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DOUBLE SUMMABILITY FACTOR THEOREMS AND APPLICATIONS

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(*communicated by L. Leindler*)

Abstract. We obtain sufficient conditions for the series $\sum\sum a_{mn}$, which is absolutely summable of order k by a double triangular matrix method A, to be such that $\sum\sum a_{mn}\lambda_{mn}$ is absolutely summable of order k by a double triangular matrix B. As corollaries we obtain a number of inclusion theorems.

Let $\sum\sum a_{mn}$ be a doubly infinite series with partial sums s_{mn}. Denote by A the doubly infinite matrix with entries $a_{mnij}, 0 \leqslant i \leqslant m, 0 \leqslant j \leqslant n$. We define the mn-th term of the A-transform of a sequence $\{s_{mn}\}$ by

$$A_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{mnij}s_{ij}. \tag{1}$$

For any double sequence $\{u_{mn}\}$ we define

$$\Delta_{11}u_{mn} = u_{mn} - u_{m+1,n} - u_{m,n+1} + u_{m+1,n+1}.$$

For any four-fold sequence $\{a_{mnij}\}$ we define

$$\Delta_{11}a_{mnij} = a_{mnij} - a_{m+1,n,i,j} - a_{m,n+1,i,j} + a_{m+1,n+1,i,j},$$
$$\Delta_{ij}a_{mnij} = a_{mnij} - a_{m,n,i+1,j} - a_{m,n,i,j+1} + a_{m,n,i+1,j+1},$$
$$\Delta_{0j}a_{mnij} = a_{mnij} - a_{m,n,i,j+1},$$

and

$$\Delta_{i0}a_{mnij} = a_{mnij} - a_{m,n,i+1,j}.$$

For any arbitrary double lower triangular matrix A, we shall say that the series $\sum\sum a_{mn}$ is absolutely A-summable of order $k \geqslant 1$ if

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|\Delta_{11}A_{m-1,n-1}|^k < \infty. \tag{2}$$

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Associated with A are two matrices $\bar{A}$ and $\hat{A}$ defined by

$$\bar{a}_{mnij} = \sum_{\mu=i}^{m} \sum_{\nu=j}^{n} a_{mn\mu\nu}, \qquad 0 \leqslant i \leqslant m, 0 \leqslant j, \leqslant n, m, n = 0, 1, \dots,$$

and

$$\hat{a}_{m-1,n-1,i,j} = \Delta_{11} \bar{a}_{m-1,n-1,i,j}, \qquad m, n = 1, 2, \dots, 0 \leqslant i \leqslant m, 0 \leqslant j \leqslant n.$$

It is easily verified that $\hat{a}_{00} = \bar{a}_{00} = a_{00}$. Using this notation,

$$Y_{mn} := \Delta_{11} A_{m-1,n-1} = \sum_{i=0}^{m} \sum_{j=0}^{n} \hat{a}_{m-1,n-1,i,j} a_{ij} \lambda_{ij}. \tag{3}$$

Let B be a doubly infinite matrix with entries $b_{mnij}, 0 \leqslant i \leqslant m, 0 \leqslant j \leqslant n$. B is called factorable if there exist two lower triangular matrices C and D such that $B = C \circ D$; i.e., $b_{mnij} = c_{mi} d_{nj}$.

If B is factorable and has an inverse, then the inverse of B, written B' has entries $b'_{mnij} = c'_{mi} d'_{nj}$, where c'_{mi} and d'_{nj} are the entries of the inverses of C and D, respectively.

The purpose of this paper is to establish a summability factor theorem for a pair of double triangles. We obtain as corollaries some inclusion theorems for special cases of double triangles.

Theorem 1 of this paper represents the first time that two arbitrary double triangles have been used in a summability factor theorem. Theorem 1 also represents one of the most general such summability factor theorems that one can expect to obtain. By setting each $\lambda_{mn} = 1$ we obtain a number of inclusion theorems.

THEOREM 1. *Let $\{\lambda_{mn}\}$ be a doubly sequence of constants and let A and B be doubly infinite triangles, B factorable, satisfying*

(*i*) $|b_{mnmn}| = O(|a_{mnmn}|)$,

(*ii*) $(b_{ijij} - b_{i+1,j,i,j}) = O(b_{i0i0} b_{i+1,j,i+1,j})$,
$(b_{ijij} - b_{i.,j+1,i,j}) = O(b_{0j0j} b_{i,j+1,i,j+1})$,

(*iii*) $\displaystyle\sum_{i=0}^{m} \sum_{j=0}^{n} |\Delta_{ij}(\hat{a}_{m-1,n-1,i,j} \lambda_{ij})| = O(|a_{mnmn}|)$,

