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ON A PROBLEM OF UNIVALENCE OF FUNCTIONS SATISFYING A DIFFERENTIAL INEQUALITY

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Abstract. Let $\mathcal{H}_\alpha(\beta)$ denote the class of normalized functions f, analytic in the unit disc E, which satisfy the condition

$$\operatorname{Re}\left[(1-\alpha)f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]>\beta,\ z\in E,$$

where α and β are pre-assigned real numbers. H. S. Al-Amiri and M. O. Reade, in 1975, have shown that for $\alpha \leqslant 0$ and also for $\alpha = 1$, the functions in $\mathcal{H}_\alpha(0)$ are univalent in E. In 2005, V. Singh, S. Singh and S. Gupta proved that for $0 < \alpha < 1$, functions in $\mathcal{H}_\alpha(\alpha)$ are also univalent. In the present note, we establish that functions in $\mathcal{H}_\alpha(\beta)$ are univalent for all real numbers α and β satisfying $\alpha \leqslant \beta < 1$ and that the result is sharp in the sense that the constant β cannot be replaced by any real number less than α.

1. Introduction

Let $\mathcal{A}$ be the class of functions f, analytic in $E=\{z:|z|<1\}$ and normalized by the conditions $f(0)=f'(0)-1=0$. Denote by K, the class of functions f, with $f'(0)\neq 0$, which are convex (univalent) in E i.e. which satisfy

$$\operatorname{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]>0,\ z\in E.$$

A function $f\in\mathcal{A}$ is said to be close-to-convex if there is a real number $\alpha, -\pi/2<\alpha<\pi/2$, and a convex function g (not necessarily normalized) such that

$$\operatorname{Re}\left[e^{i\alpha}\frac{f'(z)}{g'(z)}\right]>0,\ z\in E.$$

It is well-known that every close-to-convex function is univalent. In 1934/35, Noshiro [4] and Warchawski [6] obtained a simple but interesting criterion for univalence of analytic functions. They proved that if an analytic function f satisfies $\operatorname{Re}f'(z)>0$ for all z in E, then f is univalent in E .

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For real numbers α and β, let

$$I(\alpha,f(z)) = (1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)$$

and a class $\mathcal{H}_\alpha(\beta)$ be defined as under:

$$\mathcal{H}_\alpha(\beta) = \{f \in \mathcal{A} : \mathrm{Re}\, I(\alpha,f(z)) > \beta, z \in E\}. \tag{1}$$

In fact, the class $\mathcal{H}_\alpha(0)$ was first studied by Al-Amiri and Reade [2], in 1975. They established that for $\alpha \leqslant 0$, each function f in $\mathcal{H}_\alpha(0)$ satisfies $\mathrm{Re}\, f'(z) > 0$ in E and so, is univalent in E. They were unable to settle the question of univalence for $\alpha > 0$ except for $\alpha = 1$ when, obviously, f is convex. Ahuja and Silverman [1] noticed that the convex function $f(z) = z/(1-z)$ is not in $\mathcal{H}_\alpha(0)$ for any real α, $\alpha \neq 1$. In fact

$$\mathrm{Re}\{I(\alpha,f(z))\} = \mathrm{Re}\left[\frac{1-\alpha}{(1-z)^2} + \alpha\frac{1+z}{1-z}\right] = -\frac{(1-\alpha)\cos\theta}{2(1-\cos\theta)},\ z = e^{i\theta} \neq 1,$$

which is negative for $\theta = \theta_0 = \pi/3$ when $\alpha < 1$ and for $\theta = \theta_0 = 2\pi/3$ when $\alpha > 1$. Thus $\mathcal{H}_1(0) \not\subset \mathcal{H}_\alpha(0), \alpha \neq 1$ and even for convex functions f, $\mathrm{Re} f'(z)$ need not be positive in E.

Recently, this problem was pursued by V. Singh, S. Singh and S. Gupta [5] and they established that for $0 < \alpha < 1$, the class $\mathcal{H}_\alpha(\alpha)$ consists of univalent functions. They also showed that the functions f in $\mathcal{H}_\alpha(1/2)$ satisfy $\mathrm{Re} f'(z) > 1/2$ for all z in E and for all $\alpha \geqslant 0$.

In the present note, we prove that if $f \in \mathcal{H}_\alpha(\beta)$, then $\mathrm{Re} f'(z) > 0$ in E for all real numbers α and β satisfying $\alpha \leqslant \beta < 1$. Further, it will be shown that our result contains the result of Singh, Singh and Gupta [5] and improves the result of Al-Amiri and Reade [2]. We claim that our result is the best possible one in the sense that β cannot be replaced by any real number less than α. We use the following celebrated lemma of Miller [3] to prove our result.

