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ON THE HYERS–ULAM–RASSIAS STABILITY OF A n–DIMENSIONAL QUADRATIC FUNCTIONAL EQUATION

DONG SEUNG KANG AND HAHNG-YUN CHU

(*communicated by Th. M. Rassias*)

Abstract. Let $n \geqslant 2$ be an integer. In this paper, we investigate the generalized Hyers-Ulam-Rassias stability of a n-dimensional quadratic functional equation on Banach spaces and Banach modules over a Banach algebra;

$$(4-n)f(\sum_{j=1}^{n} x_j) + \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j) = 4\sum_{i=1}^{n} f(x_i),$$

where the function θ is defined by $\theta(i,j) = \begin{cases} 1 & \text{if } i \neq j \\ -1 & \text{if } i = j \end{cases}$.

1. Introduction

In 1940, the problem of stability of functional equations was originated by Ulam [17] as follows: Under what condition does there exist an additive mapping near an approximately additive mapping?

The first partial solution to Ulam's question was provided by D. H. Hyers [7]. Let X and Y are Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Hyers showed that if a function $f : X \to Y$ satisfies the following inequality

$$\| f(x+y) - f(x) - f(y) \| \leqslant \epsilon$$

for all $\epsilon \geqslant 0$ and for all $x, y \in X$, then the limit

$$a(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in X$ and $a : X \to Y$ is the unique additive function such that

$$\| f(x) - a(x) \| \leqslant \epsilon$$

for any $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then a is linear.

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Hyers's theorem was generalized in various directions. In particular, Th. M. Rassias [11] considered a generalized version of the theorem of Hyers which permitted the Cauchy difference to become unbounded. He proved the following theorem by using a direct method: if a function $f : X \to Y$ satisfies the following inequality

$$\| f(x+y) - f(x) - f(y) \| \leqslant \theta(||x||^p + ||x||^p)$$

for some $\theta \geqslant 0$, $0 \leqslant p < 1$, and for all $x, y \in X$, then there exists a unique additive function such that

$$\| f(x) - a(x) \| \leqslant \frac{2\theta}{2-2^p} ||x||^p$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then a is linear. Găvruta [6] generalized the Rassias's result above.

The quadratic function $f(x) = cx^2$ $(c \in \mathbb{R})$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

Hence this question is called the quadratic functional equation, and every solution of the quadratic equation (1.1) is called a quadratic function.

A Hyers-Ulam stability theorem for the quadratic functional equation (1.1) was proved by Skof [16] for functions $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an abelian group. In [3], Czerwik proved the Hyers-Ulam-Rassias stability of the quadratic functional equation. Several functional equations have been investigated; see [4-6], [12-15]. Recently, Park [9] proved the Cauchy-Rassias stability of the given functional equation in Banach spaces where the mapping must be even.

In [10], the generalized Hyer-Ulam-Rassias stability problem for generalized A-quadratic mappings, defined in [8], in Banach modules over a Banach $*$-algebra has been solved. Furthermore, Bae and Park [1] have proved the Hyer-Ulam-Rassias stability problem in Banach modules over a Banach C^*-algebra.

In this paper, we are going to introduce another kind of n-dimensional quadratic mapping and to investigate the generalized Hyers-Ulam-Rassias stability of a n-dimensional quadratic functional equation as follows:

$$(4-n)f(\sum_{j=1}^{n} x_j) + \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j) = 4\sum_{i=1}^{n} f(x_i), \tag{1.2}$$

for all $x_1, \cdots, x_n \in X$, where the function θ is defined by

$$\theta(i,j) = \begin{cases} 1 & \text{if } i \neq j \\ -1 & \text{if } i = j \end{cases}.$$

2. A n-dimensional quadratic mapping

Throughout this paper, the function θ is defined by

$$\theta(i,j) = \begin{cases} 1 & \text{if } i \neq j \\ -1 & \text{if } i = j \end{cases}.$$

THEOREM 2.1. *Let X, Y be vector spaces. The given mapping $f : X \to Y$ defined by*

$$(4-n)f(\sum_{j=1}^{n} x_j) + \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j) = 4\sum_{i=1}^{n} f(x_i),$$

for all $x_1, \cdots, x_n \in X$. Then f has the following properties:
(1) $f(0) = 0$
(2) $f(x) = f(-x)$, *for all $x \in X$.*
(3) f *is a quadratic mapping.*

Proof. (1) By letting $x_k = 0\,(k = 1, \cdots, n)$, we have

$$(4-n)f(0) + nf(0) = 4nf(0).$$

Since $n \geqslant 2$, $f(0) = 0$.