(*iv*) $\displaystyle\sum_{m=i+1}^{\infty} \sum_{n=j+1}^{\infty} (mn|a_{mnmn}|)^{k-1} |\Delta_{ij}(\hat{a}_{m-1,n-1,i,j} \lambda_{ij})| = O((ij)^{k-1} |a_{ijij}|^{k})$,

(*v*) $\displaystyle\sum_{i=0}^{m} \sum_{j=0}^{n} |b_{i0i0}| |\Delta_{0j}(\hat{a}_{m-1,n-1,i+1,j} \lambda_{i+1,j})| = O(|a_{mnmn}|)$,

(*vi*) $\displaystyle\sum_{m=i+1}^{\infty} \sum_{n=j+1}^{\infty} (mn|a_{mnmn}|)^{k-1} |\Delta_{0j}(\hat{a}_{m-1,n-1,i+1,j} \lambda_{i+1,j})| = O((ij|a_{ijij}|)^{k-1} |b_{0j0j}|)$,

(*vii*) $\displaystyle\sum_{i=0}^{m} \sum_{j=0}^{n} |b_{0j0j}| |\Delta_{i0}(\hat{a}_{m-1,n-1,i,j+1} \lambda_{i.j+1}| = O(|a_{mnmn}|)$,

(*viii*) $\displaystyle\sum_{m=i+1}^{\infty} \sum_{n=j+1}^{\infty} (mn|a_{mnmn}|)^{k-1} |\Delta_{i0}(\hat{a}_{m-1,n-1,k,j+1} \lambda_{i,j+1})| = O((ij|a_{ijij}|)^{k-1} |b_{i0i0}|)$,

(ix) $\sum_{i=0}^{m}\sum_{j=0}^{n}|a_{ijij}||\hat{a}_{m-1,n-1,i+1,j+1}\lambda_{i+1,j+1}| = O(|a_{mnmn}|)$,

(x) $\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\hat{a}_{m-1,n-1,i+1,j+1}\lambda_{i+1,j+1}| = O((ij|a_{ijij}|)^{k-1})$,

(xi) $\sum_{i=0}^{m}\sum_{j=0}^{n}|b_{i0i0}||\Delta_{0j}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})| = O(|a_{mnmn}|)$,

(xii) $\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\Delta_{0j}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})| = O((ij)|a_{ijij}|)^{k-1}|b_{ojoj}|)$,

(xiii) $\sum_{i=0}^{m}\sum_{j=0}^{n}|b_{0j0j}||\Delta_{i0}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})| = O(|a_{mnmn}|)$,

(xiv) $\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\Delta_{i0}(\hat{a}_{m-1.n-1,i,j}\lambda_{ij})| = O((ij|a_{ijij}|)^{k-1}|b_{i0i0}|)$,

(xv) $\sum_{i=0}^{m}\sum_{j=0}^{n}|a_{ijij}||(\hat{a}_{m-1,n-1,i,j+1}\lambda_{i,j+1})| = O(|a_{mnmn}|)$,

(xvi) $\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\hat{a}_{m-1,n-1,i,j+1}\lambda_{i,j+1}| = O((ij|a_{ijij}|)^{k-1}|b_{0j0j}|^{-k})$,

(xvii) $\sum_{i=0}^{m}\sum_{j=0}^{n}|a_{ijij}||\hat{a}_{m-1,n-1,i+1,j}\lambda_{i+1,j}| = O(|a_{mnmn}|)$,

(xviii) $\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\hat{a}_{m-1,n-1,i+1,j}\lambda_{i+1,j}| = O((ij|a_{ijij}|)^{k-1}|b_{ioio}|^{-k})$,

(xix) $\sum_{i=0}^{m}\sum_{j=0}^{n}|a_{ijij}||\hat{a}_{m-1,n-1,i,j}\lambda_{ij}| = O(|a_{mnmn}|)$,

(xx) $\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\hat{a}_{m-1,n-1,i,j}\lambda_{ij}| = O(ij|a_{ijij}|)^{k-1})$,

(xxi) $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(ij)^{k-1}\left|\sum_{r=0}^{i-2}\hat{b}_{i-1,j-1,r,j}X_{rj}\right|^{k} = O(1)$,

(xxii) $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(ij)^{k-1}\left|\sum_{s=0}^{j-1}\hat{b}_{i-1,j-1,i,s}X_{is}\right|^{k} = O(1)$,

(xxiii) $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(ij)^{k-1}\left|\sum_{r=0}^{i-1}\sum_{s=0}^{j-2}\hat{b}_{i-1.j-1,r,s}X_{rs}\right|^{k} = O(1)$.

Then $\sum\sum a_{ij}$ *summable* $|A|_k$ *implies that* $\sum\sum a_{ij}\lambda_{ij}$ *is summable* $|B|_k$, *where* X_{mn} *is as defined in* (5).