LEMMA 1.1. *Let $\mathbb{D}$ be a subset of $\mathbb{C} \times \mathbb{C}$ ($\mathbb{C}$ is the complex plane) and let $\phi : \mathbb{D} \to \mathbb{C}$ be a complex function. For $u = u_1 + iu_2$, $v = v_1 + iv_2$ (u_1, u_2, v_1, v_2 are reals), let ϕ satisfy the following conditions:*

(i) $\phi(u,v)$ is continuous in $\mathbb{D}$;

(ii) $(1,0) \in \mathbb{D}$ and $\mathrm{Re}\phi(1,0) > 0$; and

(iii) $\mathrm{Re}\,\{\phi(iu_2,v_1)\} \leqslant 0$ for all $(iu_2,v_1) \in \mathbb{D}$ such that $v_1 \leqslant -(1+u_2^2)/2$.

Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be regular in the unit disc E, such that $(p(z), zp'(z)) \in \mathbb{D}$ for all $z \in E$. If

$$\mathrm{Re}[\phi(p(z), zp'(z))] > 0,\ z \in E,$$

then $\mathrm{Re}\, p(z) > 0,\ z \in E$.

2. Main result

THEOREM 2.1. *Let α and β be real numbers such that $\alpha \leqslant \beta < 1$. Assume that an analytic function $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left[(1-\alpha)f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > \beta,\ z \in E. \tag{2}$$

Then $\operatorname{Re} f'(z) > 0$ in E. So, f is close-to-convex and hence univalent in E. The result is sharp in the sense that the constant β on the right hand side of (2) cannot be replaced by a constant smaller than α.

Proof. Let $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ be analytic in E such that for all $z \in E$,

$$f'(z) = p(z) \tag{3}$$

Then,

$$(1-\alpha)f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) = (1-\alpha)p(z)+\alpha\left[1+\frac{zp'(z)}{p(z)}\right].$$

Thus, condition (2) is equivalent to

$$\operatorname{Re}\left[\frac{1-\alpha}{1-\beta}p(z)+\frac{\alpha}{1-\beta}\frac{zp'(z)}{p(z)}+\frac{\alpha-\beta}{1-\beta}\right] > 0,\ z \in E. \tag{4}$$

If $\mathbb{D} = (\mathbb{C}\backslash\{0\}) \times \mathbb{C}$, define $\Phi(u,v) : \mathbb{D} \to \mathbb{C}$ as under:

$$\Phi(u,v) = \frac{1-\alpha}{1-\beta}u + \frac{\alpha}{1-\beta}\frac{v}{u} + \frac{\alpha-\beta}{1-\beta}.$$

Then $\Phi(u,v)$ is continuous in $\mathbb{D}$, $(1,0) \in \mathbb{D}$ and $\operatorname{Re}\Phi(1,0) = 1 > 0$. Further, in view of (4), we get $\operatorname{Re}\Phi(p(z), zp'(z)) > 0,\ z \in E$. Let $u = u_1 + iu_2, v = v_1 + iv_2$, where u_1, u_2, v_1 and v_2 are all reals. Then, for $(iu_2, v_1) \in \mathbb{D}$, with $v_1 \leqslant -\frac{1+u_2^2}{2}$, we have

$$\begin{aligned}\operatorname{Re}\Phi(iu_2, v_1) &= \operatorname{Re}\left[\frac{1-\alpha}{1-\beta}u_2 i + \frac{\alpha}{1-\beta}\frac{v_1}{u_2 i} + \frac{\alpha-\beta}{1-\beta}\right]\\ &= \frac{\alpha-\beta}{1-\beta}\\ &\leqslant 0.\end{aligned}$$

In view of (3) and Lemma 1.1, proof now follows.

To show that the constant β on the right hand side of (2) is the best possible one, we consider the function $f_0(z) = -z - 2\log(1-z)$. It can be easily verified that for all $z \in \partial E$ (boudary of E), except $z = \pm 1$,

$$\operatorname{Re}\left[(1-\alpha)f_0'(z)+\alpha\left(1+\frac{zf_0''(z)}{f_0'(z)}\right)\right] = \alpha. \tag{5}$$

Moreover,

$$\operatorname{Re}\left[(1-\alpha)f_0'(z)+\alpha\left(1+\frac{zf_0''(z)}{f_0'(z)}\right)\right]=1$$

at $z=0$. Since $\alpha<1$, so by minimum principle for harmonic functions, we conclude that $f_0\in H_\alpha(\alpha)$. Further, since $\operatorname{Re} f_0'(z)>0$ in E, so the constant β on the right side of (2) cannot be replaced by a constant smaller than α. This completes the proof of our theorem. □

REMARK 2.1. Taking $\beta=\alpha$, it is obvious that Theorem 2.1 completely contains Theorem 1 proved in [5]. Moreover, it improves the result of Al-Amiri and M. O. Reade [2] as shown by taking the function f_0 in Theorem 2.1 above. For example, writing $\alpha=-1$ in (5), we observe that at $z=i$,

$$\operatorname{Re}\left[2f_0'(z)-\left(1+\frac{zf_0''(z)}{f_0'(z)}\right)\right]=-1.$$

Thus, the function $f_0(z)$ fails to satisfy the condition of univalence laid down by Al-Amiri and Reade in [2], although $\operatorname{Re} f_0'(z)>0$ in E.

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