(2) Let $x_1 = x$ and $x_k = 0\,(k = 2, \cdots, n)$. Then

$$(4-n)f(x) + f(-x) + (n-1)f(x) = 4f(x).$$

Thus $f(x) = f(-x)$, for all $x \in X$.

(3) Letting $x_1 = x$, $x_2 = y$, and $x_k = 0\,(k = 3, \cdots, n)$, we have

$$(4-n)f(x+y) + f(-x+y) + f(x-y) + (n-2)f(x+y) = 4f(x) + 4f(y) + (n-2)f(0).$$

By (1) and (2), we may conclude that

$$f(x+y) + f(x-y) = 2f(x) + 2f(y).$$

Thus f is quadratic. □

3. Stability of a n-dimensional quadratic mapping with zero terms

Throughout this section, let X be a normed vector space with norm $||\cdot||$ and Y a Banach space with norm $\|\cdot\|$. Let $n \geqslant 2$ be integer.

For the given mapping $f : X \to Y$, we define

$$Df(x_1, \cdots, x_n) := (4-n)f(\sum_{j=1}^{n} x_j) + \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j) - 4\sum_{i=1}^{n} f(x_i), \tag{3.1}$$

for all $x_1, \cdots, x_n \in X$.

THEOREM 3.1. *Let $n \geqslant 2$, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\widetilde{\phi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} 4^{-j}\phi(2^j x_1, \cdots, 2^j x_n) < \infty, \tag{3.2}$$

$$\| Df(x_1, \cdots, x_n) \| \leqslant \phi(x_1, \cdots, x_n), \tag{3.3}$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leqslant \frac{1}{8}\widetilde{\phi}(x, x, 0, \cdots, 0), \tag{3.4}$$

for all $x \in X$.

Proof. Letting $x_1 = x_2 = x$ and $x_k = 0\,(k = 3, \cdots, n)$ in (3.3), we have

$$\| f(x) - \frac{1}{4}f(2x) \| \leqslant \frac{1}{8}\phi(x, x, 0, \cdots, 0), \tag{3.5}$$

for all $x \in X$. Assume

$$\| f(x) - (\frac{1}{4})^r f(2^r x) \| \leqslant \frac{1}{8}\sum_{j=0}^{r-1}(\frac{1}{4})^j \phi(2^j x, 2^j x, 0, \cdots, 0), \tag{3.6}$$

for all $x \in X$. Now, letting $x = 2x$ in the equation (3.6), we have

$$\| f(2x) - (\frac{1}{4})^r f(2^{r+1} x) \| \leqslant \frac{1}{8}\sum_{j=1}^{r}(\frac{1}{4})^{j-1} \phi(2^j x, 2^j x, 0, \cdots, 0), \tag{3.7}$$

for all $x \in X$.

Then (3.5) and (3.7) imply that

$$\begin{aligned} \| f(x) - (\frac{1}{4})^{r+1} f(2^{r+1} x) \| &\leqslant \| f(x) - \frac{1}{4}f(2x) \| + \| \frac{1}{4}f(2x) - (\frac{1}{4})^{r+1} f(2^{r+1} x) \| \\ &\leqslant \frac{1}{8}\sum_{j=0}^{r}(\frac{1}{4})^j \phi(2^j x, 2^j x, 0, \cdots, 0), \end{aligned}$$

for all $x \in X$. Hence

$$\| f(x) - (\frac{1}{4})^{r+1} f(2^{r+1} x) \| \leqslant \frac{1}{8}\sum_{j=0}^{r}(\frac{1}{4})^j \phi(2^j x, 2^j x, 0, \cdots, 0), \tag{3.8}$$

for all $x \in X$ and for all positive integer r.