Proof. Let t_{mn} denote the mn-th term of the B-transform of a sequence $\{s_{mn}\}$.

Then

$$t_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n} b_{mnij} s_{ij}, \tag{4}$$

and

$$X_{mn} := \Delta_{11} t_{m-1,n-1} = \sum_{i=0}^{m}\sum_{j=0}^{n} \hat{b}_{m-1,n-1,i,j} a_{ij}. \tag{5}$$

Since $\hat{B}'$ is a double triangle, we may solve (5) for a_{mn} to get

$$a_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n} \hat{b}'_{m-1,n-1,i,j} X_{ij} \tag{6}$$

Substituting (6) into (3) gives

$$\begin{aligned}
Y_{mn} &= \sum_{i=0}^{m}\sum_{j=0}^{n} \hat{a}_{m-1,n-1,i,j}\lambda_{ij} \sum_{r=0}^{i}\sum_{s=0}^{j} \hat{b}'_{i-1,j-1,r,s} X_{rs} \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} \hat{a}_{m-1,n-1,i,j}\lambda_{ij} \Bigg(\hat{b}'_{i-1,j-1,i,j} X_{ij} + \sum_{r=0}^{i-1} \hat{b}'_{i-1,j-1,r,j} X_{rj} \\
&\quad + \sum_{s=0}^{j-1} \hat{b}'_{i-1,j-1,i,s} X_{is} + \sum_{r=0}^{i-1}\sum_{s=0}^{j-2} \hat{b}'_{i-1,j-1,r,s} X_{rs} \Bigg) \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} \hat{a}_{m-1,n-1,i,j}\lambda_{ij} \Big[\hat{b}'_{i-1,j-1,i,j} X_{ij} + \hat{b}'_{i-1,j-1,i-1,j} X_{i-1,j} + \hat{b}'_{i-1,j-1,i,j-1} X_{i.j-1} \\
&\quad + \hat{b}'_{i-1,j-1,i-1,j-1} X_{i-1,j-1} + \sum_{r=0}^{i-2} \hat{b}'_{i-1,j-1,r,j} X_{rj} + \sum_{s=0}^{j-2} \hat{b}'_{i-1,j-1,i,s} X_{is} \\
&\quad + \sum_{r=0}^{i-2}\sum_{s=0}^{j-2} \hat{b}'_{i-1,j-1,r,s} X_{r,s} + \sum_{r=0}^{i-2} \hat{b}'_{i-1,j-1,r,j-1} X_{r,j-1} + \sum_{s=0}^{j-2} \hat{b}'_{i-1,j-1,i-1,s} X_{i-1,s} \Big]
\end{aligned}$$

Using the substitutions $\mu = i-1$ in the second and nineth sums, $\nu = j-1$ in the third and eighth sums, and both substitutions in the fourth sum, we have

$$\begin{aligned}
Y_{mn} = &\sum_{i=0}^{m}\sum_{j=0}^{n} \hat{a}_{m-1,n-1,i,j}\lambda_{ij} \hat{b}'_{i-1,j-1,i,j} X_{ij} \\
&+ \sum_{\mu=-1}^{m-1}\sum_{j=0}^{n} \hat{a}_{m-1,n-1,\mu+1,j}\lambda_{\mu+1,j} \hat{b}'_{\mu,j-1,\mu,j} X_{\mu,j} \\
&+ \sum_{i=0}^{m}\sum_{\nu=-1}^{n-1} \hat{a}_{m-1,n-1,i,\nu+1}\lambda_{i,\nu+1} \hat{b}'_{i-1,\nu,i,\nu} X_{i\nu}
\end{aligned}$$

$$+\sum_{\mu=-1}^{m-1}\sum_{\nu=-1}^{n-1}\hat{a}_{m-1,n-1,\mu+1,\nu+1}\lambda_{\mu+1,\nu+1}\hat{b}_{\mu,\nu,\mu,\nu}X_{\mu,\nu}$$

$$+\sum_{i=0}^{m}\sum_{j=0}^{n}\hat{a}_{m-1,n-1,i,j}\lambda_{ij}\left(\sum_{r=0}^{i-2}\hat{b}'_{i-1,j-1,r,j}X_{rj}+\sum_{s=0}^{j-2}\hat{b}'_{i-1,j-1,i,s}X_{is}+\sum_{r=0}^{i-2}\sum_{s=0}^{j-2}\hat{b}'_{i-1,j-1,r,s}X_{rs}\right)$$

$$+\sum_{i=0}^{m}\sum_{\nu=-1}^{n-1}\hat{a}_{m-1,n-1,i,\nu+1}\lambda_{i,\nu+1}\sum_{r=0}^{i-2}\hat{b}'_{i-1,\nu,r,\nu}X_{r\nu}$$

$$+\sum_{\mu=-1}^{m-1}\sum_{j=0}^{n}\hat{a}_{m-1,n-1,\mu+1,j}\lambda_{\mu+1,j}\sum_{s=0}^{j-2}\hat{b}'_{\mu,j-1,\mu,s}X_{\mu s}.$$

