x_1\leqslant\cdots\leqslant x_n$ as $x_{\frac{n+1}{2}}$, if n is odd and as $\frac{1}{2}\left(x_{\frac{n}{2}}+x_{\frac{n}{2}+1}\right)$, if n is even) is not regular (check e.g. the case, when $n=2$ and $x_0=1$, $x_1=2$, $x_2=3$). Though one can define the empirical median as a setvalued function, namely, for $x_1\leqslant\cdots\leqslant x_n$ as $x_{\frac{n+1}{2}}$, if n is odd and as the closed interval $[x_{\frac{n}{2}},x_{\frac{n}{2}+1}]$, if n is even. This function is not a decision function in the above sense. One could however obtain this function also via a similar generalized least squares method as above by taking the function $d(x,y)=|x-y|$. Now the mapping $y\mapsto d(x_1,y)+\cdots+d(x_n,y)$ is not necessarily strongly convex and thus the set of all points of minimum is an interval, (see also Lemma 2.1), which is just the (set-valued) median of $x_1,\dots,x_n$. We will discuss this example further in Section 3.

Our aim is to characterize the properties of decision functions in terms of the properties of their decision generating functions. In the sequel, we intend to investigate their continuity and monotonicity properties.

2. Convex functions

In this section we give an overview of some properties of convex functions that we need later. The following known Lemma characterizes the points of minimum of a convex function (see, e.g. [11]).

LEMMA 2.1. *Let $f:I\to\mathbb{R}$ be a convex, non-monotone function. Then the set of* *all points of minimum of f is an $[s,t]$ compact interval. Furthermore, it holds that*

$$\begin{aligned} f'_-(x) < 0 &\quad \textit{and} \quad f'_+(x) < 0\,, \textit{if} \quad x < s, \\ f'_-(x) \leqslant 0 &\quad \textit{and} \quad f'_+(x) \geqslant 0\,, \textit{if} \quad s \leqslant x \leqslant t, \\ f'_-(x) > 0 &\quad \textit{and} \quad f'_+(x) > 0\,, \textit{if} \quad x > t, \end{aligned}$$

where f'_+ and f'_- denote the left and right derivatives of f, respectively.

DEFINITION 2.1. We call the above interval $[s,t]$ *minimality interval*, i.e., the set of all points of minimum of a convex function f and denote it by $\operatorname{minint} f$ or by $\operatorname{minint}_{x\in I} f(x)$.

LEMMA 2.2. *Let $f_m : I \to \mathbb{R}$ be a sequence of non-monotone convex functions and let $f : I \to \mathbb{R}$ be a non-monotone function such that $\lim_{m\to\infty} f_m(x) = f(x)$ for every $x \in I$. Then f is convex. Furthermore, it holds that*

$$\overline{\lim_{m\to\infty}} \operatorname{minint} f_m \subseteq \operatorname{minint} f. \tag{2.1}$$

Proof. The convexity of f is trivial. To verify equation (2.1), let $\varepsilon_0 > 0$ be arbitrary and we show that there exists a $\delta_0 > 0$ such that

$$\operatorname{minint} f_m \subseteq \operatorname{minint} f + [-\varepsilon_0, \varepsilon_0] \tag{2.2}$$

holds for every $m > \delta_0$. Denote $[s,t] = \operatorname{minint} f$ and $\varepsilon = \min\{f(s-\varepsilon_0) - f(s),\ f(t+\varepsilon_0) - f(t)\}$. Because of the pointwise convergence of the sequence f_m, there exists a δ_0 such that

$$\begin{aligned} |\, f_m(s-\varepsilon_0) - f(s-\varepsilon_0)\,| &< \frac{\varepsilon}{2}, \quad |\, f_m(s) - f(s)\,| < \frac{\varepsilon}{2}, \\ |\, f_m(t+\varepsilon_0) - f(t+\varepsilon_0)\,| &< \frac{\varepsilon}{2}, \quad |\, f_m(t) - f(t)\,| < \frac{\varepsilon}{2} \end{aligned}$$

hold for every $m > \delta_0$. Thus

$$f_m(s-\varepsilon_0) - f_m(s) > f(s-\varepsilon_0) - f(s) - \varepsilon \geqslant \varepsilon - \varepsilon = 0.$$

Analogously,

$$f_m(t+\varepsilon_0) - f_m(t) > 0.$$

From the convexity of f_m follows equation (2.2) for every $m > \delta_0$. Thus (2.1) really holds.

COROLLARY 2.3. *Let $f_m : I \to \mathbb{R}$ be a sequence of non-monotone strongly convex functions and let $f : I \to \mathbb{R}$ be a non-monotone strongly convex function such that $\lim_{m\to\infty} f_m(x) = f(x)$ for every $x \in I$. Then it holds that*

$$\lim_{m\to\infty} \operatorname{argmin} f_m = \operatorname{argmin} f.$$

Let us introduce the following relation on $\mathcal{P}(I)$: let $J_1 \preceq J_2$ for $J_1, J_2 \subseteq I$, if $\sup J_1 \leqslant \inf J_2$. The relation $\succeq$ can be defined analogously. If, for example, $J_1 = \{x\}$, we will shortly write $x \preceq J_2$.