LEMMA 3.2. *For any positive integer m,*

$$\| (\frac{1}{4})^{m+1} f(2^{m+1} x) - (\frac{1}{4})^m f(2^m x) \| \leqslant (\frac{1}{4})^m \frac{1}{8}\phi(2^m x, 2^m x, 0, \cdots, 0),$$

for all $x \in X$.

Proof. Letting $x = 2^m x$ in (3.5), we get the desired result. □

Now, we will show that the sequence $\{2^{-2m} f(2^m x)\}$ is a Cauchy sequence in a Banach space Y. For all integers $r > m > 0$,

$$\begin{aligned} &\| (\frac{1}{4})^r f(2^r x) - (\frac{1}{4})^m f(2^m x) \| \\ &\quad \leqslant \| (\frac{1}{4})^r f(2^r x) - (\frac{1}{4})^{r-1} f(2^{r-1} x) \| + \cdots + \| (\frac{1}{4})^{m+1} f(2^{m+1} x) - (\frac{1}{4})^m f(2^m x) \| \\ &\quad \leqslant \frac{1}{8}\sum_{j=m}^{r-1}(\frac{1}{4})^j \phi(2^j x, 2^j x, 0, \cdots, 0) \end{aligned}$$

for all $x \in X$. As $r \to \infty$, we may conclude that the sequence $\{2^{-2m}f(2^m x)\}$ is a Cauchy sequence. Hence the sequence $\{2^{-2m}f(2^m x)\}$ converges in Y for all $x \in X$. Thus we may define a mapping $Q : X \to Y$ via

$$Q(x) = \lim_{m\to\infty} 2^{-2m}f(2^m x),$$

for all $x \in X$. By (3.1), (3.2), and (3.3),

$$\begin{aligned}\| DQ(x_1,\cdots,x_n) \| &= \lim_{m\to\infty} (\frac{1}{4})^m \| Df(2^m x_1,\cdots,2^m x_n) \| \\ &\leqslant \lim_{m\to\infty} (\frac{1}{4})^m \phi(2^m x_1,\cdots,2^m x_n) = 0,\end{aligned}$$

for all $x_1,\cdots,x_n \in X$. That is, $DQ(x_1,\cdots,x_n) = 0$. By Theorem 2.1, the mapping $Q : X \to Y$ is quadratic. Also, letting $m = 0$ and passing the limit $r \to \infty$, we get the (3.4).

Note that

$$\begin{aligned} Q(2^j x) &= \lim_{m\to\infty} 2^{-2m}f(2^m(2^j x)) \\ &= 2^{2j} \lim_{m\to\infty} 2^{-2(m+j)}f(2^{m+j}x) \\ &= 2^{2j}Q(x).\end{aligned}$$

Now, let $Q' : X \to Y$ be another n-dimensional quadratic mapping satisfying (3.4). Then by previous note, we have

$$\begin{aligned}\| Q(x) - Q'(x) \| &= 2^{-2j} \| Q(2^j x) - Q'(2^j x) \| \\ &\leqslant 2^{-2j}\Big(\| Q(2^j x) - f(2^j x) \| + \| Q'(2^j x) - f(2^j x) \| \Big) \\ &\leqslant \frac{2\cdot 2^{-2j}}{8}\widetilde{\phi}(2^j x, 2^j x, 0,\cdots,0),\end{aligned}$$

for all $x \in X$. As $j \to \infty$, we may conclude that $Q(x) = Q'(x)$, for all $x \in X$. Thus such a n-dimensional quadratic mapping $Q : X \to Y$ is unique. $\square$

THEOREM 3.3. *Let $n \geqslant 2$, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^n \to [0,\infty)$ such that*

$$\widetilde{\phi}(x_1,\cdots,x_n) := \sum_{j=0}^{\infty} 4^j \phi(2^{-j}x_1,\cdots,2^{-j}x_n) < \infty,$$

$$\| Df(x_1,\cdots,x_n) \| \leqslant \phi(x_1,\cdots,x_n),$$

for all $x_1,\cdots,x_n \in X$. Then there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leqslant \frac{1}{8}\widetilde{\phi}(x,x,0,\cdots,0),$$

for all $x \in X$.