If we use i and j as the indices in all sums, since $\hat{a}_{m-1,n-1,m+1,j}=\hat{a}_{m-1,n-1,i,n+1}=\hat{a}_{m-1,n-1,m+1,j+1}=\hat{a}_{m-1.n-1,i+1,n+1}=0$, we have

$$\begin{aligned}
Y_{mn}=&\sum_{i=0}^{m}\sum_{j=0}^{n}\hat{a}_{m-1,n-1,i,j}\lambda_{ij}\hat{b}'_{i-1,j-1,i,j}X_{ij}\\
&+\sum_{i=0}^{m}\sum_{j=0}^{n}\hat{a}_{m-1,n-1,i+1,j}\lambda_{i+1,j}\hat{b}'_{i,j-1,i,j}X_{ij}\\
&+\sum_{i=0}^{m}\sum_{j=0}^{n}\hat{a}_{m-1,n-1,i,j+1}\lambda_{i,j+1}\hat{b}'_{i-1,j,i,j}X_{ij}\\
&+\sum_{i=0}^{m}\sum_{j=0}^{n}\hat{a}_{m-1,n-1,i+1,j+1}\lambda_{i+1,j+1}\hat{b}'_{ijij}X_{ij}\\
&+\sum_{i=0}^{m}\sum_{j=0}^{n}\hat{a}_{m-1,n-1,i,j}\lambda_{ij}\sum_{r=0}^{i-2}\hat{b}'_{i-1,j-1,r,j}X_{rj}\\
&+\sum_{i=0}^{m}\sum_{j=0}^{n}\hat{a}_{m-1,n-1,i,j}\lambda_{i,j}\sum_{s=0}^{j-2}\hat{b}'_{i-1,j-1,i,s}X_{is}\\
&+\sum_{i=0}^{m}\sum_{j=0}^{n}\hat{a}_{m-1,n-1,i,j+1}\lambda_{i,j+1}\sum_{r=0}^{i-2}\hat{b}'_{i-1,j,r,j}X_{rj}\\
&+\sum_{i=0}^{m}\sum_{j=0}^{n}\hat{a}_{m-1,n-1,i+1,j}\lambda_{i+1,j}\sum_{s=0}^{j-2}\hat{b}'_{i,j-1,i,s}X_{is}\\
&+\sum_{i=0}^{m}\sum_{j=0}^{n}\hat{a}_{m-1,n-1,i,j}\lambda_{ij}\sum_{r=0}^{i-2}\sum_{s=0}^{j-2}\hat{b}'_{i-1,j-1,r,s}X_{rs}.
\end{aligned}$$

Since B is factorable, we have

$$\begin{aligned}
\hat{b}_{m-1,n-1,i,j} &= \bar{b}_{m-1,n-1,i,j} - \bar{b}_{m,n-1,i,j} - \bar{b}_{m-1,n,i,j} + \bar{b}_{mnij} \\
&= \sum_{r=i}^{m-1}\sum_{s=j}^{n-1} b_{m-1,n-1,r,s} - \sum_{r=i}^{m}\sum_{s=j}^{n-1} b_{m,n-1,r,s} - \sum_{r=i}^{m-1}\sum_{s=j}^{n} b_{m-1,n,r,s} + \sum_{r=i}^{m}\sum_{s=j}^{n} b_{mnrs} \\
&= \sum_{r=i}^{m-1} c_{m-1,r}\sum_{s=j}^{n-1} d_{n-1,s} - \sum_{r=i}^{n} c_{mr}\sum_{s=j}^{n-1} d_{n-1,s} - \sum_{r=i}^{m-1} c_{m-1,r}\sum_{s=j}^{n} d_{ns} + \sum_{r=i}^{m} c_{mr}\sum_{s=j}^{n} d_{ns} \\
&= \bar{c}_{m-1.i}\bar{d}_{n-1,j} - \bar{c}_{mi}\bar{d}_{n-1,j} - \bar{c}_{m-1,i}\bar{d}_{nj} + \bar{c}_{mi}\bar{d}_{nj} \\
&= (\bar{c}_{m-1,i} - \bar{c}_{mi})(\bar{d}_{n-1,j} - \bar{d}_{nj}) = \hat{c}_{mi}\hat{d}_{nj}.
\end{aligned} \tag{7}$$