LEMMA 2.4. *Let* $f_1, f_2 \colon I \to \mathbb{R}$ *be non-monotone convex functions such that*

$$\mathrm{minint} f_1 \preceq \mathrm{minint} f_2.$$

Then it holds that

$$\mathrm{minint} f_1 \preceq \mathrm{minint}(\alpha \cdot f_1 + \beta \cdot f_2) \preceq \mathrm{minint} f_2$$

for every $\alpha, \beta \in \mathbb{R}_+$.

Proof. From the convexity it follows that $\beta \cdot f_2$ decreases strictly on the interval $]\inf I, \min(\mathrm{minint} f_2)[$, while $\alpha \cdot f_1$ decreases on the interval $]\inf I, \max(\mathrm{minint} f_1)[$ and thus $\alpha \cdot f_1 + \beta \cdot f_2$ decreases strictly on the interval $]\inf I, \max(\mathrm{minint} f_1)[$. Analogously $\alpha \cdot f_1 + \beta \cdot f_2$ increases strictly on the interval $]\min(\mathrm{minint} f_2), \sup I[$. From the convexity of $\alpha \cdot f_1 + \beta \cdot f_2$ follows the statement of our Lemma.

3. Weighted decision functions

In this section we introduce the notion of a weighted decision function and describe its basic properties. Denote

$$\widetilde{D}_d \begin{pmatrix} x_1 & , & \dots & , & x_n \\ m_1 & , & \dots & , & m_n \end{pmatrix} = D_d\Big(\underbrace{x_1, \dots, x_1}_{m_1}, \dots, \underbrace{x_n, \dots, x_n}_{m_n}\Big)$$

for $m_1, \dots, m_n \in \mathbb{N}$ and for a decision generating function $d \in \delta(I)$. Multiplying a convex function by a positive number, the points of minimum remain unchanged, so it is trivially true that

$$\operatorname*{argmin}_{y \in I}(m_1 \cdot d(x_1, y) + \dots + m_n \cdot d(x_n, y)) = \operatorname*{argmin}_{y \in I}\left(\frac{m_1}{k} \cdot d(x_1, y) + \dots + \frac{m_n}{k} \cdot d(x_n, y)\right)$$

for $m_1, \dots, m_n, k \in \mathbb{N}$. This validates the definition of the weighted decision function $\widetilde{D}_d$ for nonnegative rational weights as follows:

$$\widetilde{D}_d \begin{pmatrix} x_1 & , & \dots & , & x_n \\ \frac{m_1}{k} & , & \dots & , & \frac{m_n}{k} \end{pmatrix} = \widetilde{D}_d \begin{pmatrix} x_1 & , & \dots & , & x_n \\ m_1 & , & \dots & , & m_n \end{pmatrix}.$$

To extend the weighted decision function also for irrational weights, it could be difficult to use the above approach. Instead, let us consider the function defined by

$$y \mapsto r_1 \cdot d(x_1, y) + \dots + r_n \cdot d(x_n, y)$$

(for fixed $n \in \mathbb{N}$, $x_1, \dots, x_n \in I$ and $(r_1, \dots, r_n) \in \mathbb{R}_0^n$). This function is convex, but not necessarily strongly convex. So it does not necessarily have a unique point of minimum, but it always has a minimality interval. This minimality interval seems to be a natural candidate for the value of the weighted decision function $\widetilde{D}_d$ at the point $(x_1, \dots, x_n, r_1, \dots, r_n)$. So we define the weighted decision function $\widetilde{D}_d$ as a set-valued function as follows:

DEFINITION 3.1. Let $d \in \delta(I)$ be a decision generating function. Then the function

$$\widetilde{D}_d \colon \bigcup_{i=1}^{\infty} I^i \times \mathbb{R}_0^i \to \mathcal{P}(I),$$

defined by

$$\widetilde{D}_d \begin{pmatrix} x_1 & , & \dots & , & x_n \\ r_1 & , & \dots & , & r_n \end{pmatrix} = \operatorname*{minint}_{y \in I}(r_1 \cdot d(x_1, y) + \dots + r_n \cdot d(x_n, y))$$

for every $n \in \mathbb{N}$, $x_1, \dots, x_n \in I$ and $(r_1, \dots, r_n) \in \mathbb{R}_0^n$ is called a *weighted decision function*.

(For rational weights this definition naturally coincides with the concept of the weighted decision function introduced above for rational weights.)

According to Lemma 2.1, for every $x_1, \dots, x_n \in I$ and $(r_1, \dots, r_n) \in \mathbb{R}_0^n$ the minimality interval of the function $y \mapsto r_1 \cdot d(x_1, y) + \dots + r_n \cdot d(x_n, y)$ can be described as follows. The solutions y of the system of inequalities

$$r_1 \cdot d^-(x_1, y) + \dots + r_n \cdot d^-(x_n, y) \leqslant 0$$
$$r_1 \cdot d^+(x_1, y) + \dots + r_n \cdot d^+(x_n, y) \geqslant 0$$

form an interval $[s, t]$ such that

$$[s, t] = \widetilde{D}_d \begin{pmatrix} x_1 & , & \dots & , & x_n \\ r_1 & , & \dots & , & r_n \end{pmatrix}.$$

Here d^+ denotes the right derivative, while d^- denotes the left derivative of d in its second variable.

