Mathematical
Inequalities
& Applications
Volume 10, Number 1 (2007), 49–55

AN ADDENDUM OF AN EQUIVALENCE THEOREM

L. LEINDLER

(*communicated by S. Owa*)

Abstract. In a previous paper we showed that the two conditions which guarantee that a factor sequence should be a Weyl multiplier for a certain property of a given orthogonal series is equivalent to only one assumption. In the present paper it is verified that the additional condition prescribed on the Weyl multipliers is also necessary to the equivalence.

1. Introduction

There are several theorems declaring that if a monotone increasing sequence $\{\omega_n\}$ of positive numbers satisfying a certain condition and

$$\sum_{n=1}^{\infty} a_n^2\, \omega_n < \infty, \tag{1.1}$$

then for a decided orthonormal system $\{\psi_n(x)\}$ the series

$$\sum_{n=1}^{\infty} a_n\, \psi_n(x)$$

has an appropriate property. Then the sequence $\{\omega_n\}$ is called Weyl multiplier for the given property and the considered orthonormal system $\{\psi_n\}$.

For example a sequence $\{\omega_n\}$ is a Weyl multiplier for unconditional convergence for the Haar system if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n\,\omega_n} < \infty, \tag{1.2}$$

see P. L. Ul'janov [9, Ch. 9]. Similar best possible type results for the Walsh system are due to K. Tandori [8] and S. Nakata [4], [5].

For general orthonormal systems $\{\varphi_n(x)\}$ we recall a classical theorem of W. Orlicz [6].

Mathematics subject classification (2000): 26D15, 40A05, 42C15.

Key words and phrases: inequalities for sums, Weyl multipliers, orthogonal series, equivalence theorems.

This research was partially supported by the Hungarian National Foundation for Scientific Research under Grant # T042462.

If $\{\nu_n\}$ is an increasing sequence of indices with $\log \nu_{n+1} = O(\log \nu_n)$ and there exists a positive monotone increasing function $\lambda(x)$ such that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda(\nu_n)} < \infty, \tag{1.3}$$

then the condition

$$\sum_{n=1}^{\infty} a_n^2 \lambda(n) \log^2 n < \infty \tag{1.4}$$

implies the convergence almost everywhere of any orthogonal series

$$\sum_{n=1}^{\infty} a_n \varphi_n(x) \tag{1.5}$$

with every arrangement of its terms.

In brief, the factors $\omega_n := \lambda(n) \log^2 n$ are Weyl multipliers for unconditional convergence for the orthogonal series (1.5).

K. Tandori [7] developed this result verifying that if

$$\sum_{k=0}^{\infty} \left\{ \sum_{n=\nu_k+1}^{\nu_{k+1}} a_n^2 \log^2 n \right\}^{1/2} < \infty, \tag{1.6}$$

then the series (1.5) is unconditionally convergent; and if $|a_n| \geqslant |a_{n+1}|$ then (1.6) is not only sufficient but it is necessary as well.

In [1] we showed that the conditions (1.6) and

$$\sum_{n=1}^{\infty} \frac{1}{n} \left\{ \sum_{k=n+1}^{\infty} a_k^2 \right\}^{1/2} < \infty \tag{1.7}$$

are equivalent.

It was natural to ask: What is the relation between (1.7) and the Orlicz's conditions?

In the same paper we verified that the conditions

$$\sum_{n=0}^{\infty} 2^{2n}/\rho_{2^{2^n}} < \infty \tag{1.8}$$

and

$$\sum_{n=0}^{\infty} a_n^2 \rho_n < \infty \tag{1.9}$$

with an increasing sequence $\{\rho_n\}$ are already equivalent to our simple condition (1.7) and thus to (1.6), too.

It is easy to see that the conditions (1.8) and (1.9) claim only a little bit less than (1.3) and (1.4) do, but they equivalent to (1.7), consequently to (1.6), and therefore if $\{|a_n|\}$ is monotonic, then they are best possible.

The equivalence of (1.7) and the pair of (1.8) and (1.9) was the initiation of proving a very general equivalence result [2, Hilfssatz] whose special case is the following theorem.