Proof. Similar to the proof of Theorem 3.1, if the x is replaced by $\frac{1}{2}x$ (not $2x$) in the proof of Theorem 3.1, then we have the desired results. □

COROLLARY 3.4. *Let $p \neq 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\| Df(x_1, \cdots, x_n) \| \leqslant \theta \sum_{i=1}^{n} ||x_i||^p ,$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leqslant \frac{\theta}{|4 - 2^p|} ||x||^p ,$$

for all $x \in X$.

Proof. Let

$$\phi(x_1, \cdots, x_n) = \theta \sum_{i=1}^{n} ||x_i||^p ,$$

and then apply to Theorem 3.1 when $p < 2$, or apply to Theorem 3.3 when $p > 2$. □

THEOREM 3.5. *Let $n \geqslant 2$, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\widetilde{\phi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} 9^{-j} \phi(3^j x_1, \cdots, 3^j x_n) < \infty , \tag{3.9}$$

$$\| Df(x_1, \cdots, x_n) \| \leqslant \phi(x_1, \cdots, x_n) , \tag{3.10}$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leqslant \frac{1}{9} \widetilde{\phi}(x, x, x, 0, \cdots, 0) , \tag{3.11}$$

for all $x \in X$.

Proof. Letting $x_1 = x_2 = x_3 = x$ and $x_k = 0\,(k = 4, \cdots, n)$ in (3.10), we have

$$\| f(x) - \frac{1}{9} f(3x) \| \leqslant \frac{1}{9} \phi(x, x, x, 0, \cdots, 0) , \tag{3.12}$$

for all $x \in X$.

Replacing x by $3x$, inductively, we have the following equation,

$$\| f(x) - (\frac{1}{9})^r f(3^r x) \| \leqslant \frac{1}{9} \sum_{j=0}^{r-1} (\frac{1}{9})^j \phi(3^j x, 3^j x, 3^j x, 0, \cdots, 0) , \tag{3.13}$$

for all $x \in X$ and all positive integer r. For the remaining proof see Theorem 3.1, since the equation (3.13) is similar to the equation (3.8). □

THEOREM 3.6. *Let $n \geqslant 2$, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\widetilde{\phi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} 9^j \phi(3^{-j}x_1, \cdots, 3^{-j}x_n) < \infty,$$

$$\| Df(x_1, \cdots, x_n) \| \leqslant \phi(x_1, \cdots, x_n),$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leqslant \frac{1}{9}\widetilde{\phi}(x, x, x, 0, \cdots, 0),$$

for all $x \in X$.

Proof. Similar to the proof of Theorem 3.1, if the x is replaced by $\frac{1}{3}x$ (not $3x$) in the proof of Theorem 3.1, then we have the desired results. □

COROLLARY 3.7. *Let $p \neq 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\| Df(x_1, \cdots, x_n) \| \leqslant \theta \sum_{i=1}^{n} ||x_i||^p,$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leqslant \frac{\theta}{|3 - 3^{p-1}|} ||x||^p,$$

for all $x \in X$.

Proof. Let

$$\phi(x_1, \cdots, x_n) = \theta \sum_{i=1}^{n} ||x_i||^p,$$

and then apply to Theroem 3.5 when $p < 2$, or apply to Theroem 3.6 when $p > 2$. □

THEOREM 3.8. *Let $n \geqslant 2$, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\widetilde{\phi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} 4^{-j} \phi(2^j x_1, \cdots, 2^j x_n) < \infty, \tag{3.14}$$

$$\| Df(x_1, \cdots, x_n) \| \leqslant \phi(x_1, \cdots, x_n), \tag{3.15}$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leqslant \frac{1}{16}\widetilde{\phi}(x, x, x, x, 0, \cdots, 0), \tag{3.16}$$

for all $x \in X$.