EXMPLE 3.2. Let H be a positive Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ and let further be given an injective function $q \colon I \to H$. Then the function $d_{\text{med}} \colon I \times I \to \mathbb{R}$, $d_{\text{med}}(x, y) = q(x)|x - y|$ is a decision generating function. The positivity and convexity properties are obvious. For the strong convexity it suffices $d^-_{\text{med}}(x_1, y) + \dots + d^-_{\text{med}}(x_n, y) \neq 0$ to hold for every $x_1, \dots, x_n, y \in I$, i.e., $\varepsilon_1 q(x_1) + \dots + \varepsilon_n q(x_n) \neq 0$, where $\varepsilon_i = 1$, if $y > x_i$ and $\varepsilon_i = -1$, if $y \leqslant x_i$. This condition is indeed satisfied in case of a Hamel basis.

With the weighted decision function $\widetilde{D}_{d_{\text{med}}}$ generated by d_{med} one obtains at the point $(x_1, \dots, x_n, \frac{1}{q(x_1)}, \dots, \frac{1}{q(x_n)})$ the (set-valued) median of $x_1, \dots, x_n$. So we gain the (set-valued) median as a weighted decision function with suitable weights.

The following Lemmata describe the properties of the weighted decision function. The first Lemma can be easily verified.

LEMMA 3.1. *Let $d \in \delta(I)$ be a decision generating function. Then the weighted decision function $\widetilde{D}_d$ generated by d is a mean according to the relation $\preceq$, i.e., for every $n \in \mathbb{N}$, $x_1 \dots, x_n \in I$, $(r_1, \dots, r_n) \in \mathbb{R}_0^n$ it holds that*

$$\min\{x_1, \dots, x_n\} \preceq \widetilde{D}_d \begin{pmatrix} x_1 & , & \dots & , & x_n \\ r_1 & , & \dots & , & r_n \end{pmatrix} \preceq \max\{x_1, \dots, x_n\}.$$

The second Lemma describes a kind of monotonicity property of the weighted decision function.

LEMMA 3.2. *Let $d \in \delta(I)$ be a decision generating function. Moreover, let $x_1, \dots, x_n \in I$, $(s_1, \dots, s_n)$, $(r_1, \dots, r_n) \in \mathbb{R}_0^n$ be fixed such that*

$$\widetilde{D}_d \begin{pmatrix} x_1 & ,\dots, & x_n \\ r_1 & ,\dots, & r_n \end{pmatrix} \preceq \widetilde{D}_d \begin{pmatrix} x_1 & ,\dots, & x_n \\ s_1 & ,\dots, & s_n \end{pmatrix}.$$

Then the mapping $[0,1] \to \mathcal{P}(I)$ defined by

$$t \mapsto \widetilde{D}_d \begin{pmatrix} x_1 & ,\dots, & x_n \\ t \cdot r_1 + (1-t)\cdot s_1 & ,\dots, & t \cdot r_n + (1-t)\cdot s_n \end{pmatrix}$$

is decreasing according to the relation $\preceq$, i.e., if $0 \leqslant t_1 < t_2 \leqslant 1$, then

$$\widetilde{D}_d \begin{pmatrix} x_1 & ,\dots, & x_n \\ t_2 \cdot r_1 + (1-t_2)\cdot s_1 & ,\dots, & t_2 \cdot r_n + (1-t_2)\cdot s_n \end{pmatrix}$$
$$\preceq \widetilde{D}_d \begin{pmatrix} x_1 & ,\dots, & x_n \\ t_1 \cdot r_1 + (1-t_1)\cdot s_1 & ,\dots, & t_1 \cdot r_n + (1-t_1)\cdot s_n \end{pmatrix}.$$

Proof. Let $0 < t_1 < t_2 < 1$. According to Lemma 2.4,

$$\widetilde{D}_d \begin{pmatrix} x_1 & ,\dots, & x_n \\ r_1 & ,\dots, & r_n \end{pmatrix} = \operatorname*{minint}_{y\in I} (r_1 \cdot d(x_1,y) + \dots + r_n \cdot d(x_n,y))$$
$$\preceq \operatorname*{minint}_{y\in I} \big(t_1 \cdot \big(r_1 \cdot d(x_1,y) + \dots + r_n \cdot d(x_n,y)\big) + (1-t_1)\big(s_1 \cdot d(x_1,y) + \dots + s_n \cdot d(x_n,y)\big)\big)$$
$$\preceq \operatorname*{minint}_{y\in I} \big(s_1 \cdot d(x_1,y) + \dots + s_n \cdot d(x_n,y)\big) = \widetilde{D}_d \begin{pmatrix} x_1 & ,\dots, & x_n \\ s_1 & ,\dots, & s_n \end{pmatrix}.$$

Thus we can apply Lemma 2.4 again for

$$f_1(y) = r_1 \cdot d(x_1,y) + \dots + r_n \cdot d(x_n,y)$$

and

$$f_2(y) = (t_1 \cdot r_1 + (1-t_1)\cdot s_1)\cdot d(x_1,y) + \dots + (t_1 \cdot r_n + (1-t_1)\cdot s_n)\cdot d(x_n,y).$$

The statement of the Lemma follows immediately by choosing $\alpha = \frac{t_2-t_1}{1-t_1}$ and $\beta = \frac{1-t_2}{1-t_1}$.

