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FUZZY RANDOM KOROVKIN THEORY AND INEQUALITIES

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Abstract. We introduce and study the fuzzy random positive linear operators acting on fuzzy random continuous functions. We establish a series of fuzzy random Shisha–Mond type inequalities of L^q-type $1 \leqslant q < \infty$ and related fuzzy random Korovkin type theorems, regarding the fuzzy random q-mean convergence of fuzzy random positive linear operators to the fuzzy random unit operator for various cases. All convergences are with rates and are given via the above fuzzy random inequalities involving the fuzzy random modulus of continuity of the engaged fuzzy random function. The assumptions for the Korovkin theorems are minimal and of natural realization, fulfilled by almost all example fuzzy random positive linear operators. The astonishing fact is that the real Korovkin test functions assumptions are enough for the conclusions of our fuzzy random Korovkin theory. We give at the end applications.

1. Introduction

Motivation for this work are [1], [4], [5], [8], [3], [11] and [16]. This type of work for fuzzy stochastic processes is new to our knowledge. We introduce the concept of fuzzy random positive linear operator and we prove our results for a very large general class of such operators. Most of the summation and integration operators fall into this class. To do that we are greatly helped by the fuzzy Riesz representation theorem developed in [5]. The surprising fact is that the basic assumptions of real Korovkin theory for the test functions 1, id, id^2 carry over here and they are the only ones needed. Of course a natural realization condition is required in the fuzzy random setting to prove the fuzzy random q-mean convergence. But first we establish a series of fuzzy random Shisha–Mond type inequalities for various important cases. These contain the fuzzy random modulus of continuity of the involved function.

So this paper is basically the study with rates and quantitatively for the fuzzy random q-mean convergence of a sequence of very general and abstract fuzzy random positive linear operators to the fuzzy random unit operator. Linearity and positivity here are the analogs of the real case. Finally we give applications to fuzzy random Bernstein operators.

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2. Background

We start with

DEFINITION 1. (See [18]) Let $\mu\colon \mathbb{R} \to [0,1]$ with the following properties:

(i) is *normal*, i.e., $\exists x_0 \in \mathbb{R};\ \mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda)y) \geqslant \min\{\mu(x), \mu(y)\}$, $\forall x, y \in \mathbb{R}$, $\forall \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

(iii) μ is *upper semicontinuous* on $\mathbb{R}$, i.e., $\forall x_0 \in \mathbb{R}$ and $\forall \varepsilon > 0$, $\exists$ neighborhood $V(x_0)$: $\mu(x) \leqslant \mu(x_0) + \varepsilon$, $\forall x \in V(x_0)$.

(iv) The set $\overline{\operatorname{supp}(\mu)}$ is compact in $\mathbb{R}$ (where $\operatorname{supp}(\mu) := \{x \in \mathbb{R};\ \mu(x) > 0\}$).

We call μ a *fuzzy real number*. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g., $\mathcal{X}_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\mathcal{X}_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leqslant 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define $[\mu]^r := \{x \in \mathbb{R}:\ \mu(x) \geqslant r\}$ and

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) > 0\}}.$$

Then it is well known [12] that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$. For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \quad \forall r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda[u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., [13]). Notice $1 \odot u = u$ and it holds $u \oplus v = v \oplus u$, $\lambda \odot u = u \odot \lambda$. If $0 \leqslant r_1 \leqslant r_2 \leqslant 1$ then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r = [u_-^{(r)}, u_+^{(r)}]$, where $u_-^{(r)} \leqslant u_+^{(r)}$, $u_-^{(r)}$, $u_+^{(r)} \in \mathbb{R}$, $\forall r \in [0,1]$. Based on [12] we can then identify any $u \in \mathbb{R}_{\mathcal{F}}$ with the parametrized representation $\left\{(u_-^{(r)}, u_+^{(r)}) \mid 0 \leqslant r \leqslant 1\right\}$. We denote $u \precsim v$ iff $u_-^{(r)} \leqslant v_-^{(r)}$ and $u_+^{(r)} \leqslant v_+^{(r)}$, for all $r \in [0,1]$. Define

$$D\colon \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+$$

by

$$D(u,v) := \sup_{r \in [0,1]} \max\{|u_-^{(r)} - v_-^{(r)}|, |u_+^{(r)} - v_+^{(r)}|\},$$

where $[v]^r = [v_-^{(r)}, v_+^{(r)}]$; $u, v \in \mathbb{R}_{\mathcal{F}}$. We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$. Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [17], with the properties

$$D(u \oplus w, v \oplus w) = D(u,v), \quad \forall u, v, w \in \mathbb{R}_{\mathcal{F}},$$
$$D(k \odot u, k \odot v) = |k| D(u,v), \quad \forall u, v \in \mathbb{R}_{\mathcal{F}},\ \forall k \in \mathbb{R},$$
$$D(u \oplus v, w \oplus e) \leqslant D(u,w) + D(v,e), \quad \forall u, v, w, e \in \mathbb{R}_{\mathcal{F}}.$$

We need the following lemmas.

LEMMA 1. (See [7]) *For any $a, b \in \mathbb{R}$: $ab \geqslant 0$ and any $u \in \mathbb{R}_{\mathcal{F}}$ we have*

$$D(a \odot u, b \odot u) \leqslant |a-b| \cdot D(u, \tilde{o}), \tag{1}$$

where $\tilde{o} \in \mathbb{R}_{\mathcal{F}}$ is defined by $\tilde{o} := \mathcal{X}_{\{0\}}$.

LEMMA 2. (See [7])

(*i*) *If we denote* $\tilde{o} := \mathcal{X}_{\{0\}}$, *then* $\tilde{o} \in \mathbb{R}_{\mathcal{F}}$ *is the neutral element with respect to* $\oplus$, *i.e.,* $u \oplus \tilde{o} = \tilde{o} \oplus u = u$, $\forall u \in \mathbb{R}_{\mathcal{F}}$.

(*ii*) *With respect to* $\tilde{o}$, *none of* $u \in \mathbb{R}_{\mathcal{F}}$, $u \neq \tilde{o}$ *has opposite in* $\mathbb{R}_{\mathcal{F}}$.

(*iii*) *Let* $a, b \in \mathbb{R}$: $a \cdot b \geqslant 0$, *and any* $u \in \mathbb{R}_{\mathcal{F}}$, *we have* $(a+b) \odot u = a \odot u \oplus b \odot u$. *For general* $a, b \in \mathbb{R}$, *the above property is false.*

(*iv*) *For any* $\lambda \in \mathbb{R}$ *and any* $u, v \in \mathbb{R}_{\mathcal{F}}$, *we have* $\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot v$.

(*v*) *For any* $\lambda, \mu \in \mathbb{R}$ *and* $u \in \mathbb{R}_{\mathcal{F}}$, *we have* $\lambda \odot (\mu \odot u) = (\lambda \cdot \mu) \odot u$.

(*vi*) *If we denote* $\|u\|_{\mathcal{F}} := D(u, \tilde{o})$, $\forall u \in \mathbb{R}_{\mathcal{F}}$, *then* $\|\cdot\|_{\mathcal{F}}$ *has the properties of a usual norm on* $\mathbb{R}_{\mathcal{F}}$, *i.e.,*

$$\|u\|_{\mathcal{F}} = 0 \text{ iff } u = \tilde{o}, \|\lambda \odot u\|_{\mathcal{F}} = |\lambda| \cdot \|u\|_{\mathcal{F}},$$
$$\|u \oplus v\|_{\mathcal{F}} \leqslant \|u\|_{\mathcal{F}} + \|v\|_{\mathcal{F}}, \ \|u\|_{\mathcal{F}} - \|v\|_{\mathcal{F}} \leqslant D(u, v).$$

Notice that $(\mathbb{R}_{\mathcal{F}}, \oplus, \odot)$ is *not* a linear space over $\mathbb{R}$, and consequently $(\mathbb{R}_{\mathcal{F}}, \|\cdot\|_{\mathcal{F}})$ is *not* a normed space.

We need the following definitions.

DEFINITION 2. (See also [11, Definition 13. 16, p. 654]). Let $(X, \mathcal{B}, P)$ be a probability space. A *fuzzy-random variable is a* $\mathcal{B}$*-measurable mapping* $g: X \to \mathbb{R}_{\mathcal{F}}$, i.e., for any open set $U \subseteq \mathbb{R}_{\mathcal{F}}$, in the topology of $\mathbb{R}_{\mathcal{F}}$ generated by the metric D, we have

$$g^{-1}(U) = \{s \in X; \ g(s) \in U\} \in \mathcal{B}. \tag{2}$$

The set of all fuzzy-random variables is denoted by $\mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$. Let $g_n, g \in \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $n \in \mathbb{N}$, and $0 < q < +\infty$. We say,

$$g_n(s) \xrightarrow[n \to +\infty]{\text{“} q \text{-mean”}} g(s),$$

if

$$\lim_{n \to +\infty} \int_X \big(D(g_n(s), g(s))\big)^q P(ds) = 0. \tag{3}$$

DEFINITION 3. (See [11, p. 654, Definition 13. 17].) Let $(T, \mathcal{T})$ be a topological space. A mapping $f: T \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$ will be called *fuzzy-random function* (or *fuzzy-stochastic process*) on T. We denote $f(t)(s) = f(t, s)$, $t \in T$, $s \in X$.

REMARK 1. (See [11, p. 655].) Any usual fuzzy real function $f: T \to \mathbb{R}_{\mathcal{F}}$ can be identified with the degenerate fuzzy-random function $f(t, s) = f(t)$, $\forall t \in T$, $s \in X$.

REMARK 2. (See [11, p. 655].) Fuzzy-random functions that coincide with probability one, for each $t \in T$, will be considered equivalent.

REMARK 3. (See [11, p. 655].) Let $f, g: T \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$. Then, $f \oplus g$ and $k \odot f$ are defined pointwise, i.e.,

$$(f \oplus g)(t, s) = f(t, s) \oplus g(t, s),$$
$$(k \odot f)(t, s) = k \odot f(t, s), \quad t \in T, \quad s \in X, \quad k \in \mathbb{R}.$$

DEFINITION 4. (See also [11, Definition 13. 18, pp. 655–656].) For a fuzzy-random function $f:[a,b] \to \mathcal{L}_{\mathcal{F}}(X,\mathcal{B},P)$, we define the (first) fuzzy-random modulus of continuity

$$\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} = \sup\left\{\left(\int_X D^q\big(f(x,s),f(y,s)\big)P(ds)\right)^{1/q};\ x,y\in[a,b],\ |x-y|\leqslant\delta\right\}, \tag{4}$$
$$0<\delta,\quad 1\leqslant q<\infty.$$

DEFINITION 5. Here, $1 \leqslant q < \infty$. Let $f:[a,b] \to \mathcal{L}_{\mathcal{F}}(X,\mathcal{B},P)$ be a fuzzy random function. We call f a (q-mean) uniformly continuous fuzzy random function over $[a,b]$ iff $\forall\varepsilon>0\ \exists\delta>0$: whenever $|x-y|\leqslant\delta$, $x,y\in[a,b]$, implies that

$$\int_X \big(D(f(x,s),f(y,s))\big)^q P(ds) \leqslant \varepsilon. \tag{5}$$

We denote it as $f \in C_{\mathcal{F}R}^{Uq}([a,b])$.

We need

PROPOSITION 1. *Let $f \in C_{\mathcal{F}R}^{Uq}([a,b])$. Then, $\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} < \infty$, any $\delta > 0$.*

Proof. Let $\varepsilon_0 > 0$ be arbitrary, but fixed. Then, there exists $\delta_0 > 0$: $|x-y| \leqslant \delta_0$, $x,y \in [a,b]$ which implies

$$\int_X \big(D(f(x,s),f(y,s))\big)^q P(ds) \leqslant \varepsilon_0 < \infty.$$

That is, $\Omega_1^{(\mathcal{F})}(f,\delta_0)_{L^q} \leqslant \varepsilon_0^{1/q} < \infty$. Let now $\delta > 0$ arbitrary, $x,y \in [a,b]$, such that $|x-y| \leqslant \delta$. Choose $n \in \mathbb{N}$: $n\delta_0 \geqslant \delta$ and set $x_i := x + (i/n)(y-x)$, $0 \leqslant i \leqslant n$. Then,

$$\begin{aligned} &D\big(f(x,s),f(y,s)\big)\\ &\quad\leqslant D\big(f(x,s),f(x_1,s)\big) + D\big(f(x_1,s),f(x_2,s)\big) + \cdots + D\big(f(x_{n-1},s),f(y,s)\big). \end{aligned}$$

Consequently,

$$\begin{aligned} &\left(\int_X \big(D(f(x,s),f(y,s))\big)^q P(ds)\right)^{1/q}\\ &\quad\leqslant \left(\int_X \big(D(f(x,s),f(x_1,s))\big)^q P(ds)\right)^{1/q} + \cdots + \left(\int_X \big(D(f(x_{n-1},s),f(y,s))\big)^q P(ds)\right)^{1/q}\\ &\quad\leqslant n\Omega_1^{(\mathcal{F})}(f,\delta_0)_{L^q} \leqslant n\varepsilon_0^{1/q} < \infty, \end{aligned}$$

since $|x_i - x_{i+1}| \leqslant (1/n)|x-y| \leqslant (1/n)\delta \leqslant \delta_0$, $0 \leqslant i \leqslant n$. Therefore, $\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} \leqslant n\varepsilon_0^{1/q} < \infty$. □

PROPOSITION 2. *Let $f,g:[a,b] \to \mathcal{L}_{\mathcal{F}}(X,\mathcal{B},P)$ be fuzzy random functions, $[a,b] \subset \mathbb{R}$. The following hold.*

(*i*) $\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q}$ *be nonnegative and nondecreasing in* $\delta > 0$.