Proof. Letting $x_1 = x_2 = x_3 = x_4 = x$ and $x_k = 0\,(k = 5, \cdots, n)$ in (3.15), we have

$$\| f(x) - \frac{1}{4} f(2x) \| \leqslant \frac{1}{16} \phi(x, x, x, x, 0, \cdots, 0), \tag{3.17}$$

for all $x \in X$. Similarly, the remains follow from the proof of Theorem 3.1. □

THEOREM 3.9. *Let $n \geqslant 2$, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\widetilde{\phi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} 4^j \phi(2^{-j} x_1, \cdots, 2^{-j} x_n) < \infty,$$
$$\| Df(x_1, \cdots, x_n) \| \leqslant \phi(x_1, \cdots, x_n),$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leqslant \frac{1}{16} \widetilde{\phi}(x, x, x, x, 0, \cdots, 0),$$

for all $x \in X$.

Proof. Similar to the proof of Theorem 3.1, if the x is replaced by $\frac{1}{2}x$ (not $2x$) in the proof of Theorem 3.1, then we have the desired results. □

COROLLARY 3.10. *Let $p \neq 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\| Df(x_1, \cdots, x_n) \| \leqslant \theta \sum_{i=1}^{n} ||x_i||^p,$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leqslant \frac{\theta}{|4 - 2^p|} ||x||^p,$$

for all $x \in X$.

Proof. Let

$$\phi(x_1, \cdots, x_n) = \theta \sum_{i=1}^{n} ||x_i||^p,$$

and then apply to Theorem 3.8 when $p < 2$, or apply to Theorem 3.9 when $p > 2$. □

4. Stability of a n-dimensional quadratic mapping without zero terms

THEOREM 4.1. *Let $n \geqslant 2$ be even and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\widetilde{\phi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} 4^{-j} \phi(2^j x_1, \cdots, 2^j x_n) < \infty, \tag{4.1}$$

$$\| Df(x_1, \cdots, x_n) \| \leqslant \phi(x_1, \cdots, x_n), \tag{4.2}$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leqslant \frac{1}{4n} \widetilde{\phi}(x, -x, x, -x, \cdots, x, -x), \tag{4.3}$$

for all $x \in X$.

Proof. For each $k = 1, \cdots, n$, $x_k = (-1)^{k-1}x$ in (4.2), we have

$$\| nf(2x) - 4nf(x) \| \leqslant \phi(x, -x, x, -x, \cdots, x, -x),$$

for all $x \in X$. Then we write

$$\| f(x) - \frac{1}{4} f(2x) \| \leqslant \frac{1}{4n} \phi(x, -x, x, -x, \cdots, x, -x), \tag{4.4}$$

for all $x \in X$. If x is replaced by $2x$ in the equation(4.4), inductively, we have the following form

$$\| f(x) - (\frac{1}{4})^r f(2^r x) \| \leqslant \frac{1}{4n} \sum_{j=0}^{r-1} (\frac{1}{4})^j \phi(2^j x, -2^j x, 2^j x, -2^j x, \cdots, 2^j x, -2^j x), \tag{4.5}$$

for all $x \in X$ and all positive integer r. Given the equation (4.5) is similar to the equation (3.8), see proof of Theorem 3.1. □

THEOREM 4.2. *Let $n \geqslant 2$ be even and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\widetilde{\phi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} 4^j \phi(2^{-j} x_1, \cdots, 2^{-j} x_n) < \infty, \tag{4.6}$$

$$\| Df(x_1, \cdots, x_n) \| \leqslant \phi(x_1, \cdots, x_n), \tag{4.7}$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leqslant \frac{1}{4n} \widetilde{\phi}(x, -x, x, -x, \cdots, x, -x), \tag{4.8}$$

for all $x \in X$.

Proof. Similar to the proof of Theorem 4.1, if the x is replaced by $\frac{1}{2}x$ (not $2x$) in the proof of Theorem 4.1, then we have the desired results. □

COROLLARY 4.3. *Let $n \geqslant 2$ be even, let $p \neq 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and*

$$\| Df(x_1, \cdots, x_n) \| \leqslant \theta \sum_{i=1}^{n} ||x_i||^p ,$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leqslant \frac{\theta}{|4 - 2^p|} ||x||^p ,$$

for all $x \in X$.