Thus

$$\begin{aligned}
Y_{mn} &= \sum_{i=0}^{m}\sum_{j=0}^{n} \hat{a}_{m-1,n-1,i,j}\lambda_{ij}\hat{c}'_{ii}\hat{d}'_{jj}X_{ij} \\
&+ \sum_{i=0}^{m}\sum_{j=0}^{n} \hat{a}_{m-1,n-1,i+1,j}\lambda_{i+1,j}\hat{c}'_{i+1,i}\hat{d}'_{jj}X_{ij} \\
&+ \sum_{i=0}^{m}\sum_{j=0}^{n} \hat{a}_{m-1,n-1,i,j+1}\lambda_{i,j+1}\hat{c}'_{ii}\hat{d}'_{j,j+1}X_{ij} \\
&+ \sum_{i=0}^{m}\sum_{j=0}^{n} \hat{a}_{m-1,n-1,i+1,j+1}\lambda_{i+1,j+1}\hat{c}'_{i+1,i}\hat{d}'_{j+1,j}X_{ij} \\
&+ \sum_{i=0}^{m}\sum_{j=0}^{n} \hat{a}_{m-1,n-1,i,j}\lambda_{i,j}\hat{d}'_{jj}\sum_{r=0}^{i-2}\hat{c}'_{ir}X_{rj} \\
&+ \sum_{i=0}^{m}\sum_{j=0}^{n} \hat{a}_{m-1,n-1,i,j}\lambda_{ij}\hat{c}'_{i,i}\sum_{s=0}^{j-2}\hat{d}'_{js}X_{is} \\
&+ \sum_{i=0}^{m}\sum_{j=0}^{n} \hat{a}_{m-1,n-1,i,j+1}\lambda_{i,j+1}\hat{d}'_{j+1,j}\sum_{r=0}^{i-1}\hat{c}'_{i,r}X_{rj} \\
&+ \sum_{i=0}^{m}\sum_{j=0}^{n} \hat{a}_{m-1,n-1,i+1,j}\lambda_{i+1,j}\hat{c}'_{i+1,i}\sum_{s=0}^{j-2}\hat{d}'_{js}X_{is} \\
&+ \sum_{i=0}^{m}\sum_{j=0}^{n} \hat{a}_{m-1,n-1,i,j}\lambda_{i,j}\sum_{r=0}^{i-2}\sum_{s=0}^{j-2}\hat{c}'_{ir}\hat{d}'_{js}X_{rs}.
\end{aligned} \tag{8}$$

Using the facts that C and D are triangles,

$$\sum_{j=k}^{n}\hat{c}_{nj}\hat{c}'_{jk} = \delta_n^k, \quad \hat{c}'_{ii} = \frac{1}{c_{ii}}, \quad \text{and} \quad \hat{d}'_{ii} = \frac{1}{d_{ii}},$$

$$\hat{c}'_{i+1,i} = -\frac{\hat{c}_{i+1,i}}{c_{ii}c_{i+1,i+1}} = -\frac{\bar{c}_{i+1,i} - \bar{c}_{ii}}{c_{ii}c_{i+1,i+1}} = \frac{c_{ii} - c_{i+1,i} - c_{i+1,i+1}}{c_{ii}c_{i+1,i+1}}$$

$$\hat{d}'_{j+1,j} = -\frac{\hat{d}_{j+1,j}}{d_{jj}d_{j+1,j+1}} = -\frac{\bar{d}_{j+1,j} - \bar{d}_{jj}}{d_{jj}d_{j+1,j+1}} = \frac{d_{jj} - d_{j+1,j} - d_{j+1,j+1}}{d_{jj}d_{j+1,j+1}}.$$

It then follows that

$$\begin{aligned} \hat{c}'_{i+1,i}\hat{d}'_{jj} &= \frac{c_{ii} - c_{i+1,i} - c_{i+1,i+1}}{c_{ii}c_{i+1,i+1}}\frac{1}{d_{jj}} = \frac{1}{b_{ijij}}\left(\frac{c_{ii} - c_{i+1,i} - c_{i+1,i+1}}{c_{i+1,i+1}}\right) \\ \hat{c}'_{ii}\hat{d}'_{j+1,j} &= \frac{1}{c_{ii}}\frac{d_{jj} - d_{j+1,j} - d_{j+1,j+1}}{d_{jj}d_{j+1,j+1}} = \frac{1}{b_{ijij}}\left(\frac{d_{jj} - d_{j+1,j} - d_{j+1,j+1}}{d_{j+1,j+1}}\right) \\ \hat{c}'_{i+1,i}\hat{d}'_{j+1,j} &= \frac{c_{ii} - c_{i+1,i} - c_{i+1,i+1})(d_{jj} - d_{j+1,j} - d_{j+1,j+1})}{b_{ijij}c_{i+1,i+1}d_{j+1,j+1}}. \end{aligned} \tag{9}$$