THEOREM 1. *Let* $\{\lambda_n\}$ *be a monotone sequence of positive numbers and denote* $\Lambda_n := \sum_{k=1}^{n} \lambda_k^{-1}$. *Then*

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left\{ \sum_{k=n}^{\infty} a_k^2 \right\}^{1/2} < \infty \tag{1.10}$$

holds if and only if there exists a nondecreasing sequence $\{\mu_n\}$ *of positive numbers with the properties*

$$\sum_{n=1}^{\infty} \frac{\Lambda_n}{\lambda_n \mu_n} < \infty \tag{1.11}$$

and

$$\sum_{n=1}^{\infty} a_n^2 \mu_n < \infty. \tag{1.12}$$

This theorem delivers several corollaries giving the compact equivalence form with one condition instead of the two conditions of Weyl multipliers type, and conversely. E.g. instead of the conditions (1.1) and (1.2) it is sufficient to claim that

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{1/2}} \left\{ \sum_{k=n}^{\infty} a_k^2 \right\}^{1/2} < \infty. \tag{1.13}$$

The other benefit of the condition (1.13) is that the factor $n(\log n)^{1/2}$ is well-determined, but in (1.1) and (1.2) the factor $\omega_n (= \mu_n)$ is not, and thus clearly there is no exact Weyl multiplier for the Haar system.

It is easy to verify that if (1.10) holds then with

$$\mu_n := \Lambda_n A_n^{-1}, \text{ where } A_n := \left\{ \sum_{k=n}^{\infty} a_k^2 \right\}^{1/2}$$

(1.11) and (1.12) maintain. Hence it is clear that then the sequence $\{\Lambda_n/\mu_n\}$ is nonincreasing. This gives a certain explanation for the fact that such a type, but somewhat weaker assumption will appear in our new theorem, too.

One more remark before ending our introductory words.

It is clear that if $\varepsilon_n \to 0$ then with $\mu_n^* := \mu_n \varepsilon_n$ in place of μ_n the condition (1.12) is also satisfied, but it can be happened that

$$\sum_{n=1}^{\infty} \frac{\Lambda_n}{\lambda_n \mu_n^*} = \infty \tag{1.14}$$

will befall.

This raises the question: Do (1.14) and (1.12) with μ_n^* imply (1.10) for arbitrary $\{a_n\}$?

In this note we shall show that the answer is negative in other words we shall verify the necessity of (1.11).

Before formulating our theorem we give some notions and notations.

A sequence $\gamma := \{\gamma_n\}$ of positive terms is said to be quasi increasing (decreasing) if there exist a natural number $N := N(\gamma)$ and a constant $K := K(\gamma) > 1$ such that

$$K\gamma_n \geqslant \gamma_m \quad (\gamma_n \leqslant K\gamma_m)$$

holds for any $n \geqslant m \geqslant N$.

The above mentioned sequence γ is quasi geometrically increasing (decreasing) if there exist natural numbers $\nu := \nu(\gamma)$, $N := N(\gamma)$ and a constant $K := K(\gamma) \geqslant 1$ such that

$$\gamma_{n+\nu} \geqslant 2\gamma_n \text{ and } \gamma_n \leqslant K\gamma_{n+1} \quad \left(\gamma_{n+\nu} \leqslant \frac{1}{2}\gamma_n \text{ and } \gamma_{n+1} \leqslant K\gamma_n\right)$$

hold for all $n \geqslant N$.

Furthermore K and K_i will denote positive constants that are not necessarily the same at each occurrence.

2. Result

Now we formulate our result.

THEOREM 2. *Let* $\{\lambda_n\}$ *and* $\{\mu_n\}$ *be monotone nondecreasing sequences of positive terms and denote* $\Lambda_n := \sum_{k=1}^{n} \lambda_k^{-1}$. *Furthermore let us assume that the sequence* $\{\Lambda_n/\mu_n\}$ *is quasi decreasing with a constant* K_0, *and*

$$\sum_{n=1}^{\infty} \frac{\Lambda_n}{\lambda_n \mu_n} = \infty. \tag{2.1}$$

Then there exists a sequence $\{a_n\}$ *of real numbers such that the inequality* (1.12) *holds, but*

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left\{ \sum_{k=n}^{\infty} a_k^2 \right\}^{1/2} = \infty. \tag{2.2}$$

REMARK 1. Our theorem shows that the condition (1.11) is necessary in order that (1.11) and (1.12) imply (1.10) for any $\{a_n\}$.