Proof. Let

$$\phi(x_1, \cdots, x_n) = \theta \sum_{i=1}^{n} ||x_i||^p ,$$

and then apply to Theorem 4.1 when $p < 2$, or apply to Theorem 4.2 when $p > 2$. □

THEOREM 4.4. *Let $n \geqslant 2$ be odd and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\widetilde{\phi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} 3^{-j} \phi(3^j x_1, \cdots, 3^j x_n) < \infty , \tag{4.9}$$

$$\| Df(x_1, \cdots, x_n) \| \leqslant \phi(x_1, \cdots, x_n) , \tag{4.10}$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique n-dimensional quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leqslant \frac{2}{9(n-1)} \widetilde{\phi}(x, -x, x, -x, \cdots, x) , \tag{4.11}$$

for all $x \in X$.

Proof. For each $k = 1, \cdots, n$, $x_k = (-1)^{k-1} x$ in (4.10), we have

$$\| f(x) - \frac{1}{9} f(3x) \| \leqslant \frac{2}{9(n-1)} \phi(x, -x, x, \cdots, -x, x) , \tag{4.12}$$

for all $x \in X$. replacing x by $3x$ in (4.12), inductively, we have the following form

$$\| f(x) - (\frac{1}{9})^r f(3^r x) \| \leqslant \frac{2}{9(n-1)} \sum_{j=0}^{r-1} (\frac{1}{9})^j \phi(3^j x, -3^j x, 3^j x, \cdots, -3^j x, 3^j x) , \tag{4.13}$$

for all $x \in X$. The remains are similar to the proof of Theorem 4.1. □

THEOREM 4.5. *Let $n \geqslant 2$ be odd and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^n \to [0, \infty)$ such that*

$$\widetilde{\phi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} 3^j \phi(3^{-j} x_1, \cdots, 3^{-j} x_n) < \infty , \tag{4.14}$$

$$\| Df(x_1,\cdots,x_n) \| \leqslant \phi(x_1,\cdots,x_n)\,, \tag{4.15}$$

for all $x_1,\cdots,x_n \in X$. *Then there exists a unique* n*-dimensional quadratic mapping* $Q : X \to Y$ *such that*

$$\| f(x) - Q(x) \| \leqslant \frac{2}{9n-1}\widetilde{\phi}(x,-x,x,-x,\cdots,x)\,, \tag{4.16}$$

for all $x \in X$.

Proof. Similar to the proof of Theorem 4.4, if the x is replaced by $\frac{1}{2}x$ (not $2x$) in the proof of Theorem 4.4, then we have the desired results. □

COROLLARY 4.6. *Let* $n \geqslant 2$ *be odd, let* $p \neq 2$ *and* θ *be positive real numbers, and let* $f : X \to Y$ *be an even mapping satisfying* $f(0) = 0$ *and*

$$\| Df(x_1,\cdots,x_n) \| \leqslant \theta \sum_{i=1}^{n} ||x_i||^p\,,$$

for all $x_1,\cdots,x_n \in X$. *Then there exists a unique* n*-dimensional quadratic mapping* $Q : X \to Y$ *such that*

$$\| f(x) - Q(x) \| \leqslant \frac{2n\theta}{n-1}\frac{1}{|9-3^p|}||x||^p\,,$$

for all $x \in X$.

Proof. Let

$$\phi(x_1,\cdots,x_n) = \theta \sum_{i=1}^{n} ||x_i||^p\,,$$

and then apply to Theorem 4.4 when $p < 2$, or apply to Theorem 4.5 when $p > 2$. □

5. Results in Banach modules over a Banach algebra

Throughout this section, let B be a unital Banach $*$-algebra with norm $|\ |$ and $B_1 = \{a \in B \mid |a| = 1\}$, let ${}_B\mathbb{B}_1$ and ${}_B\mathbb{B}_2$ be left Banach modules with norms $\|\ \|$ and $\|\ \|$, respectively, and let

$$\varphi : [{}_B\mathbb{B}_1 \setminus \{0\}]^n \to \mathbb{R}$$

be the function such that

$$\widetilde{\varphi}(x_1,\cdots,x_n) := \sum_{j=0}^{\infty} 4^{-j}\varphi(2^jx_1,\cdots,2^jx_n) < \infty\,, \tag{5.1}$$

for all $x_1,\cdots,x_n \in_B \mathbb{B}_1\setminus\{0\}$.