(*ii*) $\lim_{\delta\downarrow 0}\omega_1^{(\mathcal{F})}(f,\delta)_{L^q} = \Omega_1^{(\mathcal{F})}(f,0)_{L^q} = 0$, *iff* $f \in C_{\mathcal{F}R}^{Uq}([a,b])$.

(*iii*) $\Omega_1^{(\mathcal{F})}(f,\delta_1+\delta_2)_{L^q} \leqslant \Omega_1^{(\mathcal{F})}(f,\delta_1)_{L^q} + \Omega_1^{(\mathcal{F})}(f,\delta_2)_{L^q}$, $\delta_1,\delta_2 > 0$.

(*iv*) $\Omega_1^{(\mathcal{F})}(f,n\delta)_{L^q} \leqslant n\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q}$, $\delta > 0$, $n \in \mathbb{N}$.

(*v*) $\Omega_1^{(\mathcal{F})}(f,\lambda\delta)_{L^q} \leqslant \lceil\lambda\rceil\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} \leqslant (\lambda+1)\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q}$, $\lambda > 0$, $\delta > 0$, *where* $\lceil\cdot\rceil$ *is the ceiling of the number.*

(*vi*) $\Omega_1^{(\mathcal{F})}(f\oplus g,\delta)_{L^q} \leqslant \Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} + \Omega_1^{(\mathcal{F})}(g,\delta)_{L^q}$, $\delta > 0$. *Here,* $f\oplus g$ *is a fuzzy random function.*

(*vii*) $\Omega_1^{(\mathcal{F})}(f,\cdot)_{L^q}$ *is continuous on* $\mathbb{R}_+$, *for* $f \in C_{\mathcal{F}R}^{Uq}([a,b])$.

Proof. The proof is obvious. □

PROPOSITION 3. (See [3]) *Let* f,g *be fuzzy random variables from* $X \to \mathbb{R}_{\mathcal{F}}$. *Then, we have the following.*

(*i*) *Let* $c \in \mathbb{R}$, *then* $c\odot f$ *is a fuzzy random variable.*

(*ii*) $f\oplus g$ *is a fuzzy random variable.*

For the definition of general fuzzy integral we follow [14] next.

DEFINITION 6. Let (Ω,Σ,μ) be a complete σ-finite measure space. We call $F:\Omega\to\mathbb{R}_{\mathcal{F}}$ *measurable* iff $\forall$ closed $B\subseteq\mathbb{R}$ the function $F^{-1}(B):\Omega\to[0,1]$ defined by

$$F^{-1}(B)(\omega) := \sup_{x\in B} F(\omega)(x), \quad \text{all } \omega\in\Omega$$

is measurable, see [9], [14].

Notice here that the concept of measurability is different than the $\mathcal{B}$-measurability of Definition 2.

THEOREM 1. ([14]) *For* $F:\Omega\to\mathbb{R}_{\mathcal{F}}$, $F(\omega) = \left\{(F_-^{(r)}(\omega),F_+^{(r)}(\omega)) \mid 0\leqslant r\leqslant 1\right\}$, *the following are equivalent.*

(1) F *is measurable,*

(2) $\forall r\in[0,1]$, $F_-^{(r)}$, $F_+^{(r)}$ *are measurable.*

Following [14], *given that for each* $r\in[0,1]$, $F_-^{(r)}$, $F_+^{(r)}$ *are integrable we have that the parametrized representation*

$$\left\{\left(\int_A F_-^{(r)}\,d\mu, \int_A F_+^{(r)}\,d\mu\right) \mid 0\leqslant r\leqslant 1\right\}$$

is a fuzzy real number for each $A\in\Sigma$.

The last fact leads to

DEFINITION 7. ([14]) A measurable function $F:\Omega\to\mathbb{R}_{\mathcal{F}}$,

$$F(\omega) = \left\{(F_-^{(r)}(\omega),F_+^{(r)}(\omega)) \mid 0\leqslant r\leqslant 1\right\}$$

is called *integrable* if for each $r\in[0,1]$, $F_\pm^{(r)}$ are integrable, or equivalently, if $F_\pm^{(0)}$ are integrable. In this case, the fuzzy integral of F over $A\in\Sigma$ is defined by

$$\int_A F\,d\mu := \left\{\left(\int_A F_-^{(r)}\,d\mu, \int_A F_+^{(r)}\,d\mu\right) \mid 0\leqslant r\leqslant 1\right\}.$$

By [14], F is integrable iff $\omega\to\|F(\omega)\|_{\mathcal{F}}$ is real-valued integrable.

We need also

THEOREM 2. ([14])) *Let $F, G\colon \Omega \to \mathbb{R}_{\mathcal{F}}$ be integrable. Then*

(1) *Let $a, b \in \mathbb{R}$, then $a \odot F + b \odot G$ is integrable and for each $A \in \Sigma$,*

$$\int_A (a \odot F \oplus b \odot G)\, d\mu = a \odot \int_A F\, d\mu \oplus b \odot \int_A G\, d\mu;$$

(2) *$D(F, G)$ is a real-valued integrable function and for each $A \in \Sigma$,*

$$D\left(\int_A F\, d\mu, \int_A G\, d\mu\right) \leqslant \int_A D(F, G)\, d\mu.$$

In particular,

$$\left\| \int_A F\, d\mu \right\|_{\mathcal{F}} \leqslant \int_A \|F\|_{\mathcal{F}}\, d\mu.$$

We need

DEFINITION 8. Let U open or compact $\subseteq$ (M, d) metric space and $f\colon U \to \mathbb{R}_{\mathcal{F}}$. We say that f is *fuzzy continuous at* $x_0 \in U$ iff whenever $x_n \to x_0$, then $D(f(x_n), f(x_0)) \to 0$. If f is continuous for every $x_0 \in U$, we then call f a *fuzzy continuous real number valued function*. We denote the related space by $C_{\mathcal{F}}(U)$. Similarly one defines $C_{\mathcal{F}}([a, b])$, $[a, b] \subseteq \mathbb{R}$.

DEFINITION 9. Let $L\colon C_{\mathcal{F}}(U) \hookrightarrow C_{\mathcal{F}}(U)$, where U is open or compact $\subseteq$ (M, d) metric space, such that

$$L(c_1 \odot f \oplus c_2 \odot g) = c_1 \odot L(f) \oplus c_2 \odot L(g), \quad \forall c_1, c_2 \in \mathbb{R}.$$

We call L a *fuzzy linear operator*.

We give the following example of a fuzzy linear operator, etc.

DEFINITION 10. Let $f\colon [0, 1] \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy real function. The fuzzy algebraic polynomial defined by

$$B_n^{(\mathcal{F})}(f)(x) = \sum_{k=0}^{n}{}^{*} \binom{n}{k} x^k (1-x)^{n-k} \odot f\left(\frac{k}{n}\right), \quad \forall x \in [0, 1],$$

will be called the *fuzzy Bernstein operator*. Here $\sum^*$ stands for the fuzzy summation.

We also need

DEFINITION 11. Let $f, g\colon U \to \mathbb{R}_{\mathcal{F}}$, $U \subseteq (M, d)$ metric space. We denote $f \succsim g$, iff $f(x) \succsim g(x)$, $\forall x \in U$, iff $f_+^{(r)}(x) \geqslant g_+^{(r)}(x)$ and $f_-^{(r)}(x) \geqslant g_-^{(r)}(x)$, $\forall x \in U$, $\forall r \in [0, 1]$, iff $f_+^{(r)} \geqslant g_+^{(r)}$ and $f_-^{(r)} \geqslant g_-^{(r)}$, $\forall r \in [0, 1]$, where $[f(x)]^r = [f_-^{(r)}(x), f_+^{(r)}(x)]$.

We give

DEFINITION 12. Let $L\colon C_{\mathcal{F}}(U) \hookrightarrow C_{\mathcal{F}}(U)$ be a fuzzy linear operator, U open or compact $\subseteq$ (M, d) metric space. We say that L is *positive*, iff whenever $f, g \in C_{\mathcal{F}}(U)$ are such that $f \succsim g$ then $L(f) \succsim L(g)$, iff

$$(L(f))_+^{(r)} \geqslant (L(g))_+^{(r)}$$

and

$$(L(f))_-^{(r)} \geqslant (L(g))_-^{(r)}, \quad \forall r \in [0,1].$$

Here we denote

$$[L(f)]^r = \left[(L(f))_-^{(r)}, (L(f))_+^{(r)}\right], \quad \forall r \in [0,1].$$

An example of a fuzzy positive linear operator is the fuzzy Bernstein operator on the domain $[0,1]$, etc. For more see [4], [5], [8].

We mention

Assumption 1. (See [5]). Let L be a fuzzy positive linear operator from $C_{\mathcal{F}}(K)$, K compact $\subseteq$ (M,d) metric space, into itself. Here we *assume* that there exists a positive linear operator $\tilde{L}$ from $C(K)$ into itself with the property

$$(Lf)_{\pm}^{(r)} = \tilde{L}(f_{\pm}^{(r)}), \tag{6}$$

respectively, for all $r \in [0,1]$, $\forall f \in C_{\mathcal{F}}(K)$.

Again, as an example, we mention the fuzzy Bernstein operator and the real Bernstein operator fulfilling the above assumption on $[0,1]$, etc.

We apply the following *Fuzzy Riesz Representation Theorem.*

THEOREM 3. (See [5]) *Let L be a fuzzy positive linear operator from $C_{\mathcal{F}}(K)$ into itself as in Assumption 1, K compact $\subseteq$ (M,d) metric space. Then for each $x \in K$ there exists a unique positive finite completed Borel measure μ_x on K such that*

$$(Lf)(x) = \int_K f(t)\mu_x(dt), \quad \forall f \in C_{\mathcal{F}}(K).$$

3. Auxilliary material

In proofs we apply

REMARK 4. Let $f: [a,b] \to \mathcal{L}_{\mathcal{F}}(X,\mathcal{B},P)$, $[a,b] \subset \mathbb{R}$ be a fuzzy random function. Then by Proposition 2(v) we get

$$\Omega_1^{(\mathcal{F})}(f,|x-y|)_{L^q} \leqslant \left\lceil \frac{|x-y|}{\delta} \right\rceil \Omega_1^{(\mathcal{F})}(f,\delta)_{L^q}, \quad \forall x,y \in [a,b] \text{ any } \delta > 0. \tag{7}$$

The main function space we are going to work on in the paper is defined as follows.

DEFINITION 13. Let $(X,\mathcal{B},P)$ be a probability space, $[a,b] \subset \mathbb{R}$, and the fuzzy random function $f: [a,b] \times X \to \mathbb{R}_{\mathcal{F}}$ such that $f(t,\omega)$ is *fuzzy continuous in $t \in [a,b]$ uniformly with respect to ω in X*. I.e. $\forall \varepsilon > 0$ $\exists \delta > 0$ such that whenever $|x-y| \leqslant \delta$; $x,y \in [a,b]$, then

$$D\big(f(x,\omega),f(y,\omega)\big) \leqslant \varepsilon, \quad \forall \omega \in X.$$

We denote the space of all these functions by $C_{\mathcal{F}R}^U([a,b])$.

One can easily see that if $f \in C_{\mathcal{F}R}^U([a,b])$ then for each $\omega \in X$ we have that $f(\cdot,\omega) \in C_{\mathcal{F}}([a,b])$ and f is q-mean uniformly continuous in $t \in [a,b]$, i.e. $f \in C_{\mathcal{F}R}^{Uq}([a,b])$, any $1 \leqslant q < +\infty$, see Definition 5.

We mention

DEFINITION 14. Let $L^*\colon C^U_{\mathcal{F}R}([a,b]) \hookrightarrow C^U_{\mathcal{F}R}([a,b])$ such that

$$L^*(c_1 \odot f_1 \oplus c_2 \odot f_2) = c_1 \odot L^*(f_1) \oplus c_2 \odot L^*(f_2), \quad \forall c_1, c_2 \in \mathbb{R}.$$

We call L^* a *fuzzy random linear operator* on $C^U_{\mathcal{F}R}([a,b])$.

The following motivate our work.

EXAMPLE 1. (See [11], p. 656). For $f\colon [0,1] \to \mathcal{L}_{\mathcal{F}}(X,B,P)$, the fuzzy random polynomials defined by

$$B_n^{(\mathcal{F})}(f)(x,\omega) := \sum_{k=0}^{n}{}^{*} \binom{n}{k} x^k (1-x)^{n-k} \odot f\left(\frac{k}{n}, \omega\right), \ x \in [0,1], \ \omega \in X$$

will be called a Bernstein-type. Clearly $B_n^{(\mathcal{F})}(\cdot)(x,\omega)$ is a fuzzy random linear operator, $n \in \mathbb{N}$.

We have

THEOREM 4. (See [11], p. 656) *For* $f\colon [0,1] \to L_{\mathcal{F}}(X,\mathcal{B},P)$ *we have the estimate*

$$\int_X D\big(B_n^{(\mathcal{F})}(f)(x,\omega), f(x,\omega)\big) P(d\omega) \leqslant \frac{3}{2} \Omega_1^{(\mathcal{F})}\left(f; \frac{1}{\sqrt{n}}\right)_{L^1}, \tag{8}$$

$\forall x \in [0,1]$, $n \in \mathbb{N}$. *If, moreover,* f *satisfies the condition*

$$\lim_{\delta \downarrow 0} \Omega_1^{(\mathcal{F})}(f,\delta)_{L_1} = 0,$$

then

$$B_n^{(\mathcal{F})}(f)(x,\omega) \xrightarrow[n \to +\infty]{\text{“1-mean”}} f(x,\omega),$$

uniformly with respect to $x \in [0,1]$.