In the case, when $t_1 = 0$ or $t_2 = 1$, the assertion can be obtained analogously.

COROLLARY 3.3. *Let $d \in \delta(I)$ be a decision generating function. For every fixed $x_1, x_2 \in I$ with $x_1 < x_2$ the mapping $[0,1] \to \mathcal{P}\big([x_1,x_2]\big)$ defined by*

$$t \mapsto \widetilde{D}_d \begin{pmatrix} x_1 & , & x_2 \\ t & , & 1-t \end{pmatrix}$$

is decreasing according to the relation $\preceq$.

Proof. The statement follows from Lemma 3.2 with $n = 2$, $(r_1, r_2) = (1,0)$ and $(s_1, s_2) = (0,1)$.

The following Lemma describes a continuity property of the weighted decision function.

THEOREM 3.4. *Let $d \in \delta(I)$ be a decision generating function. The mapping defined by*

$$(r_1,\dots,r_n) \mapsto \widetilde{D}_d \begin{pmatrix} x_1 & ,\dots, & x_n \\ r_1 & ,\dots, & r_n \end{pmatrix}$$

is upper semi-continuous for every fixed $x_1,\dots,x_n \in I$.

Proof. Let $(x_1,\dots,x_n) \in I^n$ and $(r_1,\dots,r_n) \in \mathbb{R}_0^n$ be arbitrary fixed and let $(r_1^m,\dots,r_n^m)$ be a sequence in $\mathbb{R}_0^n$ with $\lim_{m\to\infty}(r_1^m,\dots,r_n^m) = (r_1,\dots,r_n)$. The upper semi-continuity at the point $(r_1,\dots,r_n)$ follows from Lemma 2.2 by choosing

$$f_m(y) = r_1^m \cdot d(x_1,y) + \cdots + r_n^m \cdot d(x_n,y)$$

and

$$f(y) = r_1 \cdot d(x_1,y) + \cdots + r_n \cdot d(x_n,y).$$

For the weighted decision function we can verify properties analogous to those of the decision function in Definition 1.1.

THEOREM 3.5. *Let $d \in \delta(I)$ be a decision generating function. Then it holds that*

(i) *$\widetilde{D}_d$ is symmetric, i.e.,*

$$\widetilde{D}_d \begin{pmatrix} x_1 & ,\dots, & x_n \\ r_1 & ,\dots, & r_n \end{pmatrix} = \widetilde{D}_d \begin{pmatrix} x_{p_1} & ,\dots, & x_{p_n} \\ r_{p_1} & ,\dots, & r_{p_n} \end{pmatrix}$$

holds for every $n \in \mathbb{N}$, $x_1,\dots,x_n \in I$, $(r_1,\dots,r_n) \in \mathbb{R}_0^n$ and for every permutation $p\colon \{1,\dots,n\} \to \{1,\dots,n\}$;

(ii) *reflexive, i.e.,*

$$\widetilde{D}_d \begin{pmatrix} x & ,\dots, & x \\ r_1 & ,\dots, & r_n \end{pmatrix} = x$$

holds for every $n \in \mathbb{N}$, $x \in I$ and $(r_1,\dots,r_n) \in \mathbb{R}_0^n$;

(iii) *internal, i.e., if*

$$\widetilde{D}_d \begin{pmatrix} x_1 & ,\dots, & x_n \\ r_1 & ,\dots, & r_n \end{pmatrix} \preceq \widetilde{D}_d \begin{pmatrix} x_{n+1} & ,\dots, & x_{n+m} \\ r_{n+1} & ,\dots, & r_{n+m} \end{pmatrix},$$

then

$$\widetilde{D}_d \begin{pmatrix} x_1 & ,\dots, & x_n \\ r_1 & ,\dots, & r_n \end{pmatrix} \preceq \widetilde{D}_d \begin{pmatrix} x_1 & ,\dots, & x_{n+m} \\ r_1 & ,\dots, & r_{n+m} \end{pmatrix} \preceq \widetilde{D}_d \begin{pmatrix} x_{n+1} & ,\dots, & x_{n+m} \\ r_{n+1} & ,\dots, & r_{n+m} \end{pmatrix}$$

holds for every $n,m \in \mathbb{N}$, $x_1,\dots,x_{n+m} \in I$, $(r_1,\dots,r_n) \in \mathbb{R}_0^n$ and $(r_{n+1},\dots,r_{n+m}) \in \mathbb{R}_0^m$;

(iv) *regular, i.e.,*

$$\overline{\lim_{k\to\infty}}\, \widetilde{D}_d \begin{pmatrix} x_0 & , & x_1 & ,\dots, & x_n \\ r_0 & , & k\cdot r_1 & ,\dots, & k\cdot r_n \end{pmatrix} \subseteq \widetilde{D}_d \begin{pmatrix} x_1 & ,\dots, & x_n \\ r_1 & ,\dots, & r_n \end{pmatrix}$$

holds for every $n \in \mathbb{N}$, $x_0,x_1,\dots,x_n \in I$, $r_0 \in \mathbb{R}_+$ and $(r_1,\dots,r_n) \in \mathbb{R}_0^n$.