Substituting (9) into (8) gives

$$\begin{aligned} Y_{mn} = {} & \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\hat{a}_{m-1,n-1,i,j}}{b_{ijij}}\lambda_{ij}X_{ij} \\ & + \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\hat{a}_{m-1,n-1,i+1,j}}{b_{ijij}}\lambda_{i+1,j}\left(\frac{c_{ii} - c_{i+1,i}}{c_{i+1,i+1}} - 1\right)X_{ij} \\ & + \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\hat{a}_{m-1,n-1,i,j+1}}{b_{ijij}}\lambda_{i,j+1}\left(\frac{d_{jj} - d_{j+1,j}}{d_{j+1,j+1}} - 1\right)X_{ij} \\ & + \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\hat{a}_{m-1,n-1,i+1,j+1}}{b_{ijij}}\lambda_{i+1,j+1}\left[\left(\frac{c_{ii} - c_{i+1,i}}{c_{i+1,i+1}}\right)\left(\frac{d_{jj} - d_{j+1,j}}{d_{j+1,j+1}}\right)\right. \\ & \left. - \frac{c_{ii} - c_{i+1,i}}{c_{i+1,i+1}} - \frac{d_{jj} - d_{j+1,j}}{d_{j+1,j+1}} + 1\right]X_{ij} \\ & + \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\hat{a}_{m-1,n-1,i,j}}{d_{jj}}\lambda_{ij}\sum_{r=0}^{i-2}\hat{c}'_{ir}X_{rj} \\ & + \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\hat{a}_{m-1,n-1,i,j}}{c_{ii}}\lambda_{ij}\sum_{s=0}^{j-2}\hat{d}'_{js}X_{is} \\ & + \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\hat{a}_{m-1,n-1,i,j+1}}{d_{jj}}\lambda_{i,j+1}\left(\frac{d_{jj} - d_{j+1,j}}{d_{j+1,j+1}} - 1\right)\sum_{r=0}^{i-2}\hat{c}'_{ir}X_{rj} \\ & + \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\hat{a}_{m-1,n-1,i+1,j}}{c_{ii}}\lambda_{i+1,j}\left(\frac{c_{ii} - c_{i+1,i}}{c_{i+1,i+1}} - 1\right)\sum_{s=0}^{j-2}\hat{d}'_{js}X_{is} \\ & + \sum_{i=0}^{m}\sum_{j=0}^{n}\hat{a}_{m-1,n-1,i,j}\lambda_{ij}\sum_{r=0}^{i-2}\sum_{s=0}^{j-2}\hat{c}'_{ir}\hat{d}'_{js}X_{rs}. \end{aligned} \tag{10}$$

Adding first, third, fifth and nineth terms of (10) gives

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\Delta_{ij}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})}{b_{ijij}}X_{ij}. \tag{11}$$

Adding terms 2 and 7 of (10) gives

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\Delta_{0j}(\hat{a}_{m-1,n-1,i+1,j}\lambda_{i+1,j})}{b_{ijij}}\left(\frac{c_{ii}-c_{i+1,i}}{c_{i+1,i+1}}\right)X_{ij}. \tag{12}$$

Adding terms 4 and 8 of (10) gives

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\Delta_{i0}(\hat{a}_{m-1,n-1,i,j+1}\lambda_{i,j+1})}{b_{ijij}}\left(\frac{d_{jj}-d_{j+1,j}}{d_{j+1,j+1}}\right)X_{ij}. \tag{13}$$

Combining terms 10 and 13 of (10) gives

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\Delta_{0j}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})}{djj}\sum_{r=0}^{j-2}\hat{c}'_{ir}X_{rj}. \tag{14}$$

Combining terms 11 and 15 of (10) gives

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\Delta_{i0}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})}{c_{ii}}\sum_{s=0}^{i-2}\hat{d}'_{js}X_{is}. \tag{15}$$