We mention

DEFINITION 15. Let $L^*\colon C^U_{\mathcal{F}R}([a,b]) \hookrightarrow C^U_{\mathcal{F}R}([a,b])$ be a fuzzy random linear operator. We call L^* a *positive* fuzzy random linear operator iff whenever we have $f, g \in C^U_{\mathcal{F}R}([a,b])$ such that $f \succsim g$, i.e. $f(x,\omega) \succsim g(x,\omega)$ for all $(x,\omega) \in [a,b] \times X$ then $L^*f \succsim L^*g$, i.e. $(L^*f)(x,\omega) \succsim (L^*g)(x,\omega)$ for all $(x,\omega) \in [a,b] \times X$, iff $(L^*f)_+^{(r)}(x,\omega) \geqslant (L^*g)_+^{(r)}(x,\omega)$ and

$$(L^*f)_-^{(r)}(x,\omega) \geqslant (L^*g)_-^{(r)}(x,\omega), \quad \forall r \in [0,1], \ \forall (x,\omega) \in [a,b] \times X.$$

Here we denote

$$[L^*(f)(x,\omega)]^r = \left[(L^*f)_-^{(r)}(x,\omega), (L^*f)_+^{(r)}(x,\omega)\right], \quad \forall r \in [0,1], \forall (x,\omega) \in [a,b] \times X.$$

An example of a positive fuzzy random linear operator is $B_n^{(\mathcal{F})}(\cdot)(x,\omega)$, etc.

We give the useful

REMARK 5. Let L be a fuzzy positive linear operator from $C_{\mathcal{F}}([a,b])$ into itself. We *assume* that there exists a positive linear operator $\tilde{L}$ from $C([a,b])$ into itself with the property

$$(Lf)_\pm^{(r)} = \tilde{L}(f_\pm^{(r)}), \tag{9}$$

respectively, $\forall r \in [0,1]$, $\forall f \in C_{\mathcal{F}}([a,b])$. Then by Theorem 3, $\forall t \in [a,b]$ there exists a unique positive finite completed Borel measure μ_t on $[a,b]$ such that

$$(Lf)(t) = \int_{[a,b]} f(s)\mu_t(ds), \quad \forall f \in C_{\mathcal{F}}([a,b]). \tag{10}$$

Consequently for $f \in C^U_{\mathcal{F}R}([a,b])$ and since $f(\cdot,\omega) \in C_{\mathcal{F}}([a,b])$, $\forall \omega \in X$, we get that

$$L\big(f(\cdot,\omega)\big)(t) = \int_{[a,b]} f(s,\omega)\mu_t(ds), \quad \forall t \in [a,b], \ \forall \omega \in X. \tag{11}$$

Of course here by (9) we have

$$\big(L(f(\cdot,\omega))\big)^{(r)}_{\pm}(t) = \tilde{L}\big(f^{(r)}_{\pm}(\cdot,\omega)\big)(t), \quad \forall t \in [a,b], \, \forall \omega \in X, \, \forall r \in [0,1]. \tag{12}$$

We call

$$m_t := \mu_t([a,b]) \geqslant 0. \tag{13}$$

By setting

$$M(f)(t,\omega) := L\big(f(\cdot,\omega)\big)(t), \tag{14}$$

that is

$$M(f)(t,\omega) = \int_{[a,b]} f(s,\omega)\mu_t(ds), \tag{15}$$

from Theorem 2 (1) we have that

$$M(c_1 \odot f \oplus c_2 \odot g)(t,\omega) = c_1 \odot M(f)(t,\omega) \oplus c_2 \odot M(g)(t,\omega), \tag{16}$$

$\forall (t,\omega) \in [a,b] \times X$, $\forall f,g \in C^U_{\mathcal{F}R}([a,b])$, $\forall c_1, c_2 \in \mathbb{R}$.

Let $C_{\mathcal{F}R}([a,b]) := \{f : [a,b] \times X \to \mathbb{R}_{\mathcal{F}} :$ such that $f(t,\omega)$ is fuzzy continuous in $t \in [a,b]$ and $\mathcal{B}$-measurable in $\omega \in X\}$. Additionally we assume here that $M(f)(t,\omega)$ is $\mathcal{B}$-measurable in $\omega \in X$. Then

$$M(f) \in C_{\mathcal{F}R}([a,b]), \quad \forall f \in C^U_{\mathcal{F}R}([a,b]).$$

That is M is a *fuzzy random linear operator* from $C^U_{\mathcal{F}R}([a,b])$ into $C_{\mathcal{F}R}([a,b])$. Thus by (12) we have

$$(M(f))^{(r)}_{\pm}(t,\omega) = \big(L(f(\cdot,\omega))\big)^{(r)}_{\pm}(t) = \tilde{L}\big(f^{(r)}_{\pm}(\cdot,\omega)\big)(t). \tag{17}$$

Let $f,g \in C^U_{\mathcal{F}}([a,b])$ such that $f \succsim g$ iff $f^{(r)}_{-} \geqslant g^{(r)}_{-}$ and $f^{(r)}_{+} \geqslant g^{(r)}_{+}$, $\forall r \in [0,1]$. Then

$$\tilde{L}\big(f^{(r)}_{-}(\cdot,\omega)\big)(t) \geqslant \tilde{L}\big(g^{(r)}_{-}(\cdot,\omega)\big)(t)$$

and

$$\tilde{L}\big(f^{(r)}_{+}(\cdot,\omega)\big)(t) \geqslant \tilde{L}\big(g^{(r)}_{+}(\cdot,\omega)\big)(t).$$

That is $(M(f))^{(r)}_{-} \geqslant (M(g))^{(r)}_{-}$ and $(M(f))^{(r)}_{+} \geqslant (M(g))^{(r)}_{+}$, $\forall r \in [0,1]$. Hence M is a *positive* fuzzy random linear operator.

For example we notice that

$$B_n^{(\mathcal{F})}\big(f(\cdot,\omega)\big)(x) = \sum_{k=0}^{n}{}^{*} \binom{n}{k} x^k(1-x)^{n-k} \odot f\left(\frac{k}{n},\omega\right) = B_n^{(\mathcal{F})}(f)(x,\omega), \tag{18}$$

$\forall x \in [0,1]$, $\forall \omega \in X$, $\forall f \in C^U_{\mathcal{F}R}([0,1])$, the last fulfills all the above theory. So fuzzy operators like L, M are quite common, e.g. summation, integral operators in the fuzzy sense, therefore we study their approximation properties next.

Clearly, by Theorem 5 of [5], any positive linear operator $\tilde{L}$ from $C([a,b])$ into itself induces a unique positive fuzzy linear operator L acting on $C_{\mathcal{F}}([a,b])$, which in turn generates by (14) a unique positive fuzzy random linear operator M acting on $C^U_{\mathcal{F}R}([a,b])$, so the class of M's is very rich.

4. Main results

We will use the following

PROPOSITION 4. *Let $(X,\mathcal{B},P)$ be a probability space, $[a,b]\subset\mathbb{R}$, $f\in C^U_{\mathcal{F}R}([a,b])$. Let L a fuzzy positive linear operator from $C_{\mathcal{F}}([a,b])$ into itself for which there exists a positive linear operator $\tilde{L}$ from $C([a,b])$ into itself such that*

$$(Lg)^{(r)}_{\pm} = \tilde{L}(g^{(r)}_{\pm}), \tag{19}$$

respectively, $\forall r \in [0,1]$, $\forall g \in C_{\mathcal{F}}([a,b])$. We consider the positive fuzzy random linear operator M acting on $C^U_{\mathcal{F}R}([a,b])$ defined by

$$M(f)(t,\omega) := L\big(f(\cdot,\omega)\big)(t), \quad \forall (t,\omega)\in[a,b]\times X,\ \forall f \in C^U_{\mathcal{F}R}([a,b]). \tag{20}$$

We assume that $M(f)(t,\omega)$ is $\mathcal{B}$-measurable in $\omega\in X$. That is $M(f)\in C_{\mathcal{F}R}([a,b])$. Then

$$\begin{aligned} D\big(M(f)(t,\omega),f(t,\omega)\big) \leqslant{}& \int_{[a,b]} D\big(f(s,\omega),f(t,\omega)\big)\mu_t(ds) \\ &+ |m_t-1|D\big(f(t,\omega),\tilde{o}\big), \quad \forall (t,\omega)\in[a,b]\times X, \end{aligned} \tag{21}$$

where μ_t is as in (10) and m_t as in (13).

Proof. We observe that the $\mathcal{B}$-measurable function [See Remark 13. 39, p. 654, [11]]

$$\begin{aligned} D\big(M(f)(t,\omega),f(t,\omega)\big) &\overset{(15)}{=} D\left(\int_{[a,b]} f(s,\omega)\mu_t(ds), f(t,\omega)\right) \\ &\leqslant D\left(\int_{[a,b]} f(s,\omega)\mu_t(ds), f(t,\omega)\odot m_t\right) + D\big(f(t,\omega)\odot m_t, f(t,\omega)\big) \end{aligned}$$

$$= D\left(\int_{[a,b]} f(s,\omega)\mu_t(ds), \int_{[a,b]} f(t,\omega)\mu_t(ds)\right) + D\big(f(t,\omega)\odot m_t, f(t,\omega)\big)$$

$$\leqslant \text{(by Theorem 2(2) and Lemma 1)}$$

$$\int_{[a,b]} D\big(f(s,\omega), f(t,\omega)\big)\mu_t(ds) + |m_t - 1| D\big(f(t,\omega), \tilde{o}\big).$$

Here notice that

$$\begin{aligned} f(t,\omega)\odot m_t &= \left\{\left(m_t(f(t,\omega))_-^{(r)}, m_t(f(t,\omega))_+^{(r)}\right) \mid 0 \leqslant r \leqslant 1\right\} \\ &= \left\{\left(\int_{[a,b]} \big(f(t,\omega)\big)_-^{(r)} d\mu_t, \int_{[a,b]} \big(f(t,\omega)\big)_+^{(r)} d\mu_t\right) \mid 0 \leqslant r \leqslant 1\right\} \\ &= \int_{[a,b]} f(t,\omega)\mu_t(ds). \qquad \square \end{aligned}$$

By Remark 2 of [5] we trivially see that

$$m_t = \tilde{L}(1)(t) \geqslant 0, \quad \forall t \in [a,b]. \tag{22}$$

We give our first main result.

THEOREM 5. *Assume all terms and assumptions of Proposition 4 and*

$$\int_X D\big(f(t,\omega), \tilde{o}\big)\, dP(\omega) < \infty, \quad \forall t \in [a,b].$$

Then

$$\begin{aligned} \int_X D\big(M(f)(t,\omega), f(t,\omega)\big) dP(\omega) &\leqslant |\tilde{L}(1)(t) - 1| \left(\int_X D\big(f(t,\omega), \tilde{o}\big) dP(\omega)\right) \\ &+ \left(\tilde{L}(1)(t) + \sqrt{\tilde{L}(1)(t)}\right) \Omega_1^{(\mathcal{F})}\big(f, (\tilde{L}((\cdot - t)^2)(t))^{1/2}\big)_{L^1}, \quad \forall t \in [a,b], \end{aligned} \tag{23}$$

and

$$\begin{aligned} &\sup_{t\in[a,b]} \left(\int_X D\big(M(f)(t,\omega), f(t,\omega)\big) dP(\omega)\right) \\ &\quad \leqslant \|\tilde{L}(1) - 1\|_\infty \sup_{t\in[a,b]} \left(\int_X D\big(f(t,\omega), \tilde{o}\big) dP(\omega)\right) \\ &\quad + \|\tilde{L}(1) + \sqrt{\tilde{L}(1)}\|_\infty \Omega_1^{(\mathcal{F})}\big(f, \|\tilde{L}((\cdot - t)^2)(t)\|_\infty^{1/2}\big)_{L^1}. \end{aligned} \tag{24}$$

Proof. Integrating (21) we get

$$\begin{aligned} &\int_X D\big(M(f)(t,\omega), f(t,\omega)\big) dP(\omega) \\ &\leqslant \int_X \left(\int_{[a,b]} D\big(f(s,\omega), f(t,\omega)\big)\mu_t(ds)\right) dP(\omega) + |m_t - 1| \left(\int_X D\big(f(t,\omega), \tilde{o}\big) dP(\omega)\right) \end{aligned}$$

(by $D \geqslant 0$ and the facts that $D\big(f(s,\omega), f(t,\omega)\big)$ is continuous in $s \in [a,b]$, by Lemma 1 of [2], also it is a real random variable in ω, by Remark 13. 39 of [11], p. 654 and thus by Proposition $3.3(i)$ of [3] it is jointly measurable in (s,ω), and then being able to use Tonelli–Fubini's theorem, p. 104 of [10] and thus see both double integrals make sense)

$$= \int_{[a,b]} \left(\int_X D\big(f(s,\omega), f(t,\omega)\big) dP(\omega) \right) \mu_t(ds) + |m_t - 1| \left(\int_X D\big(f(t,\omega), \tilde{o}\big) dP(\omega) \right)$$

$(h > 0)$

$$\leqslant \int_{[a,b]} \Omega_1^{(\mathcal{F})} \left(f, \frac{|s-t|}{h} h \right)_{L^1} \mu_t(ds) + |m_t - 1| \left(\int_X D\big(f(t,\omega), \tilde{o}\big) dP(\omega) \right) \quad \text{(by (7))}$$

$$\leqslant \Omega_1^{(\mathcal{F})}(f,h)_{L^1} \int_{[a,b]} \left\lceil \frac{|s-t|}{h} \right\rceil \mu_t(ds) + |m_t - 1| \left(\int_X D\big(f(t,\omega), \tilde{o}\big) dP(\omega) \right)$$

$$\leqslant |m_t - 1| \left(\int_X D\big(f(t,\omega), \tilde{o}\big) dP(\omega) \right) + \left(\int_{[a,b]} \left(1 + \frac{|s-t|}{h} \right) \mu_t(ds) \right) \Omega_1^{(\mathcal{F})}(f,h)_{L^1}$$

$$= |m_t - 1| \left(\int_X D\big(f(t,\omega), \tilde{o}\big) dP(\omega) \right) + \left(m_t + \frac{1}{h} \int_{[a,b]} |s-t| \mu_t(ds) \right) \Omega_1^{(\mathcal{F})}(f,h)_{L^1}$$

(by Cauchy–Schwarz inequality)

$$\leqslant |m_t - 1| \left(\int_X D\big(f(t,\omega), \tilde{o}\big) dP(\omega) \right) + \left(m_t + \frac{1}{h} \sqrt{m_t} \left(\int_{[a,b]} (s-t)^2 \mu_t(ds) \right)^{1/2} \right) \Omega_1^{(\mathcal{F})}(f,h)_{L^1}$$

(by choosing

$$h := \left(\int_{[a,b]} (s-t)^2 \mu_t(ds) \right)^{1/2} = \big(\tilde{L}((\cdot - t)^2)(t) \big)^{1/2} > 0,$$

for > 0 it is enough to assume $\mu_t([a,b] - \{t\}) > 0$)

$$\leqslant |m_t - 1| \left(\int_X D\big(f(t,\omega), \tilde{o}\big) dP(\omega) \right) + (m_t + \sqrt{m_t}) \Omega_1^{(\mathcal{F})} \left(f, \big(\tilde{L}((\cdot - t)^2)(t) \big)^{1/2} \right)_{L^1},$$

by using (22) we have established (23). One can easily see that if

$$\tilde{L}((\cdot - t)^2)(t) = 0$$

then again (23) is valid. Clearly by Remark 13.39, p. 654, [11] $D(f(t,\omega), \tilde{o})$ is a real random variable in $\omega \in X$, for each $t \in [a,b]$.