DEFINITION 5.1. An n-dimensional quadratic mapping

$$Q :_B \mathbb{B}_1 \to_B \mathbb{B}_2$$

is called *n-dimensional $B-$quadratic* if $Q(ax) = a^2Q(x)$ for all $a \in B$ and all $x \in_B \mathbb{B}_1$.

DEFINITION 5.2. For $a \in B$, let $b = aa^*, a^*a,$ or $(aa^* + a^*a)/2$. An n-dimensional quadratic mapping $Q :_B \mathbb{B}_1 \to_B \mathbb{B}_2$ is called *n-dimensional $B_{sa}-$quadratic* if $Q(ax) = bQ(x)$, for all $a \in B,$ and all $x \in_B \mathbb{B}_1$.

Since Banach spaces ${}_B\mathbb{B}_1$ and ${}_B\mathbb{B}_2$ are considered as Banach modules over $B := \mathbb{C}$, the $B_{sa}-$quadratic mapping $Q :_B \mathbb{B}_1 \to_B \mathbb{B}_2$ implies $Q(ax) = |a|^2Q(x)$, for all $a \in \mathbb{C}$.

We define the *approximate remainder D_af* for a mapping $f :_B \mathbb{B}_1 \to_B \mathbb{B}_2$,

$$D_af(x_1, \cdots, x_n) := (4-n)f(\sum_{j=1}^{n} ax_j) + \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)ax_j) - 4b\sum_{i=1}^{n} f(x_i),$$

for all $x_1, \cdots, x_n \in_B \mathbb{B}_1$.

THEOREM 5.1. *Let $f :_B \mathbb{B}_1 \to_B \mathbb{B}_2$ be a mapping with $f(0) = 0$ for the case* (3.3) *which there is a mapping $\varphi :_B \mathbb{B}_1 \to \mathbb{R}$ satisfying*

$$\| D_af(x_1, \cdots, x_n) \| \leqslant \varphi(x_1, \cdots, x_n), \tag{5.2}$$

for all $a \in B_1$, $x_1, \cdots, x_n \in_B \mathbb{B}_1 \backslash \{0\}$. If either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$, for each fixed $x \in_B \mathbb{B}_1$, then there exists an unique $n-$dimensional $B_{sa}-$quadratic mapping $Q :_B \mathbb{B}_1 \to_B \mathbb{B}_2$ defined by

$$Q(x) = \lim_{m\to\infty} 2^{-2m} f(2^m x),$$

which satisfies the inequality (3.4) *for all $x \in_B \mathbb{B}_1$.*

Proof. By the same reasoning as the proof of Theorem 3.1, it follows from the inequality of the statement $a = 1$ that there exists a unique $n-$dimensional quadratic mapping $Q :_B \mathbb{B}_1 \to_B \mathbb{B}_2$ defined by

$$Q(x) = \lim_{m\to\infty} 2^{-2m} f(2^m x),$$

which satisfies the inequality (3.4) for all $x \in_B \mathbb{B}_1$. Under the assumptions that either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$, for each fixed $x \in_B \mathbb{B}_1$, by the same reasoning as the proof of [11], one can show that Q is $\mathbb{R}$-quadratic, that is , $Q(tx) = t^2Q(x)$ for all $t \in \mathbb{R}$, for all $x \in_B \mathbb{B}_1$.

Putting $x_1 = x_2 = x$ and $x_j = 0\,(j = 3, \cdots, n)$ in (5.2) and dividing the resulting inequality by 2^{2m},

$$\frac{1}{2^{2m}} \|2f(a2^{2m}x) + 2f(0) - 4b(n-2)f(0) - 8bf(2^{m-1}x)\| \leqslant \frac{1}{2^{2m}} \phi(2^{m-1}x, 2^{m-1}x, 0, \cdots, 0),$$

for all $x_1, \cdots, x_n \in_B \mathbb{B}_1$. By the definition of Q,

$$Q(ax) = \lim_{m\to\infty} \frac{1}{2^{2m}} f(2^m ax) = \lim_{m\to\infty} b\frac{1}{2^{2m-2}} f(2^{m-1}x) = bQ(x).$$

for every $x \in_B \mathbb{B}_1$, for every $a \in B(|a| = 1)$. For $a \in B \setminus \{0\}$,

$$Q(ax) = Q(|a|\frac{a}{|a|}x) = |a|^2 Q(\frac{a}{|a|}x) = |a|^2 \frac{b}{|a|^2} Q(x) = bQ(x),$$

for all $x \in_B \mathbb{B}_1$. Thus Q is $n-$ dimensional $B_{sa}-$ quadratic, which completes the proof. □