Proof. The symmetry and reflexivity properties are trivial.

The internality property follows from Lemma 2.4 by choosing

$$f_1(y) = r_1 \cdot d(x_1, y) + \cdots + r_n \cdot d(x_n, y),$$
$$f_2(y) = r_{n+1} \cdot d(x_{n+1}, y) + \cdots + r_{n+m} \cdot d(x_{n+m}, y),$$

and $\alpha = \beta = 1$.

At last we show the regularity property. It holds that

$$\widetilde{D}_d \begin{pmatrix} x_0 & , & x_1 & , \dots , & x_n \\ r_0 & , & k \cdot r_1 & , \dots , & k \cdot r_n \end{pmatrix} = \widetilde{D}_d \begin{pmatrix} x_0 & , & x_1 & , \dots , & x_n \\ \frac{r_0}{k} & , & r_1 & , \dots , & r_n \end{pmatrix}.$$

According to Theorem 3.4, the weighted decision function is upper semi-continuous in the weights. Since $(\frac{r_0}{k}, r_1, \dots, r_n) \to (0, r_1, \dots, r_n)$ as $k \to \infty$, it holds that

$$\overline{\lim_{k\to\infty}}\, \widetilde{D}_d \begin{pmatrix} x_0 & , & x_1 & , \dots , & x_n \\ \frac{r_0}{k} & , & r_1 & , \dots , & r_n \end{pmatrix} \subseteq \widetilde{D}_d \begin{pmatrix} x_0 & , & x_1 & , \dots , & x_n \\ 0 & , & r_1 & , \dots , & r_n \end{pmatrix},$$

i.e.,

$$\overline{\lim_{k\to\infty}}\, \widetilde{D}_d \begin{pmatrix} x_0 & , & x_1 & , \dots , & x_n \\ \frac{r_0}{k} & , & r_1 & , \dots , & r_n \end{pmatrix} \subseteq \widetilde{D}_d \begin{pmatrix} x_1 & , \dots , & x_n \\ r_1 & , \dots , & r_n \end{pmatrix}.$$

That is what we wanted to prove.

4. Continuity properties of decision functions

We give a sufficient condition for the continuity of the decision function, which is at the same time a sufficient condition for the upper semi-continuity of the weighted decision function.

THEOREM 4.1. *Let $d \in \delta(I)$ be a decision generating function, D_d the decision function, $\widetilde{D}_d$ the weighted decision function generated by d. Let us assume that for every fixed $y \in I$ the mapping $x \mapsto d(x, y)$ is continuous. Then the following statements hold:*

(i) *D_d is continuous, (i.e., the mapping $(x_1, \dots, x_n) \mapsto D_d(x_1, \dots, x_n)$ is continuous for every $n \in \mathbb{N}$);*

(ii) *$\widetilde{D}_d$ is upper semi-continuous, (i.e., the mapping $(x_1, \dots, x_n, r_1, \dots, r_n) \mapsto \widetilde{D}_d \begin{pmatrix} x_1 & , \dots , & x_n \\ r_1 & , \dots , & r_n \end{pmatrix}$ is upper semi-continuous for every $n \in \mathbb{N}$).*

Proof. Let $(x_1, \dots, x_n) \in I^n$ and $(r_1, \dots, r_n) \in \mathbb{R}_0^n$ be arbitrary fixed. Let further $(x_1^m, \dots, x_n^m) \in I^n$ and $(r_1^m, \dots, r_n^m) \in \mathbb{R}_0^n$ be sequences such that $(x_1^m, \dots, x_n^m) \to (x_1, \dots, x_n)$ and $(r_1^m, \dots, r_n^m) \to (r_1, \dots, r_n)$ as $m \to \infty$. Then (i) follows from Corollary 2.3 by choosing

$$f_m(y) = d(x_1^m, y) + \cdots + d(x_n^m, y)$$

and

$$f(y) = d(x_1, y) + \cdots + d(x_n, y).$$

(ii) follows from Lemma 2.2 by choosing

$$f_m(y) = r_1^m \cdot d(x_1^m, y) + \cdots + r_n^m \cdot d(x_n^m, y)$$

and

$$f(y) = r_1 \cdot d(x_1, y) + \cdots + r_n \cdot d(x_n, y).$$

EXAMPLE 4.1 . It is obvious that all the decision functions mentioned in Section 1. are continuous. This also follows immediately from Theorem 4.1. It should be mentioned that the condition of the theorem is sufficient but, in general, not necessary for the upper semi-continuity of the weighted decision function. The set-valued median is upper semi-continuous, while d_{med} of Example 3.2 is not continuous in x.

5. Monotonicity of decision functions

In this section we give a necessary and sufficient condition for the monotonicity of the decision function which is at the same time a necessary and sufficient condition for the monotonicity of the weighted decision function.