Next we notice that

$$|D(f(x,\omega), \tilde{o}) - D(f(y,\omega), \tilde{o})| \leqslant D(f(x,\omega), f(y,\omega)) \ \ \forall x, y \in [a,b], \ \ \forall \omega \in X.$$

Hence $\forall \varepsilon > 0 \ \ \exists \delta > 0$ such that whenever $x, y \in [a,b]$ with $|x - y| \leqslant \delta$ then

$$\left| \int_X D\big(f(x,\omega), \tilde{o}\big) P(d\omega) - \int_X D\big(f(y,\omega), \tilde{o}\big) P(d\omega) \right| \leqslant \int_X D\big(f(x,\omega), f(y,\omega)\big) P(d\omega) \leqslant \varepsilon,$$

because $f \in C^{U1}_{\mathcal{F}R}([a,b])$ by $f \in C^{U}_{\mathcal{F}R}([a,b])$. Therefore the function

$$F(x) := \int_X D\big(f(x,\omega),\tilde{o}\big)P(d\omega),$$

is continuous in $x \in [a,b]$ and hence is bounded, i.e. $\|F(x)\|_\infty < \infty$, making (24) valid. □

We need the following

PROPOSITION 5. *All here as in Proposition* 4 *and*

$$\int_X \big(D(f(t,\omega),\tilde{o})\big)^q P(d\omega) < \infty, \quad q > 1, \ \forall t \in [a,b].$$

Then

$$\begin{aligned}\left(\int_X \big(D(M(f)(t,\omega),f(t,\omega)\big)^q dP(\omega)\right)^{1/q} &\leqslant |m_t - 1| \left(\int_X \big(D(f(t,\omega),\tilde{o})\big)^q P(d\omega)\right)^{1/q} \\ &+ m_t^{1-\frac{1}{q}} \left(\int_{[a,b]} \left(1 + \frac{|s-t|}{h}\right)^q d\mu_t(s)\right)^{1/q} \Omega_1^{(\mathcal{F})}(f,h)_{L^q}, \ h > 0, \ \forall t \in [a,b].\end{aligned} \tag{25}$$

Proof. Let $q > 1$ then by (21) we have

$$\begin{aligned}&\left(\int_X D\big(M(f)(t,\omega),f(t,\omega)\big)^q dP(\omega)\right)^{1/q} \\ &\leqslant \left(\int_X \left(\int_{[a,b]} D\big(f(s,\omega),f(t,\omega)\big)\mu_t(ds)\right)^q P(d\omega)\right)^{1/q} + \theta =: (*),\end{aligned}$$

where

$$\theta := |m_t - 1| \left(\int_X \big(D(f(t,\omega),\tilde{o})\big)^q P(d\omega)\right)^{1/q}. \tag{26}$$

Let $p > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Hence by Hölder's inequality we have

$$(*) \leqslant m_t^{1/p} \left(\int_X \left(\int_{[a,b]} D^q\big(f(s,\omega),f(t,\omega)\big)\mu_t(ds)\right) P(d\omega)\right)^{1/q} + \theta$$

(by Tonelli-Fubini's theorem as in the proof of Theorem 5)

$$= m_t^{1/p} \left(\int_{[a,b]} \left(\int_X D^q\big(f(s,\omega),f(t,\omega)\big)P(d\omega)\right)\mu_t(ds)\right)^{1/q} + \theta$$

(let $h > 0$)

$$\leqslant m_t^{1/p} \left(\int_{[a,b]} \left(\Omega_1^{(\mathcal{F})}\left(f, \frac{|s-t|}{h} h\right)_{L^q}\right)^q \mu_t(ds)\right)^{1/q} + \theta$$

$$\overset{(7)}{\leqslant} m_t^{1/p} \left(\int_{[a,b]} \left(1 + \frac{|s-t|}{h}\right)^q \mu_t(ds)\right)^{1/q} \Omega_1^{(\mathcal{F})}(f,h)_{L^q} + \theta. \qquad \square$$

We examine two cases and we give

THEOREM 6. *Here we assume all as in Proposition* 5.

1) *Let* $q \in \mathbb{N} - \{1\}$. *Then*

$$\left(\int_X D^q\big(M(f)(t,\omega), f(t,\omega)\big)P(d\omega)\right)^{1/q} \leqslant |\tilde{L}(1)(t)-1|\left(\int_X D^q\big(f(t,\omega),\tilde{o}\big)P(d\omega)\right)^{1/q} + (\tilde{L}(1)(t))^{1-\frac{1}{q}}\left(\sum_{k=0}^{q}\binom{q}{k}(\tilde{L}(1)(t))^{1-\frac{k}{q}}\right)^{1/q}\Omega_1^{(\mathcal{F})}\left(f,((\tilde{L}(1\cdot -t|^q))(t))^{1/q}\right)_{L^q}, \tag{27}$$

$\forall t \in [a,b]$.

2) *Let* $q>1$ *real. Then*

$$\left(\int_X D^q\big(M(f)(t,\omega), f(t,\omega)\big)P(d\omega)\right)^{1/q} \leqslant |\tilde{L}(1)(t)-1|\left(\int_X D^q\big(f(t,\omega),\tilde{o}\big)P(d\omega)\right)^{1/q} + 2^{1-\frac{1}{q}}(\tilde{L}(1)(t))^{1-\frac{1}{q}}(\tilde{L}(1)(t)+1)^{1/q}\Omega_1^{(\mathcal{F})}\big(f,((\tilde{L}(|\cdot -t|^q))(t))^{1/q}\big)_{L^q}, \tag{28}$$

$\forall t \in [a,b]$.

When $q \in \mathbb{N}-\{1\}$ *then* (27) *is sharper than* (28). *Furthermore we have*

3) *Let* $q \in \mathbb{N} - \{1\}$. *Then*

$$\sup_{t\in[a,b]}\left(\int_X D^q\big(M(f)(t,\omega), f(t,\omega)\big)P(d\omega)\right)^{1/q} \leqslant \|\tilde{L}(1)-1\|_\infty \sup_{t\in[a,b]}\left(\int_X D^q\big(f(t,\omega),\tilde{o}\big)P(d\omega)\right)^{1/q} + \|\tilde{L}(1)\|_\infty^{1-\frac{1}{q}}\left(\left\|\sum_{k=0}^{q}\binom{q}{k}(\tilde{L}(1))^{1-\frac{k}{q}}\right\|_\infty\right)^{1/q}\Omega_1^{(\mathcal{F})}\big(f,\|(\tilde{L}(|\cdot -t|^q))(t)\|_\infty^{1/q}\big)_{L^q}. \tag{29}$$

4) *Let* $q>1$ *real. Then*

$$\sup_{t\in[a,b]}\left(\int_X D^q\big(M(f)(t,\omega), f(t,\omega)\big)P(d\omega)\right)^{1/q} \leqslant \|\tilde{L}(1)-1\|_\infty \sup_{t\in[a,b]}\left(\int_X D^q\big(f(t,\omega),\tilde{o}\big)P(d\omega)\right)^{1/q} + 2^{1-\frac{1}{q}}\|\tilde{L}(1)\|_\infty^{1-\frac{1}{q}}\|\tilde{L}(1)+1\|_\infty^{1/q}\Omega_1^{(\mathcal{F})}\big(f,\|(\tilde{L}(|\cdot -t|^q))(t)\|_\infty^{1/q}\big)_{L^q}. \tag{30}$$

When $q \in \mathbb{N}-\{1\}$ *inequality* (29) *is sharper than* (30).

Proof. 1) We notice that

$$\int_{[a,b]} \left(1+\frac{|s-t|}{h}\right)^q d\mu_t(s) = \int_{[a,b]} \left(\sum_{k=0}^{q} \binom{q}{k} \frac{|s-t|^k}{h^k}\right) d\mu_t(s)$$
$$= m_t + \sum_{k=1}^{q-1} \binom{q}{k} \frac{1}{h^k} \int_{[a,b]} |s-t|^k d\mu_t(s) + \frac{1}{h^q} \int_{[a,b]} |s-t|^q d\mu_t(s).$$

Also we see for $k=1,\dots,q-1$ that

$$\int_{[a,b]} |s-t|^k d\mu_t(s) \leqslant m_t^{1-\frac{k}{q}} \left(\int_{[a,b]} |s-t|^q d\mu_t(s)\right)^{k/q}$$

Hence

$$\int_{[a,b]} \left(1+\frac{|s-t|}{h}\right)^q d\mu_t(s)$$
$$\leqslant m_t + \sum_{k=1}^{q-1} \binom{q}{k} \frac{1}{h^k} m_t^{1-\frac{k}{q}} \left(\int_{[a,b]} |s-t|^q d\mu_t(s)\right)^{k/q} + \frac{1}{h^q} \int_{[a,b]} |s-t|^q d\mu_t(s)$$
$$= \sum_{k=0}^{q} \binom{q}{k} \frac{m_t^{1-(k/q)}}{h^k} \left(\int_{[a,b]} |s-t|^q d\mu_t(s)\right)^{k/q}$$

(by choosing

$$h := \left(\int_{[a,b]} |s-t|^q d\mu_t(s)\right)^{1/q}$$
$$= \left((\tilde{L}(|\cdot - t|^q))(t)\right)^{1/q} > 0) \tag{31}$$
$$= \sum_{k=0}^{q} \binom{q}{k} m_t^{1-\frac{k}{q}}.$$

I.e. we got that

$$\int_{[a,b]} \left(1+\frac{|s-t|}{h}\right)^q d\mu_t(s) \leqslant \sum_{k=0}^{q} \binom{q}{k} m_t^{1-\frac{k}{q}}.$$

Hence proving (27) with the use of (25). The inequality (27) is true easily if our choice is easily

$$h^q := \int_{[a,b]} |s-t|^q d\mu_t(s) = 0.$$

2) The function x^q is convex for $x \geqslant 0$, $q > 1$. Therefore

$$\left(\frac{1+\frac{|s-t|}{h}}{2}\right)^q \leqslant \frac{1+\frac{|s-t|^q}{h^q}}{2}, \quad h > 0.$$

I.e.

$$\left(1+\frac{|s-t|}{h}\right)^q \leqslant 2^{q-1} \left(1+\frac{|s-t|^q}{h^q}\right), \quad \forall s,t \in [a,b].$$

Hence

$$\left(\int_{[a,b]}\left(1+\frac{|s-t|}{h}\right)^q d\mu_t(s)\right)^{1/q} \leqslant 2^{1-\frac{1}{q}}\left(\int_{[a,b]}\left(1+\frac{|s-t|^q}{h^q}\right)d\mu_t(s)\right)^{1/q}$$
$$= 2^{1-\frac{1}{q}}\left(m_t+\frac{1}{h^q}\left(\int_{[a,b]}|s-t|^q d\mu_t(s)\right)\right)^{1/q}$$

(by choosing again

$$h := \left(\int_{[a,b]}|s-t|^q d\mu_t(s)\right)^{1/q} > 0)$$
$$= 2^{1-\frac{1}{q}}(m_t+1)^{1/q}.$$

I.e. we got that

$$\left(\int_{[a,b]}\left(1+\frac{|s-t|}{h}\right)^q d\mu_t(s)\right)^{1/q} \leqslant 2^{1-\frac{1}{q}}(m_t+1)^{1/q}.$$

Using (25) and the last estimate we obtain (28). Again if our above choice is $h=0$ then (28) is still valid.

When $q \in \mathbb{N}-\{1\}$ and $m_t > 0$ we would like to prove that

$$\left(\sum_{k=0}^{q}\binom{q}{k} m_t^{1-\frac{k}{q}}\right)^{1/q} \leqslant 2^{1-\frac{1}{q}}(m_t+1)^{1/q}, \tag{32}$$

hence (27) is better than (28). Notice that (32) is trivially true when $m_t = 0$.

Equivalently we need valid

$$\sum_{k=0}^{q}\binom{q}{k} m_t^{1-\frac{k}{q}} \leqslant 2^{q-1}(m_t+1)$$
$$\Leftrightarrow$$
$$\sum_{k=0}^{q}\binom{q}{k} m_t^{-k/q} \leqslant 2^{q-1}(1+m_t^{-1}).$$

By calling $z := m_t^{-1} > 0$, equivalently we need true

$$\sum_{k=0}^{q}\binom{q}{k} z^{k/q} \leqslant 2^{q-1}(1+z)$$
$$\Leftrightarrow$$
$$(1+z^{1/q})^q \leqslant 2^{q-1}(1+z).$$

The last is true by the convexity of z^q, $z \geqslant 0$, $q \in \mathbb{N}-\{1\}$. If $m_t = 0$, then both (27) and (28) are trivially the same.