Substituting (11) - (15) into (10) gives

$$\begin{aligned}
Y_{mn} = &\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\Delta_{ij}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})}{b_{ijij}}X_{ij}\\
&+\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\Delta_{0j}(\hat{a}_{m-1,n-1,i+1,j}\lambda_{i+1,j})}{b_{ijij}}\left(\frac{c_{ii}-c_{i+1,i}}{c_{i+1,i+1}}\right)X_{ij}\\
&+\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\Delta_{i0}(\hat{a}_{m-1,n-1,i,j+1}\lambda_{i,j+1})}{b_{ijij}}\left(\frac{d_{jj}-d_{j+1,j}}{d_{j+1,j+1}}\right)X_{ij}\\
&+\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{(\hat{a}_{m-1,n-1,i+1,j+1}\lambda_{i+1,j+1})}{b_{ijij}}\left(\frac{c_{ii}-c_{i+1,i}}{c_{i+1,i+1}}\right)\left(\frac{d_{jj}-d_{j+1,j}}{d_{j+1,j+1}}\right)X_{ij}\\
&+\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\Delta_{0j}(\hat{a}_{m-1,n-1,i,j}\lambda_{i,j})}{d_{jj}}\sum_{r=0}^{i-2}\hat{c}'_{ir}X_{rj}\\
&+\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\Delta_{i0}(\hat{a}_{m-1,n-1,i,j}\lambda_{i,j})}{c_{ii}}\sum_{s=0}^{j-2}\hat{d}'_{js}X_{is}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{(\hat{a}_{m-1,n-1,i,j+1}\lambda_{i,j+1})}{d_{jj}}\Big(\frac{d_{jj}-d_{j+1,j}}{d_{j+1,j+1}}\Big)\sum_{r=0}^{i-2}\hat{c}_{ir}'X_{rj}\\
&+\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{(\hat{a}_{m-1,n-1,i+1,j}\lambda_{i+1,j})}{c_{ii}}\Big(\frac{c_{ii}-c_{i+1,i}}{c_{i+1,i+1}}\Big)\sum_{s=0}^{j-2}\hat{d}_{js}'X_{is}\\
&+\sum_{i=0}^{m}\sum_{j=0}^{n}\hat{a}_{m-1,n-1,i,j}\lambda_{i,j}\sum_{r=0}^{i-2}\sum_{s=0}^{j-2}\hat{c}_{ir}'\hat{d}_{js}'X_{rs}.
\end{aligned}
\tag{16}
$$

From condition (ii),

$$
\begin{aligned}
\frac{1}{b_{ijij}}\Big(\frac{c_{ii}-c_{i+1,i}}{c_{i+1,i+1}}\Big)\frac{d_{jj}}{d_{jj}}&=\frac{1}{b_{ijij}}\Big(\frac{b_{ijij}-b_{i+1,j,i,j}}{b_{i+1.ji+1,j}}\Big)\\
&=\frac{1}{b_{ijij}}O(b_{i0i0})=O\Big(\frac{c_{ii}d_{00}}{c_{ii}d_{jj}}\Big)=O\Big(\frac{1}{b_{0j0j}}\Big).
\end{aligned}
\tag{17}
$$

Similarly,

$$
\frac{1}{b_{ijij}}\Big(\frac{d_{jj}-dj+1,j}{d_{j+1,j+1}}\Big)=O\Big(\frac{1}{b_{i0i0}}\Big).
\tag{18}
$$

Using (17), (18) and (7) in (16) gives

$$
\begin{aligned}
Y_{mn}=&\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\Delta_{ij}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})}{b_{ijij}}X_{ij}\\
&+O(1)\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\Delta_{0j}(\hat{a}_{m-1,n-1,i+1,j}\lambda_{i+1,j})}{b_{ojoj}}X_{ij}\\
&+O(1)\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\Delta_{i0}(\hat{a}_{m-1,n-1,i,j+1}\lambda_{i,j+1})}{b_{i0i0}}X_{ij}\\
&+O(1)\sum_{i=0}^{m}\sum_{j=0}^{n}(\hat{a}_{m-1,n-1,i+1,j+1}\lambda_{i+1,j+1})X_{ij}\\
&+O(1)\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\Delta_{0j}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})}{b_{ojoj}}\sum_{r=0}^{i-2}\hat{b}_{i-1,j-1,r,j}'X_{rj}\\
&+O(1)\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\Delta_{i0}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})}{b_{i0i0}}\sum_{s=0}^{j-1}\hat{b}_{i-1,j-1,i,s}'X_{is}\\
&+\sum_{i=0}^{m}\sum_{j=0}^{n}(\hat{a}_{m-1,n-1,i,j+1}\lambda_{i,j+1})(b_{0j0j})\sum_{r=0}^{i-2}\hat{c}_{ir}'X_{rj}
\end{aligned}
$$

$$+\sum_{i=0}^{m}\sum_{j=0}^{n}(\hat{a}_{m-1,n-1,i+1,j}\lambda_{i+1,j})(b_{i0i0})\sum_{s=0}^{j-2}\hat{d}'_{js}X_{is}$$

$$+O(1)\sum_{i=0}^{m}\sum_{j=0}^{n}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})\sum_{r=0}^{i-2}\sum_{s=0}^{j-2}\hat{b}'_{i-1,j-1,r,s}X_{rs}$$

$$=\sum_{i=1}^{9}I_i \quad \text{say.}$$

To prove the theorem it is sufficient, by Minkowski's theorem to show that

$$\sum_{n=1}^{\infty}n^{k-1}|I_r|^k<\infty, \quad r=1,2,\dots 9.$$

Substituting into (2) and using Hölder's inequality, (i), (iii) and (iv),

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|I_1|^k=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left|\frac{\Delta_{ij}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})}{b_{ijij}}X_{ij}\right|^k$$