DEFINITION 5.1 . A function $D\colon \cup_{i=1}^{\infty} I^i \to I$ is called *monotone (increasing)*, if

$$D(x_1, \ldots, x_i, \ldots, x_n) \leqslant D(x_1, \ldots, \tilde{x}_i, \ldots, x_n)$$

holds for every $n, i \in \mathbb{N}$, $i \leqslant n$, and $x_1, \ldots, x_n, \tilde{x}_i \in I$ with $x_i \leqslant \tilde{x}_i$.

(A decision function can never be monotone decreasing because of the reflexivity property described in Definition 1.1 .)

Our main result is the following:

THEOREM 5.1. *Let $d \in \delta(I)$ be a decision generating function. The following statements are equivalent:*

(i)

$$d(x_1, y_1) + d(x_2, y_2) \leqslant d(x_1, y_2) + d(x_2, y_1) \tag{5.1}$$

holds for every $x_1, x_2, y_1, y_2 \in I$ with $x_1 \leqslant x_2$, $y_1 \leqslant y_2$;

(ii) *D_d is monotone;*

(iii)

$$\min \widetilde{D}_d \begin{pmatrix} x_1 & ,\ldots, & x_i & ,\ldots, & x_n \\ r_1 & ,\ldots, & r_i & ,\ldots, & r_n \end{pmatrix} \leqslant \min \widetilde{D}_d \begin{pmatrix} x_1 & ,\ldots, & \tilde{x}_i & ,\ldots, & x_n \\ r_1 & ,\ldots, & r_i & ,\ldots, & r_n \end{pmatrix},$$

$$\max \widetilde{D}_d \begin{pmatrix} x_1 & ,\ldots, & x_i & ,\ldots, & x_n \\ r_1 & ,\ldots, & r_i & ,\ldots, & r_n \end{pmatrix} \leqslant \max \widetilde{D}_d \begin{pmatrix} x_1 & ,\ldots, & \tilde{x}_i & ,\ldots, & x_n \\ r_1 & ,\ldots, & r_i & ,\ldots, & r_n \end{pmatrix}$$

hold for every $n, i \in \mathbb{N}$, $i \leqslant n$, $(r_1, \ldots, r_n) \in \mathbb{R}_0^n$ and $x_1, \ldots, x_n, \tilde{x}_i \in I$ with $x_i \leqslant \tilde{x}_i$.

REMARK 5.1 . If the partial derivative $\partial_2 d(x,y)$ exists for all $x, y \in I$, then (5.1) is equivalent to that the mapping $x \mapsto \partial_2 d(x,y)$ is monotone decreasing for all $y \in I$. If, furthermore, the partial derivative $\partial_1\partial_2 d(x,y)$ exists for all $x, y \in I$, then (5.1) is equivalent to the inequality $\partial_1\partial_2 d(x,y) \leqslant 0$ for all $x, y \in I$.

To prove this Theorem, we will need the following Lemma that describes the connection between the monotonicity of the decision function and the monotonicity of the weighted decision function of two variables.

LEMMA 5.2. *Let $d \in \delta(I)$ be a decision generating function and let us assume that the decision function D_d generated by d is monotone. Let $r \in [0,1]$ and $x_1, \tilde{x}_1, x_2 \in I$ with $x_1 \leqslant \tilde{x}_1$. Then it holds that*

$$\min \widetilde{D}_d \begin{pmatrix} x_1 & , & x_2 \\ r & , & 1-r \end{pmatrix} \leqslant \min \widetilde{D}_d \begin{pmatrix} \tilde{x}_1 & , & x_2 \\ r & , & 1-r \end{pmatrix},$$

$$\max \widetilde{D}_d \begin{pmatrix} x_1 & , & x_2 \\ r & , & 1-r \end{pmatrix} \leqslant \max \widetilde{D}_d \begin{pmatrix} \tilde{x}_1 & , & x_2 \\ r & , & 1-r \end{pmatrix}.$$

Proof. Let $r = \frac{n}{m} \in \mathbb{Q}$, $(n \in \mathbb{N}_0,\ m \in \mathbb{N},\ m \geqslant n)$. Since D_d is monotone, it holds that

$$\widetilde{D}_d \begin{pmatrix} x_1 & , & x_2 \\ \frac{n}{m} & , & 1-\frac{n}{m} \end{pmatrix} = \widetilde{D}_d \begin{pmatrix} x_1 & , & x_2 \\ n & , & m-n \end{pmatrix} = D_d(\underbrace{x_1, \dots, x_1}_{n-\text{times}}, \underbrace{x_2, \dots, x_2}_{(m-n)-\text{times}})$$

$$\leqslant D_d(\underbrace{\tilde{x}_1, \dots, \tilde{x}_1}_{n-\text{times}}, \underbrace{x_2, \dots, x_2}_{(m-n)-\text{times}}) = \widetilde{D}_d \begin{pmatrix} \tilde{x}_1 & , & x_2 \\ n & , & m-n \end{pmatrix} = \widetilde{D}_d \begin{pmatrix} \tilde{x}_1 & , & x_2 \\ \frac{n}{m} & , & 1-\frac{n}{m} \end{pmatrix}.$$

Thus, in the rational case, we see the statement of the Lemma.