It is easy to derive (29) and (30) from (27) and (28), respectively. Clearly $D^q(f(t,\omega),\tilde{o})$ is a real random variable in $\omega \in X$, $\forall t \in [a,b]$. Additionally, we notice that $\forall \varepsilon > 0$ $\exists \delta > 0$ such that whenever $x, y \in [a,b]$ with $|x-y| \leqslant \delta$ then

$$\left| \left(\int_X D^q(f(x,\omega),\tilde{o})dP(\omega) \right)^{1/q} - \left(\int_X D^q(f(y,\omega),\tilde{o})dP(\omega) \right)^{1/q} \right|$$
$$\leqslant \left(\int_X D^q(f(x,\omega),f(y,\omega))dP(\omega) \right)^{1/q} \leqslant \varepsilon, \text{ by } f \in C^U_{\mathcal{F}R}([a,b]).$$

Hence proving that the function

$$G(x) := \left(\int_X D^q(f(x,\omega),\tilde{o})dP(\omega) \right)^{1/q},$$

is continuous in $x \in [a,b]$. Therefore $\|G\|_\infty < \infty$, making the inequalities (29), (30) valid. $\square$

Similar general results using a different initial estimate follow.

LEMMA 3. *Let $f : [a,b] \to \mathcal{L}_F(X,\mathcal{B},P)$ be fuzzy random function, $1 \leqslant q < \infty$, $\delta > 0$. Then*

$$\Omega_1^{(\mathcal{F})}(f,|x-y|)_{L^q} \leqslant \left(1 + \frac{(x-y)^2}{\delta^2}\right) \Omega_1^{(\mathcal{F})}(f,\delta)_{L^q}, \quad \forall x,y \in [a,b]. \tag{33}$$

Proof. We have by (7) that

$$\Omega_1^{(\mathcal{F})}(f,|x-y|)_{L^q} \leqslant \left(1 + \frac{|x-y|}{\delta}\right) \Omega_1^{(\mathcal{F})}(f,\delta)_{L^q}. \tag{34}$$

Let $|x-y| > \delta$, thus $\frac{|x-y|}{\delta} > 1$. Then

$$\text{R. H. S. (34)} \leqslant \left(1 + \frac{(x-y)^2}{\delta^2}\right) \Omega_1^{(\mathcal{F})}(f,\delta)_{L^q}.$$

Let $|x-y| \leqslant \delta$ then

$$\Omega_1^{(\mathcal{F})}(f,|x-y|) \leqslant \Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} \leqslant \left(1 + \frac{(x-y)^2}{\delta^2}\right) \Omega_1^{(\mathcal{F})}(f,\delta)_{L^q}. \qquad \square$$

Now, we present

THEOREM 7. *Assume all terms and assumptions of Proposition 4 and*

$$\int_X D\big(f(t,\omega),\tilde{o}\big)dP(\omega) < \infty, \quad \forall t \in [a,b].$$

Then

1)
$$\int_X D\big(M(f),(t,\omega),f(t,\omega)\big)dP(\omega)$$
$$\leqslant |\tilde{L}(1)(t) - 1| \left(\int_X D\big(f(t,\omega),\tilde{o}\big)dP(\omega) \right) \tag{35}$$
$$+ (\tilde{L}(1)(t)+1)\Omega_1^{(\mathcal{F})}\left(f, \left(\tilde{L}((\cdot - t)^2)(t)\right)^{1/2}\right)_{L^1}, \quad \forall t \in [a,b],$$

and

2)
$$\sup_{t\in[a,b]}\int_X D\big(M(f)(t,\omega),f(t,\omega)\big)\,dP(\omega) \leqslant \|\tilde{L}(1)-1\|_\infty \sup_{t\in[a,b]}\left(\int_X D\big(f(t,\omega),\tilde{o}\big)dP(\omega)\right) + \|\tilde{L}(1)+1\|_\infty \Omega_1^{(\mathcal{F})}\big(f,\|\tilde{L}((\cdot-t)^2)(t)\|_\infty^{1/2}\big)_{L^1}. \tag{36}$$

Proof. Initially from the proof of Theorem 5 we have

$$\begin{aligned}
&\int_X D\big(M(f)(t,\omega),f(t,\omega)\big)\,dP(\omega)\\
&\leqslant |m_t-1|\left(\int_X D\big(f(t,\omega),\tilde{o}\big)dP(\omega)\right)+\int_{[a,b]}\Omega_1^{(\mathcal{F})}(f,|s-t|)_{L^1}\mu_t(s)\\
&\quad (\text{let } h>0)\\
&\overset{(33)}{\leqslant} |m_t-1|\left(\int_X D\big(f(t,\omega),\tilde{o}\big)dP(\omega)\right)+\left(\int_{[a,b]}\left(1+\frac{(s-t)^2}{h^2}\right)\mu_t(ds)\right)\Omega_1^{(\mathcal{F})}(f,h)_{L^1}\\
&= |m_t-1|\left(\int_X D\big(f(t,\omega),\tilde{o}\big)dP(\omega)\right)+\left(m_t+\frac{1}{h^2}\int_{[a,b]}(s-t)^2\mu_t(ds)\right)\Omega_1^{(\mathcal{F})}(f,h)_{L^1}
\end{aligned}$$

taking

$$\begin{aligned}
h&:=\left(\int_{[a,b]}(s-t)^2\mu_t(dt)\right)^{1/2}=\big(\tilde{L}((\cdot-t)^2)(t)\big)^{1/2}>0\\
&= |m_t-1|\left(\int_X D\big(f(t,\omega),\tilde{o}\big)dP(\omega)\right)+(m_t+1)\Omega_1^{(\mathcal{F})}\left(f,\big(\tilde{L}((\cdot-t)^2)(t)\big)^{1/2}\right)_{L^1},
\end{aligned} \tag{37}$$

$\forall t\in[a,b]$. That is proving (35).

The above choice (37) of h if $h=0$ makes again (35) valid. Inequality (36) is now clear. □

Finally we get the very useful

THEOREM 8. *Assume all terms and assumptions of Proposition* 4 *and*

$$\int_X D\big(f(t,\omega),\tilde{o}\big)dP(\omega)<\infty,\quad \forall t\in[a,b].$$

Then

1)
$$\begin{aligned}
&\int_X D\big(M(f)(t,\omega),f(t,\omega)\big)\,dP(\omega)\\
&\leqslant |\tilde{L}(1)(t)-1|\left(\int_X D\big(f(t,\omega),\tilde{o}\big)dP(\omega)\right)\\
&\quad+\min\left\{\Big(\tilde{L}(1)(t)+\sqrt{\tilde{L}(1)(t)}\Big),\big(\tilde{L}(1)(t)+1\big)\right\}\times\\
&\quad\times\Omega_1^{(\mathcal{F})}\big(f,(\tilde{L}((\cdot-t)^2)(t))^{1/2}\big)_{L^1},\quad \forall t\in[a,b],
\end{aligned} \tag{38}$$

and

2)
$$\sup_{t\in[a,b]}\left(\int_X D\big(M(f)(t,\omega),f(t,\omega)\big)\,dP(\omega)\right)$$
$$\leqslant \|\tilde{L}(1)-1\|_\infty \sup_{t\in[a,b]}\left(\int_X D\big(f(t,\omega),\tilde{o}\big)dP(\omega)\right)$$
$$+\min\left\{\|\tilde{L}(1)+\sqrt{\tilde{L}(1)}\|_\infty, \|\tilde{L}(1)+1\|_\infty\right\}\times \tag{39}$$
$$\times\,\Omega_1^{(\mathcal{F})}\big(f, \|\tilde{L}((\cdot-t)^2)(t)\|_\infty^{1/2}\big)_{L^1}.$$

Proof. Obvious from Theorems 5 and 7. □

The corresponding results for $q>1$ follow.

THEOREM 9. *Here we assume all as in Proposition* 5. *Let* $q\in\mathbb{N}-\{1\}$. *Then*

1)
$$\left(\int_X D^q\big(M(f)(t,\omega),f(t,\omega)\big)\,dP(\omega)\right)^{1/q}$$
$$\leqslant |\tilde{L}(1)(t)-1|\left(\int_X \big(D(f(t,\omega),\tilde{o})\big)^q dP(\omega)\right)^{1/q}$$
$$+\ (\tilde{L}(1)(t))^{1-\frac{1}{q}}\left(\sum_{k=0}^{q}\binom{q}{k}\big((\tilde{L}(1))(t)\big)^{1-\frac{k}{q}}\right)^{1/q}\times \tag{40}$$
$$\times\,\Omega_1^{(\mathcal{F})}\big(f, ((\tilde{L}(\cdot-t)^{2q})(t))^{1/2q}\big)_{L^q},\quad \forall t\in[a,b].$$

And also holds

2)
$$\sup_{t\in[a,b]}\left(\int_X D\big(M(f)(t,\omega),f(t,\omega)\big)^q dP(\omega)\right)^{1/q}$$
$$\leqslant \|\tilde{L}(1)-1\|_\infty \sup_{t\in[a,b]}\left(\int_X D^q\big(f(t,\omega),\tilde{o}\big)dP(\omega)\right)^{1/q} \tag{41}$$
$$+\|\tilde{L}(1)\|_\infty^{1-\frac{1}{q}}\left(\left\|\sum_{k=0}^{q}\binom{q}{k}(\tilde{L}(1))^{1-\frac{k}{q}}\right\|_\infty\right)^{1/q}\Omega_1^{(\mathcal{F})}\big(f, \|(\tilde{L}(\cdot-t)^{2q})(t)\|_\infty^{1/2q}\big)_{L^q}.$$

Let $q>1$ *real. Then*

3)
$$\left(\int_X D\big(M(f)(t,\omega),f(t,\omega)\big)^q dP(\omega)\right)^{1/q}$$
$$\leqslant |\tilde{L}(1)(t)-1|\left(\int_X \big(D(f(t,\omega),\tilde{o})\big)^q dP(\omega)\right)^{1/q} \tag{42}$$
$$+\,2^{1-\frac{1}{q}}(\tilde{L}(1)(t))^{1-\frac{1}{q}}(\tilde{L}(1)(t)+1)^{1/q}\Omega_1^{(\mathcal{F})}\big(f, ((\tilde{L}(\cdot-t)^{2q})(t))^{1/2q}\big)_{L^q},$$

$\forall t\in[a,b]$. *And also holds*

4)
$$\sup_{t\in[a,b]}\left(\int_X D\big(M(f)(t,\omega),f(t,\omega)\big)^q dP(\omega)\right)^{1/q}$$
$$\leqslant \|\tilde{L}(1)-1\|_\infty \sup_{t\in[a,b]}\left(\int_X D^q(f(t,\omega),\tilde{o})dP(\omega)\right)^{1/q} \tag{43}$$
$$+\,2^{1-\frac{1}{q}}\|\tilde{L}(1)\|_\infty^{1-\frac{1}{q}}\|\tilde{L}(1)+1\|_\infty^{1/q}\Omega_1^{(\mathcal{F})}\big(f,\|(\tilde{L}(\cdot-t)^{2q})(t)\|_\infty^{1/2q}\big)_{L^q}.$$

When $q\in\mathbb{N}-\{1\}$ *then inequality* (40) *is sharper than* (42) *and inequality* (41) *is sharper than* (43).

Note. Later we will see that inequalities (40)–(43) and/or inequalities (27)–(30) can be used to prove "q-mean" convergence with rates of a sequence of M's to unit operator I.

Proof. Initially from the proof of Proposition 5 we get

$$\left(\int_X \big(D(M(f)(t,\omega),f(t,\omega)\big)^q dP(\omega)\right)^{1/q}$$
$$\leqslant\ \theta+\left(\int_X\left(\int_{[a,b]} D\big(f(s,\omega),f(t,\omega)\big)\mu_t(dt)\right)^q dP(\omega)\right)^{1/q}$$
(θ as in (26))
$$\leqslant\ \theta+m_t^{1-\frac{1}{q}}\left(\int_{[a,b]}\big(\Omega_1^{(\mathcal{F})}(f,|s-t|)_{L^q}\big)^q d\mu_t(s)\right)^{1/q}$$
(let $h>0$)
$$\overset{\text{(by (33))}}{\leqslant}\theta+m_t^{1-\frac{1}{q}}\left(\int_{[a,b]}\left(1+\frac{(s-t)^2}{h^2}\right)^q d\mu_t(s)\right)^{1/q}\Omega_1^{(\mathcal{F})}(f,h)_{L^q}=:(\xi).$$

1) Let $q\in\mathbb{N}-\{1\}$. We observe that

$$\int_{[a,b]}\left(1+\frac{(s-t)^2}{h^2}\right)^q d\mu_t(s)=\int_{[a,b]}\left(\sum_{k=0}^{q}\binom{q}{k}\frac{(s-t)^{2k}}{h^{2k}}\right)d\mu_t(s)$$
$$=m_t+\sum_{k=1}^{q-1}\binom{q}{k}\frac{1}{h^{2k}}\left(\int_{[a,b]}(s-t)^{2k}d\mu_t(s)\right)+\frac{1}{h^{2q}}\left(\int_{[a,b]}(s-t)^{2q}d\mu_t(s)\right).$$

For $k=1,\dots,q-1$, $\frac{q}{k}>1$ and by Hölder's inequality we have

$$\int_{[a,b]}(s-t)^{2k}d\mu_t(s)\leqslant m_t^{1-\frac{k}{q}}\left(\int_{[a,b]}(s-t)^{2q}d\mu_t(s)\right)^{k/q}.$$

Hence

$$\int_{[a,b]}\left(1+\frac{(s-t)^2}{h^2}\right)^q d\mu_t(s)\leqslant\sum_{k=0}^{q}\binom{q}{k}\frac{m_t^{1-(k/q)}}{h^{2k}}\left(\int_{[a,b]}(s-t)^{2q}d\mu_t(s)\right)^{k/q}$$

(by choosing

$$h := \left(\int_{[a,b]} (s-t)^{2q} d\mu_t(s) \right)^{1/2q} = \left((\tilde{L}((\cdot - t)^{2q}))(t) \right)^{1/2q} > 0 \Big) \leqslant \sum_{k=0}^{q} \binom{q}{k} m_t^{1-\frac{k}{q}}. \tag{44}$$

That is

$$\left(\int_{[a,b]} \left(1 + \frac{(s-t)^2}{h^2} \right)^q d\mu_t(s) \right)^{1/q} \leqslant \left(\sum_{k=0}^{q} \binom{q}{k} m_t^{1-\frac{k}{q}} \right)^{1/q}. \tag{45}$$

Thus by (44) and (45) we have

$$(\xi) \leqslant \theta + \left(\tilde{L}(1)(t)\right)^{1-\frac{1}{q}} \left(\sum_{k=0}^{q} \binom{q}{k} (\tilde{L}(1)(t))^{1-\frac{k}{q}} \right)^{1/q} \times \\ \times \ \Omega_1^{(\mathcal{F})} \big(f, ((\tilde{L}((\cdot - t)^{2q}))(t))^{1/2q} \big)_{L^q}, \ \ \forall t \in [a,b].$$

That is establishing (40). When the choice (44) of $h = 0$ then again (40) is trivially valid.