REMARK 2. It seems to be a difficult task to verify the statements of our Theorem without the assumption on the quasi monotonicity of $\{\Lambda_n/\mu_n\}$, but this requirement holds at all known theorems, see e.g. the results mentioned in [2, Folgerung I].

3. Lemma

The following lemma is known, see e.g. Lemma in [3].

LEMMA 1. *For any positive sequence* $\gamma := \{\gamma_n\}$ *the inequalities*

$$\sum_{n=m}^{\infty} \gamma_n \leqslant K\gamma_m \quad (m = 1,2,\ldots;\ K \geqslant 1),$$

or

$$\sum_{n=1}^{m} \gamma_n \leqslant K\gamma_m \quad (m = 1,2,\ldots;\ K \geqslant 1),$$

hold if and only if γ *is quasi geometrically decreasing or increasing, respectively.*

4. Proof

Without loss of the generality we can assume that $\lambda_1 \geqslant 1$ and $\mu_1 \geqslant 1$. Let us define

$$S_n := \sum_{k=1}^{n} \frac{\Lambda_k}{\lambda_k \mu_k}.$$

By (2.1) we can define a sequence $\{p_m\}$ of indices such that

$$2^m \leqslant S_{p_m} < 2^{m+1} \tag{4.1}$$

holds for all $m = 1,2,\ldots,$ furthermore $p_0 = 0$.

The Abel-Dini theorem and the assumption (2.1) imply that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n \mu_n S_n} \Lambda_n = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \sum_{n=k}^{\infty} \frac{1}{\lambda_n \mu_n S_n} = \infty. \tag{4.2}$$

Next we show that

$$\sum\nolimits_0 := \sum_{n=1}^{\infty} \frac{1}{\lambda_n S_n} \sum_{k=n}^{\infty} \frac{1}{\lambda_k \mu_k S_k} < \infty. \tag{4.3}$$

Let us divide the sum $\sum_0$ into blocks as follows:

$$\begin{aligned}
\sum\nolimits_0 &= \sum_{m=0}^{\infty} \sum_{n=p_m+1}^{p_{m+1}} \frac{1}{\lambda_n S_n} \left(\sum_{k=n}^{p_{m+1}} + \sum_{k=p_{m+1}+1}^{\infty} \right) \frac{1}{\lambda_k \mu_k S_k} \\
&\leqslant K \left\{ \sum_{m=0}^{\infty} 2^{-2m} \sum_{n=p_m+1}^{p_{m+1}} \frac{1}{\lambda_n} \sum_{k=n}^{p_{m+1}} \frac{1}{\lambda_k \mu_k} \right. \\
&\quad \left. + \sum_{m=0}^{\infty} 2^{-m} \sum_{n=p_m+1}^{p_{m+1}} \frac{1}{\lambda_n} \sum_{i=m+1}^{\infty} 2^{-i} \sum_{k=p_i+1}^{p_{i+1}} \frac{1}{\lambda_k \mu_k} \right\} \\
&=: \sum\nolimits_1 + \sum\nolimits_2 .
\end{aligned} \tag{4.4}$$

Using an Abel rearrangement and (4.1) we get that

$$\begin{aligned}\sum\nolimits_1 &\leqslant K\sum_{m=0}^{\infty}2^{-2m}\sum_{k=p_m+1}^{p_{m+1}}\frac{1}{\lambda_k\,\mu_k}\sum_{n=p_m+1}^{k}\frac{1}{\lambda_k}\\ &\leqslant K\sum_{m=0}^{\infty}2^{-2m}\sum_{k=p_m+1}^{p_{m+1}}\frac{1}{\lambda_k\,\mu_k}\Lambda_k<\infty.\end{aligned}\tag{4.5}$$

Similarly an elementary consideration and (4.1) yield

$$\begin{aligned}\sum\nolimits_2 &\leqslant K\sum_{m=0}^{\infty}2^{-m}\sum_{n=p_m+1}^{p_{m+1}}\frac{1}{\lambda_n}\sum_{i=m+1}^{\infty}2^{-i}\sum_{k=p_i+1}^{p_{i+1}}\frac{1}{\lambda_k\,\mu_k}\frac{\Lambda_k}{\Lambda_{p_i}}\\ &\leqslant K_1\sum_{m=0}^{\infty}2^{-m}\sum_{n=p_m+1}^{p_{m+1}}\frac{1}{\lambda_n}\sum_{i=m+1}^{\infty}\frac{1}{\Lambda_{p_i}}=:\sum\nolimits_3.\end{aligned}\tag{4.6}$$

Next we show that the sequence $\{\Lambda_{p_i}\}$ is quasi geometrically increasing. In order to verify this it suffices to prove that there exists a natural number ν with the property

$$\Lambda_{p_{m+\nu}}\geqslant 2\Lambda_{p_m}\tag{4.7}$$

for all $m\geqslant 1$.