COROLLARY 5.2. *Let $f :_B \mathbb{B}_1 \to_B \mathbb{B}_2$ be a mapping with $f(0) = 0$ for the case* (3.3) *which there exists mapping $\varphi :_B \mathbb{B}_1 \to \mathbb{R}$ satisfying*

$$\| b(4-n)f(\sum_{j=1}^{n} x_j) + b\sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j) - 4\sum_{i=1}^{n} f(ax_i) \| \leqslant \varphi(x_1, \cdots, x_n),$$

for all $a \in B_1$, for all $x_1, \cdots, x_n \in_B \mathbb{B}_1 \setminus \{0\}$. If either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$, for each fixed $x \in_B \mathbb{B}_1$, then there is an unique $n-$ dimensional $B_{sa}-$ quadratic mapping $Q :_B \mathbb{B}_1 \to_B \mathbb{B}_2$ which satisfies the inequality (3.4) *for all $x \in_B \mathbb{B}_1$.*

Proof. By the similar method of the proof of Theorem 5.1, one can obtain the result. □

An $n-$ dimensional quadratic mapping $Q : \mathbb{B} \to B$ is called an *$n-$ dimensional A-quadratic* mapping if $Q(ax) = aQ(x)a^*$ for all $a \in B, x \in \mathbb{B}$.

THEOREM 5.3. *Let $f :_B \mathbb{B}_1 \to_B \mathbb{B}_2$ be a mapping with $f(0) = 0$ for the case* (3.3) *which there is mapping $\psi :_B \mathbb{B}_1 \to \mathbb{R}$ satisfying*

$$\| Q(ax) - aQ(x)a^* \| \leqslant \psi(x) \text{ and } \lim_{m\to\infty} \frac{\psi(2^{2m}x)}{2^{2m}} = 0 \tag{5.3}$$

for all $a \in B_1$, $x \in_B \mathbb{B}_1$. If either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$, for each fixed $x \in_B \mathbb{B}_1$, then there exists an unique $n-$ dimensional $A-$ quadratic mapping $Q :_B \mathbb{B}_1 \to_B \mathbb{B}_2$ defined by

$$Q(x) = \lim_{m\to\infty} 2^{-2m} f(2^m x),$$

which satisfies the inequality (3.4) *for all $x \in_B \mathbb{B}_1$.*

Proof. By the same reasoning as the proof of Theorem 3.1, there exists a unique $n-$ dimensional $\mathbb{R}-$ quadratic mapping $Q :_B \mathbb{B}_1 \to_B \mathbb{B}_2$ defined by

$$Q(x) = \lim_{m\to\infty} 2^{-2m} f(2^m x),$$

which satisfies the inequality (3.4) for all $x \in_B \mathbb{B}_1$. By (5.3), for each element $a \in B_1$, $x \in_B \mathbb{B}_1$,

$$Q(ax) = aQ(x)a^*.$$

Since Q is $n-$ dimensional $\mathbb{R}$-quadratic,

$$Q(ax) = Q(|a|\frac{a}{|a|}x) = |a|^2 Q(\frac{a}{|a|}x) = |a|^2 \frac{a}{|a|} Q(x) \frac{a^*}{|a|} = aQ(x)a^*,$$

for all $a \in B(|a| \neq 0), x \in_B \mathbb{B}_1$. Thus Q is $n-$ dimensional $A-$ quadratic, as desired. □

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Dong Seung Kang
Mathematical Education
Dankook University
147, Hannam, Yongsan
Seoul, 140-714
KOREA
e-mail: dskang@dankook.ac.kr

Corresponding author Hahng-Yun Chu
School of Mathematics
Korea Institute for Advanced Study
207-43, Cheongnyangni 2
Dongdaemun, Seoul, 130-722
KOREA
e-mail: hychu@kias.re.kr