2) Let now $q > 1$ real, then again by convexity of x^q, $x \geqslant 0$ we have

$$\left(1 + \frac{(s-t)^2}{h^2} \right)^q \leqslant 2^{q-1} \left(1 + \frac{(s-t)^{2q}}{h^{2q}} \right), \ \ h > 0, \ \forall s,t \in [a,b].$$

Hence

$$\begin{aligned} (\xi) &\leqslant \theta + m_t^{1-\frac{1}{q}} 2^{1-\frac{1}{q}} \left(\int_{[a,b]} \left(1 + \frac{(s-t)^{2q}}{h^{2q}} \right) d\mu_t(s) \right)^{1/q} \Omega_1^{(\mathcal{F})}(f,h)_{L^q} \\ &= \theta + 2^{1-\frac{1}{q}} m_t^{1-\frac{1}{q}} \left[m_t + \frac{1}{h^{2q}} \int_{[a,b]} (s-t)^{2q} d\mu_t(s) \right]^{1/q} \Omega_1^{(\mathcal{F})}(f,h)_{L^q} \\ &\qquad (\text{let } h > 0 \text{ as in } (44)) \\ &= \theta + 2^{1-\frac{1}{q}} m_t^{1-\frac{1}{q}} (m_t + 1)^{1/q} \Omega_1^{(\mathcal{F})}(f,h)_{L^q} \\ &= \theta + 2^{1-\frac{1}{q}} (\tilde{L}(1)(t))^{1-\frac{1}{q}} (\tilde{L}(1)(t) + 1)^{1/q} \Omega_1^{(\mathcal{F})} \big(f, ((\tilde{L}((\cdot - t)^{2q}))(t))^{1/2q} \big)_{L^q}, \end{aligned}$$

$\forall t \in [a,b]$. That is proving (42).

When the choice (44) for $h = 0$ then inequality (42) is trivially valid. Inequalities (41) and (43) derive easily from (40) and (42), respectively, and they are valid, similarly, as inequalities (29) and (30). The comparison of inequalities is the same as in Theorem 6.

Finally we derive

THEOREM 10. *Here we assume all as in Proposition* 5. *Let* $q \in \mathbb{N} - \{1\}$. *Then*

1)
$$\left(\int_X D^q\big(M(f)(t,\omega), f(t,\omega)\big)dP(\omega)\right)^{1/q}$$
$$\leqslant |\tilde{L}(1)(t)-1| \left(\int_X D^q\big(f(t,\omega),\tilde{o}\big)dP(\omega)\right)^{1/q}$$
$$+\ (\tilde{L}(1)(t))^{1-\frac{1}{q}} \left(\sum_{k=0}^{q}\binom{q}{k}((\tilde{L}(1))(t))^{1-\frac{k}{q}}\right)^{1/q} \times \tag{46}$$
$$\times \min\Big\{\Omega_1^{(\mathcal{F})}\big(f, ((\tilde{L}(|\cdot - t|^q))(t)^{1/q}\big)_{L^q},$$
$$\Omega_1^{(\mathcal{F})}\big(f, ((\tilde{L}(\cdot - t)^{2q})(t))^{1/2q}\big)_{L^q}\Big\}, \quad \forall t \in [a,b].$$

And also holds

2)
$$\sup_{t\in[a,b]}\left(\int_X D^q\big(M(f)(t,\omega), f(t,\omega)\big)dP(\omega)\right)^{1/q}$$
$$\leqslant\ \|\tilde{L}(1)-1\|_\infty \sup_{t\in[a,b]}\left(\int_X D^q\big(f(t,\omega),\tilde{o}\big)dP(\omega)\right)^{1/q}$$
$$+\ \|\tilde{L}(1)\|_\infty^{1-\frac{1}{q}} \left(\left\|\sum_{k=0}^{q}\binom{q}{k}(\tilde{L}(1))^{1-\frac{k}{q}}\right\|_\infty\right)^{1/q} \times \tag{47}$$
$$\times\ \min\Big\{\Omega_1^{(\mathcal{F})}\big(f, \|(\tilde{L}(|\cdot - t|^q))(t)\|_\infty^{1/q}\big)_{L^q},$$
$$\Omega_1^{(\mathcal{F})}\big(f, \|(\tilde{L}(\cdot - t)^{2q})(t)\|_\infty^{1/2q}\big)_{L^q}\Big\}.$$

Let $q>1$ real. Then

3)
$$\left(\int_X D\big(M(f)(t,\omega), f(t,\omega)\big)^q dP(\omega)\right)^{1/q}$$
$$\leqslant |\tilde{L}(1)(t)-1| \left(\int_X \big(D(f(t,\omega),\tilde{o})\big)^q dP(\omega)\right)^{1/q}$$
$$+\ 2^{1-\frac{1}{q}}(\tilde{L}(1)(t))^{1-\frac{1}{q}}(\tilde{L}(1)(t)+1)^{1/q}\times \tag{48}$$
$$\times \min\Big\{\Omega_1^{(\mathcal{F})}\big(f, ((\tilde{L}(|\cdot - t|^q))(t))^{1/q}\big)_{L^q},$$
$$\Omega_1^{(\mathcal{F})}\big(f, ((\tilde{L}(\cdot - t)^{2q})(t))^{1/2q}\big)_{L^q},\Big\}, \quad \forall t \in [a,b].$$

And also holds

4)
$$\sup_{t\in[a,b]}\left(\int_X D^q\big(M(f)(t,\omega), f(t,\omega)\big)P(d\omega)\right)^{1/q}$$
$$\leqslant \|\tilde{L}(1)-1\|_\infty \sup_{t\in[a,b]}\left(\int_X D^q\big(f(t,\omega),\tilde{o}\big)P(d\omega)\right)^{1/q}$$
$$+\ 2^{1-\frac{1}{q}}\|\tilde{L}(1)\|_\infty^{1-\frac{1}{q}}\|\tilde{L}(1)+1\|_\infty^{1/q}\times \tag{49}$$
$$\times \min\Big\{\Omega_1^{(\mathcal{F})}\big(f, \|(\tilde{L}(|\cdot - t|^q))(t)\|_\infty^{1/q}\big)_{L^q},$$
$$\Omega_1^{(\mathcal{F})}\big(f, \|(\tilde{L}(\cdot - t)^{2q})(t)\|_\infty^{1/2q}\big)_{L^q}\Big\}.$$

When $q \in \mathbb{N} - \{1\}$ *then inequality* (46) *is sharper than* (48) *and* (47) *sharper than* (49).

Proof. By Theorems 6 and 9. □

We give

COROLLARY 1. *All here as in Proposition* 4 *and*

$$\int_X D^2\big(f(t,\omega),\tilde{o}\big)dP(\omega) < \infty, \quad \forall t \in [a,b].$$

Then

1)
$$\begin{aligned} &\left(\int_X D^2\big(M(f)(t,\omega),f(t,\omega)\big)P(d\omega)\right)^{1/2} \\ &\leqslant |\tilde{L}(1)(t)-1| \left(\int_X D^2\big(f(t,\omega),\tilde{o}\big)P(d\omega)\right)^{1/2} \\ &\quad + (\tilde{L}(1)(t))^{1/2}\big(\tilde{L}(1)(t)+2(\tilde{L}(1)(t))^{1/2}+1\big)^{1/2}\times \\ &\quad\times \Omega_1^{(\mathcal{F})}\big(f,((\tilde{L}((\cdot-t)^2))(t))^{1/2}\big)_{L^2}, \ \ \forall t \in [a,b]. \end{aligned} \tag{50}$$

and

2)
$$\begin{aligned} &\sup_{t\in[a,b]}\left(\int_X D^2\big(M(f)(t,\omega),f(t,\omega)\big)P(d\omega)\right)^{1/2} \\ &\leqslant \|\tilde{L}(1)-1\|_\infty \sup_{t\in[a,b]}\left(\int_X D^2\big(f(t,\omega),\tilde{o}\big)P(d\omega)\right)^{1/2} \\ &+ \|\tilde{L}(1)\|_\infty^{1/2}\|\tilde{L}(1)+2(\tilde{L}(1))^{1/2}+1\|_\infty^{1/2}\Omega_1^{(\mathcal{F})}\big(f,\|(\tilde{L}((\cdot-t)^2))(t)\|_\infty^{1/2}\big)_{L^q}. \end{aligned} \tag{51}$$

Proof. By Theorem 6, inequalities (27) and (29). □

All inequalities produced in this article are of Shisha–Mond type (see [16]) in the fuzzy-random sense. We will derive next some Fuzzy-Random Korovkin Theorems regarding the spaces of functions

$$K_q([a,b]) := \left\{f \in C^U_{\mathcal{F}R}([a,b]) \colon \int_X D^q\big(f(t,\omega),\tilde{o}\big)dP(\omega) < \infty,\ \forall t \in [a,b]\right\},$$

where $1 \leqslant q < \infty$. We observe that if $1 \leqslant k < \infty$ such that $k \leqslant q$ then

$$K_q([a,b]) \subseteq K_k([a,b]).$$

For the above purpose we need to put together the following assumptions and settings.

Assumption 2. Let $(X,\mathcal{B},P)$ be a probability space, $[a,b] \subset \mathbb{R}$, $f \in C^U_{\mathcal{F}R}([a,b])$. Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy positive linear operators from $C_{\mathcal{F}}([a,b])$ into itself for which there exists a corresponding sequence of positive linear operators $\{\tilde{L}\}_{n\in\mathbb{N}}$ from $C([a,b])$ into itself such that

$$(L_n g)_{\pm}^{(r)} = \tilde{L}_n(g_{\pm}^{(r)}), \tag{52}$$

respectively, $\forall r \in [0,1]$, $\forall n \in \mathbb{N}$, $\forall g \in C_{\mathcal{F}}([a,b])$. We then consider the sequence of positive fuzzy random linear operators $\{M_n\}_{n\in\mathbb{N}}$ from $C_{\mathcal{F}R}^{U}([a,b])$ into $C_{\mathcal{F}R}([a,b])$ defined by

$$M_n(f)(t,\omega) := L_n(f(\cdot,\omega))(t), \tag{53}$$

$\forall (t,\omega) \in [a,b]\times X$, $\forall n \in \mathbb{N}$, $\forall f \in C_{\mathcal{F}R}^{U}([a,b])$. *Here I is the fuzzy random unit operator, i.e. $I(f)(t,\omega) = f(t,\omega)$, $\forall (t,\omega) \in [a,b]\times X$. We assume also that $\{\tilde{L}_n(1)\}_{n\in\mathbb{N}}$ is bounded.*

From Theorem 8 we have

COROLLARY 2. *Here all are as in Assumption 2, and*

$$\int_X D\big(f(t,\omega),\tilde{o}\big)dP(\omega) < \infty, \quad \forall t \in [a,b].$$

Then

1)
$$\begin{aligned}
&\int_X D\big(M_n(f)(t,\omega), f(t,\omega)\big)dP(\omega) \\
&\leqslant |\tilde{L}_n(1)(t) - 1| \left(\int_X D\big(f(t,\omega),\tilde{o}\big)dP(\omega)\right) \\
&\quad + \min\left\{\left(\tilde{L}_n(1)(t) + \sqrt{\tilde{L}_n(1)(t)}\right), \left(\tilde{L}_n(1)(t)+1\right)\right\} \times \\
&\quad \times \Omega_1^{(\mathcal{F})}\big(f, (\tilde{L}_n((\cdot - t)^2)(t))^{1/2}\big)_{L^1}, \quad \forall t \in [a,b], \ \forall n \in \mathbb{N},
\end{aligned} \tag{54}$$

and

2)
$$\begin{aligned}
&\sup_{t\in[a,b]} \left(\int_X D\big(M_n(f)(t,\omega), f(t,\omega)\big)dP(\omega)\right) \\
&\leqslant \|\tilde{L}_n(1) - 1\|_\infty \sup_{t\in[a,b]} \left(\int_X D\big(f(t,\omega),\tilde{o}\big)dP(\omega)\right) \\
&\quad + \min\left\{\|\tilde{L}_n(1) + \sqrt{\tilde{L}_n(1)}\|_\infty, \|\tilde{L}_n(1)+1\|_\infty\right\} \times \\
&\quad \times \Omega_1^{(\mathcal{F})}\big(f, \|\tilde{L}_n((\cdot - t)^2)(t)\|_\infty^{1/2}\big)_{L^1}, \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{55}$$