Now let $r \in [0,1]$ be arbitrary and $(q_n) \in \mathbb{Q} \cap [0,1]$ be a sequence with $q_n \uparrow r$. Then we have proved that

$$\widetilde{D}_d \begin{pmatrix} x_1 & , & x_2 \\ q_n & , & 1-q_n \end{pmatrix} \leqslant \widetilde{D}_d \begin{pmatrix} \tilde{x}_1 & , & x_2 \\ q_n & , & 1-q_n \end{pmatrix}. \tag{5.2}$$

On the other hand, the mapping

$$r \mapsto \widetilde{D}_d \begin{pmatrix} x_1 & , & x_2 \\ r & , & 1-r \end{pmatrix}$$

is upper semi-continuous and monotone decreasing, thus it follows that

$$\widetilde{D}_d \begin{pmatrix} x_1 & , & x_2 \\ q_n & , & 1-q_n \end{pmatrix} \underset{n\to\infty}{\longrightarrow} \max \widetilde{D}_d \begin{pmatrix} x_1 & , & x_2 \\ r & , & 1-r \end{pmatrix}$$

monotone decreasingly. Analogously,

$$\widetilde{D}_d \begin{pmatrix} \tilde{x}_1 & , & x_2 \\ q_n & , & 1-q_n \end{pmatrix} \underset{n\to\infty}{\longrightarrow} \max \widetilde{D}_d \begin{pmatrix} \tilde{x}_1 & , & x_2 \\ r & , & 1-r \end{pmatrix}$$

also monotone decreasingly. Therefore, because of (5.2), it holds that

$$\max \widetilde{D}_d \begin{pmatrix} x_1 & , & x_2 \\ r & , & 1-r \end{pmatrix} \leqslant \max \widetilde{D}_d \begin{pmatrix} \tilde{x}_1 & , & x_2 \\ r & , & 1-r \end{pmatrix}.$$

The other inequality of the Lemma can be proved similarly.

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. In what follows, we will show the implications $(ii) \implies (i)$ and $(i) \implies (iii)$ only, because $(iii) \implies (ii)$ is trivial.

1. To prove the implication $(ii) \implies (i)$, let D_d be monotone increasing. Let further $x_1, x_2,\ y_1, y_2 \in I$ such that $x_1 < x_2,\ y_1 < y_2$. (If $x_1 = x_2$ or $y_1 = y_2$, then (5.1) is trivially true.) (5.1) is equivalent to the inequality

$$d(x_1, y_1) - d(x_2, y_1) \leqslant d(x_1, y_2) - d(x_2, y_2),$$

i.e., the mapping $y \mapsto d(x_1, y) - d(x_2, y)$ is monotone increasing for every $x_1 < x_2$. To justify this monotonicity it is enough to prove that the left derivative of the above mapping is nonnegative everywhere. (Indeed, from the convexity of d in its second variable follows its local Lipschitz property and thus its absolute continuity on compact subintervals of I. Therefore, one can apply Newton-Leibniz Theorem to get

$$0 \leqslant \int_{y_1}^{y_2} d^-(x_1, t) - d^-(x_2, t)dt = d(x_1, y_2) - d(x_2, y_2) - d(x_1, y_1) + d(x_2, y_1),$$

which is the desired property (i).) Thus it suffices to show the inequality

$$d^-(x_1, y) \geqslant d^-(x_2, y) \tag{5.3}$$

for every $y \in I$. If $y \in [x_1, x_2]$, then $d^-(x_1, y) \geqslant 0$ and $0 \geqslant d^-(x_2, y)$, so in this case inequality (5.3) holds trivially. Let now $y \in]\inf I, x_1[$. The case when $y \in]x_2, \sup I[$ can be treated analogously.

Let $x_0 \in I$ be fixed such that $x_0 < y$. We are looking for weights $r_1, r_2 \in [0, 1]$ such that

$$y \in \widetilde{D}_d \begin{pmatrix} x_0 & , & x_1 \\ r_1 & , & 1 - r_1 \end{pmatrix} \bigcap \widetilde{D}_d \begin{pmatrix} x_0 & , & x_2 \\ r_2 & , & 1 - r_2 \end{pmatrix}. \tag{5.4}$$

For this purpose, determine the numbers $r_1, r_2 \in \mathbb{R}$ by

$$r_i \cdot d^-(x_0, y) + (1 - r_i) \cdot d^-(x_i, y) = 0, \tag{5.5}$$

i.e., let

$$r_i = \frac{d^-(x_i, y)}{d^-(x_i, y) - d^-(x_0, y)}, \quad (i = 1, 2).$$

Since $d^-(x_i, y) < 0$ and $d^-(x_0, y) > 0$, it holds that $r_i \in]0, 1[,\ (i = 1, 2)$.