Utilizing again (4.1) and the quasi monotonicity of the sequence $\{\Lambda_n/\mu_n\}$ we obtain that

$$2^{m+\nu-1}\leqslant\sum_{k=p_m+1}^{p_{m+\nu}}\frac{1}{\lambda_k\,\mu_k}\Lambda_k\leqslant K_0\,\Lambda_{p_{m+\nu}}\frac{\Lambda_{p_m}}{\mu_{p_m}}$$

and

$$2^{m+1}>\sum_{k=1}^{p_m}\frac{1}{\lambda_k\,\mu_k}\Lambda_k\geqslant\frac{\Lambda_{p_m}}{K_0\,\mu_{p_m}}\Lambda_{p_m}.$$

These inequalities plainly imply (4.7) if $2^\nu\geqslant 8K_0^2$.

On the basis of (4.7) the sequence $\{\Lambda_{p_m}^{-1}\}$ is clearly quasi geometrically decreasing, and hereby, our Lemma yields that

$$\sum_{i=m+1}^{\infty}\frac{1}{\Lambda_{p_i}}\leqslant K\frac{1}{\Lambda_{p_{m+1}}}.$$

Putting this into (4.6) we get that $\sum_2<\infty$. Since by (4.5) $\sum_1<\infty$ also holds, thus (4.4) shows that the assertion (4.3) is true.

Now we can define the sequence $\{a_n\}$ satisfying the assertions of our theorem.

Let

$$A_n:=\left\{\sum_{k=n}^{\infty}a_k^2\right\}^{1/2}:=\sum_{k=n}^{\infty}\frac{1}{\lambda_k\,\mu_k\,S_k}.\tag{4.8}$$

Since then

$$a_n^2 = A_n^2 - A_{n+1}^2 = (A_n - A_{n+1})(A_n + A_{n+1}) \leqslant \frac{2}{\lambda_n\,\mu_n\,S_n}\sum_{k=n}^{\infty}\frac{1}{\lambda_k\,\mu_k\,S_k}.$$

This and (4.3) clearly imply (1.12).
On the other hand, (4.8) and (4.2) verify (2.2).
Consequently the proof is complete.

REFERENCES

[1] L. LEINDLER, *Über die unbedingte Konvergenz der Orthogonalreihen*, Publ. Math. Debrecen, **11**, (1964), 139–148.
[2] L. LEINDLER, *Über einen Äquivalenzsatz*, Publ. Math. Debrecen, **12**, (1965), 213–218.
[3] L. LEINDLER, *On the converses of inequality of Hardy and Littlewood*, Acta Sci. Math. (Szeged), **58**, (1993), 191–196.
[4] S. NAKATA, *On the divergence of rearranged Walsh series*, Tôhoku Math., **24**, (1972), 275–280.
[5] S. NAKATA, *On the unconditional convergence of Walsh series*, Analysis Mathematica, **5**, (1979), 201–205.
[6] W. ORLICZ, *Zur Theorie der Orthogonalreihen*, Bulletin Internat. Acad. Sci. Polonaise Cracovie (1927), 81–115.
[7] K. TANDORI, *Über die orthogonalen Funktionen.* X (*Unbedingte Konvergenz*), Acta Sci. Math. (Szeged), **23**, (1962), 185–221.
[8] 8 K. TANDORI, *Über die Divergenz der Walshschen Reihen*, Acta Sci. Math. (Szeged), **27**, (1966), 261–263.
[9] 9 P.L. UL'JANOV, *Soled and unsolved problems in the theory of trigonometric and orthogonal series*, (in Russian) Uspehi Mat. Nauk, **19**(1)(1964), 3-69; Russian Math. Surveys, **19**, (1) (1964), 1–62.

(Received October 20, 2003)

Bolyai Institute
Jozsef Attila University
Aradi vertanuk tere 1
H-6720 Szeged
HUNGARY
e-mail: leindler@math.u-szeged.hu