From Corollary 1 we get

COROLLARY 3. *Here all are as in Assumption 2, and*

$$\int_X D^2\big(f(t,\omega),\tilde{o}\big)dP(\omega) < \infty, \quad \forall t \in [a,b].$$

Then

1)
$$\left(\int_X D^2\big(M_n(f)(t,\omega),f(t,\omega)\big)P(d\omega)\right)^{1/2}$$
$$\leqslant |\tilde{L}_n(1)(t)-1|\left(\int_X D^2\big(f(t,\omega),\tilde{o}\big)P(d\omega)\right)^{1/2}$$
$$+(\tilde{L}_n(1)(t))^{1/2}\big(\tilde{L}_n(1)(t)+2(\tilde{L}_n(1)(t))^{1/2}+1\big)^{1/2}\times$$
$$\times\,\Omega_1^{(\mathcal{F})}\big(f,((\tilde{L}_n((\cdot-t)^2))(t))^{1/2}\big)_{L^2},\ \forall t\in[a,b],\ \forall n\in\mathbb{N}. \tag{56}$$

And also holds

2)
$$\sup_{t\in[a,b]}\left(\int_X D^2\big(M_n(f)(t,\omega),f(t,\omega)\big)P(d\omega)\right)^{1/2}$$
$$\leqslant \|\tilde{L}_n(1)-1\|_\infty \sup_{t\in[a,b]}\left(\int_X D^2\big(f(t,\omega),\tilde{o}\big)P(d\omega)\right)^{1/2}$$
$$+\|\tilde{L}_n(1)\|_\infty^{1/2}\|\tilde{L}_n(1)+2(\tilde{L}_n(1))^{1/2}+1\|_\infty^{1/2}\times$$
$$\times\,\Omega_1^{(\mathcal{F})}\big(f,\|(\tilde{L}_n((\cdot-t)^2))(t)\|_\infty^{1/2}\big)_{L^2},\ \ \forall n\in\mathbb{N}. \tag{57}$$

Note. One sees from [16] that

$$\big\|\big(\tilde{L}_n((\cdot-t)^2)\big)(t)\big\|_\infty\leqslant\|\tilde{L}_n(x^2)(t)-t^2\|_\infty+2c\|L_n(x)(t)-t\|_\infty+c^2\|L_n(1)(t)-1\|_\infty, \tag{58}$$

where $c:=\max(|a|,|b|)$, $\forall n\in\mathbb{N}$. Then one from the above fuzzy random Shisha–Mond type inequalities (55) and (57) derives the following basic fuzzy random Korovkin Theorems, see also [15].

THEOREM 11. *Here all are as in Assumption* 2*. Furthermore assume that*

$$\tilde{L}_n(1)\xrightarrow{u}1,\ \ \tilde{L}_n(id)\xrightarrow{u}id,\ \ \tilde{L}_n(id^2)\xrightarrow{u}id^2,$$

as $n\to\infty$*, where* u *means uniformly and* id *is the identity map. Then*

$$\lim_{n\to\infty}\left\|\int_X D\big(M_n(f)(t,\omega),f(t,\omega)\big)dP(\omega)\right\|_{\infty,t}=0,\ \ \forall f\in K_1([a,b]). \tag{59}$$

I.e.

$$M_n(f)(t,\omega)\underset{n\to\infty}{\overset{\text{“1-mean”}}{\longrightarrow}}f(t,\omega), \tag{60}$$

uniformly, $\forall f\in K_1([a,b])$*, that is uniformly* $M_n\overset{\text{“1-mean”}}{\longrightarrow}I$*, as* $n\to\infty$*, on* $K_1([a,b])$*.*

Proof. Using (55), (58) and Proposition 2(ii). □

We continue with the second basic fuzzy random Korovkin theorem.

THEOREM 12. *Here all are as in Assumption* 2*. Furthermore assume that*

$$\tilde{L}_n(1)\xrightarrow{u}1,\ \ \tilde{L}_n(id)\xrightarrow{u}id,\ \ \tilde{L}_n(id^2)\xrightarrow{u}id^2,\ \ \text{as } n\to\infty.$$

Then

$$\lim_{n\to\infty}\left\|\int_X D^2\big(M_n(f)(t,\omega),f(t,\omega)\big)P(d\omega)\right\|_{\infty,t}=0,\quad \forall f\in K_2([a,b]). \tag{61}$$

I.e.

$$M_n(f)(t,\omega)\overset{\text{“2-mean”}}{\longrightarrow} f(t,\omega), \tag{62}$$

uniformly, $\forall f\in K_2([a,b])$*, that is uniformly* $M_n \overset{\text{“2-mean”}}{\longrightarrow} I$*, as* $n\to\infty$*, on* $K_2([a,b])$*.*

Proof. From (57), (58) and Proposition $2(ii)$. □

The related general fuzzy random Korovkin theorem follows.

THEOREM 13. *Here all are as in Assumption* 2*,* $q>2$*. Furthermore we assume that*

(i) $\tilde{L}_n(1) \underset{n\to\infty}{\overset{u}{\longrightarrow}} 1$,

and

(ii) $\lim_{n\to\infty}\|(\tilde{L}_n(|\cdot-t|^q))(t)\|_\infty=0,$ (63)

or

$(ii)'$ $\lim\limits_{n\to\infty}\|(\tilde{L}_n(\cdot-t)^{2q})(t)\|_\infty=0.$

Then

$$\lim_{n\to\infty}\left\|\int_X D^q\big(M_n(f)(t,\omega),f(t,\omega)\big)P(d\omega)\right\|_{\infty,t}=0,\ \forall f\in K_q([a,b]). \tag{64}$$

I.e.

$$M_n(f)(t,\omega)\overset{\text{“}q\text{-mean”}}{\longrightarrow} f(t,\omega), \tag{65}$$

uniformly, $\forall f\in K_q([a,b])$*, that is uniformly* $M_n \overset{\text{“}q\text{-mean”}}{\longrightarrow} I$*, as* $n\to\infty$*, on* $K_q([a,b])$*.*

Proof. By (30) or (43) and Proposition $2(ii)$. In fact $(ii)'$ implies (ii). So one can use (ii) or $(ii)'$ as long as it is easier to be verified. □

The case $m_t=\tilde{L}(1)(t)=1$, $\forall t\in[a,b]$ is a very important and common one. Then all results of the paper simplify a lot as follows.

PROPOSITION 6. *All here as in Proposition* 4 *and* $m_t=1$*,* $\forall t\in[a,b]$*. Then*

$$D\big(M(f)(t,\omega),f(t,\omega)\big)\leqslant\int_{[a,b]}D\big(f(s,\omega),f(t,\omega)\big)\mu_t(ds),\ \ \forall(t,\omega)\in[a,b]\times X, \tag{66}$$

where μ_t *is as in* (10)*.*

Proof. We notice that the B-measurable function

$$D\big(M(f)(t,\omega),f(t,\omega)\big)\overset{(15)}{=}D\left(\int_{[a,b]}f(s,\omega)\mu_t(ds),\int_{[a,b]}f(t,\omega)\mu_t(ds)\right)$$

$$\leqslant \text{(by Theorem 2(2))}\ \int_{[a,b]}D\big(f(s,\omega),f(t,\omega)\big)\mu_t(ds). \qquad \square$$

Thus we obtain

THEOREM 14. *All here as in Proposition* 6. *Then*

1) $$\int_X D\big(M(f)(t,\omega), f(t,\omega)\big)\,dP(\omega) \leqslant 2\Omega_1^{(\mathcal{F})}\big(f, (\tilde{L}((\cdot-t)^2)(t))^{1/2}\big)_{L^1}, \tag{67}$$

$\forall t \in [a,b]$,

and

2) $$\sup_{t\in[a,b]} \int_X D\big(M(f)(t,\omega), f(t,\omega)\big)\,dP(\omega) \leqslant 2\Omega_1^{(\mathcal{F})}\big(f, \|\tilde{L}((\cdot-t)^2)(t)\|_\infty^{1/2}\big)_{L^1}. \tag{68}$$

Proof. By integrating (66), the proof follows, in a simpler way, as the proof of Theorem 5. □

Also we have

PROPOSITION 7. *All here as in Proposition* 6, $q > 1$. *Then*

$$\begin{aligned} &\left(\int_X \big(D(M(f)(t,\omega), f(t,\omega)\big)^q dP(\omega)\right)^{1/q} \\ &\quad\leqslant \left(\int_{[a,b]} \left(1+\frac{|s-t|}{h}\right)^q d\mu_t(s)\right)^{1/q} \Omega_1^{(\mathcal{F})}(f,h)_{L^q}, \quad h>0,\ \forall t\in[a,b]. \end{aligned} \tag{69}$$

Proof. Using (66) in exactly the same but simpler manner as in the proof of Proposition 5. □

We present

THEOREM 15. *All here as in Proposition* 6, $q > 1$. *Then*

1) $$\begin{aligned} &\left(\int_X D^q\big(M(f)(t,\omega), f(t,\omega)\big)P(d\omega)\right)^{1/q} \\ &\quad\leqslant 2\Omega_1^{(\mathcal{F})}\big(f, ((\tilde{L}(|\cdot-t|^q))(t))^{1/q}\big)_{L^q}, \quad \forall t\in[a,b], \end{aligned} \tag{70}$$

and

2) $$\begin{aligned} &\sup_{t\in[a,b]} \left(\int_X D^q\big(M(f)(t,\omega), f(t,\omega)\big)P(d\omega)\right)^{1/q} \\ &\quad\leqslant 2\Omega_1^{(\mathcal{F})}\big(f, \|(\tilde{L}(|\cdot-t|^q))(t)\|_\infty^{1/q}\big)_{L^q}. \end{aligned} \tag{71}$$

Proof. We use (69) and it follows similarly as the proof of Theorem 6. □

We also give

THEOREM 16. *Here all as in Proposition* 4*,* $m_t = 1$*,* $\forall t \in [a,b]$ *and* $q > 1$*. Then*

1)
$$\left(\int_X D^q\big(M(f)(t,\omega), f(t,\omega)\big)\,dP(\omega)\right)^{1/q} \leqslant 2\Omega_1^{(\mathcal{F})}\big(f, ((\tilde{L}(\cdot - t)^{2q})(t))^{1/2q}\big)_{L^q}, \quad \forall t \in [a,b], \tag{72}$$

and

2)
$$\sup_{t\in[a,b]}\left(\int_X D^q\big(M(f)(t,\omega), f(t,\omega)\big)\,dP(\omega)\right)^{1/q} \leqslant 2\Omega_1^{(\mathcal{F})}\big(f, \|(\tilde{L}(\cdot - t)^{2q})(t)\|_\infty^{1/2q}\big)_{L^q}. \tag{73}$$

Proof. Similar to the proof of Theorem 9. □

We derive

THEOREM 17. *Here all as in Proposition* 4*,* $m_t = 1$*,* $\forall t \in [a,b]$ *and* $q > 1$*. Then*

1)
$$\left(\int_X D^q\big(M(f)(t,\omega), f(t,\omega)\big)\,dP(\omega)\right)^{1/q} \leqslant 2\min\big\{\Omega_1^{(\mathcal{F})}\big(f, ((\tilde{L}(|\cdot - t|^q))(t))^{1/q}\big)_{L^q}, \Omega_1^{(\mathcal{F})}\big(f, ((\tilde{L}(\cdot - t)^{2q})(t))^{1/2q}\big)_{L^q}\big\}, \ \forall t \in [a,b], \tag{74}$$

and

2)
$$\sup_{t\in[a,b]}\left(\int_X D^q\big(M(f)(t,\omega), f(t,\omega)\big)\,dP(\omega)\right)^{1/q} \leqslant 2\min\big\{\Omega_1^{(\mathcal{F})}\big(f, \|(\tilde{L}(|\cdot - t|^q))(t)\|_\infty^{1/q}\big)_{L^q}, \Omega_1^{(\mathcal{F})}\big(f, \|(\tilde{L}(\cdot - t)^{2q})(t)\|_\infty^{1/2q}\big)_{L^q}\big\}. \tag{75}$$

Proof. From Theorems 15 and 16. □

We have

COROLLARY 4. *All here as in Proposition* 4*,* $m_t = 1$*,* $\forall t \in [a,b]$*. Then*

1)
$$\left(\int_X D^2\big(M(f)(t,\omega), f(t,\omega)\big)\,P(d\omega)\right)^{1/2} \leqslant 2\Omega_1^{(\mathcal{F})}\big(f, ((\tilde{L}((\cdot - t)^2))(t))^{1/2}\big)_{L^2}, \ \forall t \in [a,b], \tag{76}$$

and

2)
$$\sup_{t\in[a,b]}\left(\int_X D^2\big(M(f)(t,\omega), f(t,\omega)\big)\,P(d\omega)\right)^{1/2} \leqslant 2\Omega_1^{(\mathcal{F})}\big(f, \|(\tilde{L}((\cdot - t)^2))(t)\|_\infty^{1/2}\big)_{L^2}. \tag{77}$$

Proof. By Theorem 15, $q = 2$. □

COROLLARY 5. *Here all as in Assumption* 2*,* $m_t = 1$*,* $\forall t \in [a,b]$*. Then*

1) $$\int_X D\big(M_n(f)(t,\omega), f(t,\omega)\big)\,dP(\omega) \leqslant 2\Omega_1^{(\mathcal{F})}\big(f, (\tilde{L}_n((\cdot-t)^2)(t))^{1/2}\big)_{L^1}, \ \ \forall t \in [a,b],\ \forall n \in \mathbb{N}, \tag{78}$$

and

2) $$\sup_{t\in[a,b]}\left(\int_X D\big(M_n(f)(t,\omega), f(t,\omega)\big)\,dP(\omega)\right) \leqslant 2\Omega_1^{(\mathcal{F})}\big(f, \|(\tilde{L}_n((\cdot-t)^2))(t)\|_\infty^{1/2}\big)_{L^1}, \quad \forall n \in \mathbb{N}. \tag{79}$$

Proof. By Theorem 14. □

COROLLARY 6. *Here all as in Assumption 2, $m_t = 1$, $\forall t \in [a,b]$. Then*

1) $$\left(\int_X D^2\big(M_n(f)(t,\omega), f(t,\omega)\big)P(d\omega)\right)^{1/2} \leqslant 2\Omega_1^{(\mathcal{F})}\big(f, ((\tilde{L}_n((\cdot-t)^2))(t))^{1/2}\big)_{L^2}, \ \ \forall t \in [a,b],\ \forall n \in \mathbb{N}. \tag{80}$$

and

2) $$\sup_{t\in[a,b]}\left(\int_X D^2\big(M_n(f)(t,\omega), f(t,\omega)\big)P(d\omega)\right)^{1/2} \leqslant 2\Omega_1^{(\mathcal{F})}\big(f, \|(\tilde{L}_n((\cdot-t)^2))(t)\|_\infty^{1/2}\big)_{L^2}, \quad \forall n \in \mathbb{N}. \tag{81}$$

Proof. By Theorem 15, $q = 2$. □

We give now the following fuzzy random Korovkin Theorems for the case of $\tilde{L}(1)(t) = 0$, $\forall t \in [a,b]$.