Because of the definition of r_1 and r_2, (5.4) holds indeed. We show that $r_1 \leqslant r_2$. On the contrary, assume that $r_2 < r_1$. According to Lemma 5.2, Corollary 3.3, and (5.4),

$$y \leqslant \max \widetilde{D}_d \begin{pmatrix} x_0 & , & x_1 \\ r_1 & , & 1 - r_1 \end{pmatrix}$$

$$\leqslant \max \widetilde{D}_d \begin{pmatrix} x_0 & , & x_2 \\ r_1 & , & 1 - r_1 \end{pmatrix} \leqslant \min \widetilde{D}_d \begin{pmatrix} x_0 & , & x_2 \\ r_2 & , & 1 - r_2 \end{pmatrix} \leqslant y.$$

Thus

$$y = \max \widetilde{D}_d \begin{pmatrix} x_0 & , & x_2 \\ r_1 & , & 1 - r_1 \end{pmatrix}.$$

Therefore

$$r_1 \cdot d^-(x_0,y) + (1-r_1)\cdot d^-(x_2,y) \leqslant 0. \tag{5.6}$$

In view of $r_2 < r_1$, we have that

$$0 < r_2 \cdot d^-(x_0,y) < r_1 \cdot d^-(x_0,y)$$

and

$$(1-r_2)\cdot d^-(x_2,y) < (1-r_1)\cdot d^-(x_2,y) < 0.$$

Then, according to (5.5),

$$0 = r_2 \cdot d^-(x_0,y) + (1-r_2)\cdot d^-(x_2,y) < r_1 \cdot d^-(x_0,y) + (1-r_1)\cdot d^-(x_2,y).$$

This contradicts (5.6), which validates $r_1 \leqslant r_2$.

Now expressing $d^-(x_1,y)$ and $d^-(x_2,y)$ from (5.5), and using that

$$\frac{r_2}{r_2-1} \leqslant \frac{r_1}{r_1-1},$$

we get

$$d^-(x_2,y) = \frac{r_2}{r_2-1}\cdot d^-(x_0,y) \leqslant \frac{r_1}{r_1-1}\cdot d^-(x_0,y) = d^-(x_1,y),$$

which is the inequality we wanted to get.

2. Now we prove that (iii) follows from (i). We show the first inequality, the second can be verified analogously. Because of the symmetry of the function $\widetilde{D}_d$, it is enough to prove monotonicity in the first variable. On the contrary, assume that there are $n \in \mathbb{N}$, $x_1,\dots,x_n,\tilde{x}_1 \in I$ and $(r_1,\dots,r_n) \in \mathbb{R}_0^n$ with $x_1 < \tilde{x}_1$ such that

$$y = \min \widetilde{D}_d \begin{pmatrix} x_1, & x_2 & ,\dots, & x_n \\ r_1, & r_2 & ,\dots, & r_n \end{pmatrix} > \min \widetilde{D}_d \begin{pmatrix} \tilde{x}_1, & x_2 & ,\dots, & x_n \\ r_1, & r_2 & ,\dots, & r_n \end{pmatrix} = \tilde{y}.$$

Then, using that our points y and $\tilde{y}$ are the smallest points of minimum, we get

$$r_1 \cdot d(x_1,y) + r_2\cdot d(x_2,y) + \dots + r_n \cdot d(x_n,y) < r_1\cdot d(x_1,\tilde{y}) + r_2 \cdot d(x_2,\tilde{y}) + \dots + r_n \cdot d(x_n,\tilde{y}),$$

$$r_1 \cdot d(\tilde{x}_1,\tilde{y}) + r_2\cdot d(x_2,\tilde{y}) + \dots + r_n \cdot d(x_n,\tilde{y}) \leqslant r_1\cdot d(\tilde{x}_1,y) + r_2 \cdot d(x_2,y) + \dots + r_n \cdot d(x_n,y).$$

Summing the above inequalities, we arrive at

$$d(x_1,y) + d(\tilde{x}_1,\tilde{y}) < d(x_1,\tilde{y}) + d(\tilde{x}_1,y).$$

As we have $\tilde{y} < y$ and $x_1 < \tilde{x}_1$, the above inequality contradicts (5.1). Therefore $\widetilde{D}_d$ is monotone.

Thus the proof is complete.

EXAMPLE 5.1. Applying Theorem 5.1 and Remark 5.1 we can now easily investigate the monotonicity property of the decision functions of the foregoing examples. In case of quasi-arithmetic means it holds that $\partial_2 d_\varphi(x,y) = \varphi(y) - \varphi(x)$ for $x,y \in I$. Due to the monotone increasingness of φ, $\partial_2 d_\varphi$ is decreasing in x, therefore quasi-arithmetic means (and thus also power means) are always monotone decision functions.

To investigate the monotonicity of the Gini mean $G_{p,q}$, observe that

$$\partial_1\partial_2 d_{p,q}(x,y) = \begin{cases} \dfrac{1}{p-q}(-px^{p-1}+qx^{q-1}y^{p-q}), & \text{if} \quad p-q \neq 0, \\ x^{p-1}(p\ln y - p\ln x - 1), & \text{if} \quad p-q = 0 \end{cases}$$

is nonpositive for every $x,y > 0$ if and only if $q \leqslant 0 \leqslant p$ or $p \leqslant 0 \leqslant q$. Thus the Gini mean $G_{p,q}$ is monotone exactly when (p,q) belongs to one of these quadrants.

Acknowledgment. The author is indebted to Prof. Zs. Páles for the several valuable discussions during the preparation of this work.

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(Received July 12, 2004)

Institute of Mathematics
University of Debrecen
4010 Debrecen, Pf. 12
Hungary
e-mail: fborbala@math.klte.hu
e-mail: fazekas@mathematik.uni-karlsruhe.de