THEOREM 18. *Here all are as in Assumption 2. Furthermore assume that*

$$\tilde{L}_n(1)(t) = 1, \quad \forall t \in [a,b],\ \ \tilde{L}_n(id) \xrightarrow{u} id,\ \ \tilde{L}_n(id^2) \xrightarrow{u} id^2,\ \ \text{as } n \to \infty.$$

Then

$$\lim_{n\to\infty}\left\|\int_X D\big(M_n(f)(t,\omega), f(t,\omega)\big)\,dP(\omega)\right\|_{\infty,t} = 0,\ \ \forall f \in C^U_{\mathcal{F}R}([a,b]). \tag{82}$$

I.e.

$$M_n(f)(t,\omega) \overset{\text{“1-mean”}}{\longrightarrow} f(t,\omega), \tag{83}$$

uniformly, $\forall f \in C^U_{\mathcal{F}R}([a,b])$, that is uniformly $M_n \overset{\text{“1-mean”}}{\longrightarrow} I$, as $n \to \infty$, on $C^U_{\mathcal{F}R}([a,b])$.

Proof. Using (79) and (58) and Proposition $2(ii)$. □

We continue with

THEOREM 19. *Here all are as in Assumption 2. Furthermore assume that*

$$\tilde{L}_n(1)(t) = 1,\ \ \forall t \in [a,b],\ \ \tilde{L}_n(id) \xrightarrow{u} id,\ \ \tilde{L}_n(id^2) \xrightarrow{u} id^2,\ \ \text{as } n \to \infty.$$

Then

$$\lim_{n\to\infty}\left\|\int_X D^2\big(M_n(f)(t,\omega),f(t,\omega)\big)P(d\omega)\right\|_{\infty,t}=0,\ \ \forall f\in C^U_{\mathcal{F}R}([a,b]). \tag{84}$$

I.e.

$$M_n(f)(t,\omega)\overset{\text{“2-mean”}}{\longrightarrow}f(t,\omega), \tag{85}$$

uniformly, $\forall f \in C^U_{\mathcal{F}R}([a,b])$*, that is uniformly* $M_n \overset{\text{“2-mean”}}{\longrightarrow} I$*, as* $n\to\infty$*, on* $C^U_{\mathcal{F}R}([a,b])$*. Notice here that* $M_n \overset{\text{“2-mean”}}{\longrightarrow} I$ *implies* $M_n \overset{\text{“1-mean”}}{\longrightarrow} I$*, uniformly, i.e. Theorem* 18*.*

Proof. Using (81) and (58) and Proposition $2(ii)$. □

Finally we present

THEOREM 20. *Here all are as in Assumption* 2*,* $\tilde{L}_n(1)(t)=1$*,* $\forall t\in[a,b]$ *and* $q>2$*. We assume further that*

(i) $\lim_{n\to\infty}\|(\tilde{L}_n(|\cdot -t|^q))(t)\|_\infty=0$,
or

$(ii)'$ $\lim_{n\to\infty}\|(\tilde{L}_n(\cdot-t)^{2q})(t)\|_\infty=0.$ (86)

Then

$$\lim_{n\to\infty}\left\|\int_X D^q\big(M_n(f)(t,\omega),f(t,\omega)\big)P(d\omega)\right\|_{\infty,t}=0,\quad \forall f\in C^U_{\mathcal{F}R}([a,b]). \tag{87}$$

I.e.

$$M_n(f)(t,\omega)\overset{\text{“}q\text{-mean”}}{\longrightarrow}f(t,\omega), \tag{88}$$

uniformly, $\forall f \in C^U_{\mathcal{F}R}([a,b])$ *that is uniformly* $M_n \overset{\text{“}q\text{-mean”}}{\longrightarrow} I$*, as* $n\to\infty$*, on* $C^U_{\mathcal{F}R}([a,b])$*.*

Proof. By (71) or (73) and Propositoin $2(ii)$. □

REMARK 6.

1) Notice here from (87), that $M_n \overset{\text{“}q\text{-mean”}}{\longrightarrow} I$ implies $M_n \overset{\text{“}k\text{-mean”}}{\longrightarrow} I$, uniformly for any $1\leqslant k\leqslant q<\infty$ on $C^U_{\mathcal{F}R}([a,b])$.

2) In the case of $m_t=1$, $\forall t\in[a,b]$, all presented results here *did not require* the condition

$$\int_X D^q\big(f(t,\omega),\tilde{o}\big)P(d\omega)<\infty,\quad \forall t\in[a,b],\ \ 1\leqslant q<\infty,$$

as they did the earlier ones for general $m_t\geqslant 0$.

3) One can do related research for other domains other than $[a,b]$, e.g. $[0,+\infty)$, multivariate domains in $\mathbb{R}^k$, $k>1$ and K compact convex subset of a metric space. Of course not all results can pass through there.

5. Application

We consider here the fuzzy random Bernstein polynomials

$$B_n^{(\mathcal{F})}(f)(x,\omega) = {\sum_{k=0}^{n}}^{*} \binom{n}{k} x^k (1-x)^{n-k} \odot f\left(\frac{k}{n},\omega\right),$$

$\forall x \in [0,1]$, $\forall \omega \in X$, $\forall f \in C_{\mathcal{F}R}^{U}([0,1])$, $\forall n \in \mathbb{N}$, see (18). We apply first here (79) for

$$M_n(f)(t,\omega) = B_n^{(\mathcal{F})}(f)(t,\omega), \quad \forall (t,\omega) \in [a,b] \times X,$$

and $\tilde{L}_n = B_n$ the real Bernstein operator

$$B_n(g)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} g\left(\frac{k}{n}\right), \quad \forall g \in C([0,1]), \; \forall x \in [0,1], \; \forall n \in \mathbb{N}.$$

Clearly

$$B_n((\cdot - t)^2)(t) = \frac{t(1-t)}{n}, \quad t \in [0,1].$$

Hence

$$\left\| B_n((\cdot - t)^2)(t) \right\|_{\infty}^{1/2} \leqslant \frac{1}{2\sqrt{n}}, \quad \forall n \in \mathbb{N}.$$

Notice also that $(B_n(1))(t) = 1$, $\forall t \in [0,1]$.

Clearly here $B_n^{(\mathcal{F})}(f)(t,\omega)$ fulfill Assumption 2. Thus by (79) we obtain

$$\begin{aligned} &\sup_{t\in[0,1]} \left(\int_X D\big(B_n^{(\mathcal{F})}(f)(t,\omega), f(t,\omega)\big)\, dP(\omega) \right) \\ &\qquad \leqslant 2\Omega_1^{(\mathcal{F})}\left(f, \frac{1}{2\sqrt{n}}\right)_{L^1}, \quad \forall f \in C_{\mathcal{F}R}^{U}([0,1]), \; \forall n \in \mathbb{N}. \end{aligned} \tag{89}$$

Similarly, from (81) we obtain

$$\begin{aligned} &\sup_{t\in[0,1]} \left(\int_X D^2\big(B_n^{(\mathcal{F})}(f)(t,\omega), f(t,\omega)\big)\, dP(\omega) \right)^{1/2} \\ &\qquad \leqslant 2\Omega_1^{(\mathcal{F})}\left(f, \frac{1}{2\sqrt{n}}\right)_{L^2}, \quad \forall f \in C_{\mathcal{F}R}^{U}([0,1]), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{90}$$

Finally, from (75) for $q > 2$ we obtain

$$\begin{aligned} &\sup_{t\in[0,1]} \left(\int_X D^q\big(B_n^{(\mathcal{F})}(f)(t,\omega), f(t,\omega)\big)\, dP(\omega) \right)^{1/q} \\ &\leqslant 2\min\Big\{ \Omega_1^{(\mathcal{F})}\big(f, \|(B_n(|\cdot - t|^q))(t)\|_{\infty}^{1/q}\big)_{L^q}, \\ &\quad \Omega_1^{(\mathcal{F})}\big(f, \|(B_n(\cdot - t)^{2q})(t)\|_{\infty}^{1/2q}\big)_{L^q}, \Big\}, \; \forall f \in C_{\mathcal{F}R}^{U}([0,1]), \; \forall n \in \mathbb{N}. \end{aligned} \tag{91}$$

In particular, if f is additionally of Lipschitz type, i.e.

$$\int_X D\big(f(x,\omega), f(y,\omega)\big) P(d\omega) \leqslant \theta |x-y|, \quad \theta > 0, \; \forall x,y \in [0,1], \tag{92}$$

then

$$\Omega_1^{(\mathcal{F})}(f,\delta)_{L^1} \leqslant \theta \cdot \delta, \quad \delta > 0, \tag{93}$$

and

$$\Omega_1^{(\mathcal{F})}\left(f, \frac{1}{2\sqrt{n}}\right)_{L^1} \leqslant \frac{\theta}{2\sqrt{n}}, \quad \forall n \in \mathbb{N}. \tag{94}$$

Hence

$$\sup_{t\in[0,1]}\left(\int_X D\big(B_n^{(\mathcal{F})}(f)(t,\omega), f(t,\omega)\big)\,dP(\omega)\right) \leqslant \frac{\theta}{\sqrt{n}}, \quad \forall n \in \mathbb{N}, \tag{95}$$

$\forall f \in C^U_{\mathcal{F}R}([0,1])$ which is of Lipschitz type (92).

Inequality (95) improves the corresponding inequality from (8), since over there we only get

$$\sup_{x\in[0,1]}\left(\int_X D\big(B_n^{(\mathcal{F})}(f)(x,\omega), f(x,\omega)\big)P(d\omega)\right) \leqslant \frac{3\theta}{2\sqrt{n}}, \quad \forall n \in \mathbb{N}. \tag{96}$$

REFERENCES

[1] G. A. ANASTASSIOU, *Moments in Probability and Approximation Theory*, Pitman/Longman, #287, UK, 1993.

[2] G. A. ANASTASSIOU, *Rate of convergence of fuzzy neural network operators, univariate case*, Journal of Fuzzy Mathematics, **10**, (3) (2002), 755–780.

[3] G. A. ANASTASSIOU, *Univariate fuzzy-random neural network approximation operators*, Computers and Mathematics with Applications, **48**, (2004), 1263–1283.

[4] G. A. ANASTASSIOU, *On basic fuzzy Korovkin theory*, Studia Univ."Babes-Bolyai" Mathematica, **L**, (4) Dec.2005, 3–10.

[5] G. A. ANASTASSIOU, *Fuzzy Korovkin theorems and inequalities*, The Journal of Fuzzy Mathematics, **14**, (3) (2006).

[6] G. A. ANASTASSIOU, *On fuzzy global smoothness preservation*, The Journal of Fuzzy Mathematics, **14**, (3) 2006.

[7] G. ANASTASSIOU, S. GAL, *On a fuzzy trigonometric approximation theorem of Weierstrass-type*, Journal of Fuzzy Mathematics, Los Angeles, **9**, (3) (2001), 701–708.

[8] G. ANASTASSIOU, S. GAL, *On fuzzy trigonometric Korovkin theory*, J.Nonlinear Functional Analysis and Appl, (2005).

[9] D. BUTNARIU, *Measurability concepts of fuzzy mappings*, Fuzzy Sets and Systems, **31**, (1989), 77–82.

[10] R. DUDLEY, *Real Analysis and Probability*, Wadsworth & Brooks/Cole Mathematics Series, Pacific Grove, CA, 1989.

[11] S. GAL, *Approximation theory in Fuzzy setting*, Chapter 13, 617–666, in *Handbook of Analytic-Computational Methods in Applied Mathematics*, editor, G.A. Anastassiou, Chapman & Hall/CRC, Boca Raton, 2000.

[12] R. GOETSCHEL JR., W. VOXMAN, *Elementary fuzzy calculus*, Fuzzy Sets and Systems, **18**, (1986), 31–43.

[13] O. KALEVA, *Fuzzy differential equations*, Fuzzy Sets and Systems, **24**, (1987), 301–317.

[14] YUN KYONG KIM, BYUNG MOON GHIL, *Integrals of fuzzy-number-valued functions*, Fuzzy Sets and Systems, **86**, (1997), 213–222.

[15] P. P. KOROVKIN, *Linear Operators and Approximation Theory*, Hindustan Publ. Corp., Delhi, India, 1960.

[16] O. SHISHA, B. MOND, *The degree of convergence of sequences of linear positive operators*, Nat. Acad. of Sci. U.S., **60**, (1968), 1196–1200.

[17] CONGXIN WU, MING MA, *On embedding problem of fuzzy number space: Part 1*, Fuzzy Sets and Systems, **44**, (1991), 33–38.

[18] CONGXIN WU, ZENGTAI GONG, *On Henstock integral of fuzzy-number-valued functions (I)*, Fuzzy Sets and Systems, **120**, (2001), 523–532.

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