$$\leqslant\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left(\sum_{i=0}^{m}\sum_{j=0}^{n}|b_{ijij}|^{-1}|\Delta_{ij}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})||X_{ij}|\right)^k$$

$$\leqslant\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\sum_{i=0}^{m}\sum_{j=0}^{n}|a_{ijij}|^{-k}|\Delta_{ij}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij}||X_{ij}|^k\times$$

$$\times\left(\sum_{i=0}^{m}\sum_{j=0}^{n}|\Delta_{ij}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})|\right)^{k-1}$$

$$=O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn|a_{mnmn}|)^{k-1}\times$$

$$\times\sum_{i=0}^{m}\sum_{j=0}^{n}|a_{ijij}|^{-k}|\Delta_{ij}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij}||X_{ij}|^k$$

$$=O(1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|a_{ijij}|^{-k}|X_{ij}|^k\times$$

$$\times\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\Delta_{ij}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})|$$

$$=O(1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(ij)^{k-1}|X_{ij}|^k=O(1).$$

Using Hölder's inequality, (v), (vi) and (i),

$$\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|I_2|^k=\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left|\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\Delta_{0j}(\hat{a}_{m-1,n-1,i+1,j}\lambda_{i+1,j})}{b_{0j0j}}X_{ij}\right|^k$$

$$\leqslant \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left(\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{|b_{0j0j}|^{-1}|b_{i0i0}|}{|b_{i0i0}|}|\Delta_{0j}(\hat{a}_{m-1,n-1,i+1,j}\lambda_{i+1,j})||X_{ij}|\right)^{k}$$

$$\leqslant \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\sum_{i=0}^{m}\sum_{j=0}^{n}|b_{0j0j}|^{-k}|b_{i0i0}|^{1-k}|\Delta_{0j}(\hat{a}_{m-1,n-1,i+1,j}\lambda_{i+1,j})||X_{ij}|^{k}\times$$

$$\times\left(\sum_{i=0}^{m}\sum_{j=0}^{n}|b_{i0i0}||\Delta_{0j}(\hat{a}_{m-1,n-1,i+1,j}\lambda_{i+1,j})|\right)^{k-1}$$

$$=O(1)\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}(mn|a_{mnmn}|)^{k-1}\times$$

$$\times\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{|b_{0j0j}|^{-1}}{|b_{0j0j}|^{k-1}|b_{i0i0}|^{k-1}}|\Delta_{0j}(\hat{a}_{m-1,n-1,i+1,j}\lambda_{i+1,j})||X_{ij}|^{k}$$

$$=O(1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{|b_{0j0j}|^{-1}}{|b_{ijij}|^{k-1}}|X_{ij}|^{k}\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\Delta_{0j}(\hat{a}_{m-1,n-1,i+1,j}\lambda_{i+1,j})|$$

$$=O(1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{|a_{ijij}|^{k-1}}{|b_{ijij}|^{k-1}}|b_{0j0j}|^{-1}|b_{0j0j}||X_{ij}|^{k}(ij)^{k-1}$$

$$=O(1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(ij)^{k-1}|X_{ij}|^{k}=O(1).$$

Using Hölder's inequality, (i), (vii) and $(viii)$,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|I_3|^{k}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left|\frac{\Delta_{i0}(\hat{a}_{m-1,n-1,i,j+1}\lambda_{i,j+1})}{b_{i0i0}}X_{ij}\right|^{k}$$

$$\leqslant \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\sum_{i=0}^{m}\sum_{j=0}^{n}|b_{i0i0}|^{-k}|b_{0j0j}|^{1-k}|\Delta_{i0}(\hat{a}_{m-1,n-1,i,j+1}\lambda_{i,j+1})||X_{ij}|^{k}\times$$

$$\times\left(\sum_{i=0}^{n}\sum_{j=0}^{n}|b_{0j0j}||\Delta_{i0}(\hat{a}_{m-1,n-1,i,j+1}\lambda_{i,j+1})|\right)^{k-1}$$

$$=O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn|a_{mnmn}|)^{k-1}\sum_{i=0}^{m}\sum_{j=0}^{n}|b_{ijij}|^{1-k}|b_{i0i0}|^{-1}|\Delta_{i0}(\hat{a}_{m-1,n-1,i,j+1}\lambda_{i,j+1})||X_{ij}|^{k}$$

$$=O(1)\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{|b_{i0i0}|^{-1}}{|b_{ijij}|^{k-1}}|X_{ij}|^{k}\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\Delta_{i0}(\hat{a}_{m-1,n-1,i,j+1}\lambda_{i,j+1})|$$

$$=O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^{k-1}|X_{ij}|^{k}<\infty.$$

Using Hölder's inequality, (ix) and (x),

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|I_4|^k &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left|\sum_{i=0}^{m}\sum_{j=0}^{n}\hat{a}_{m-1,n-1,i+1,j+1}\lambda_{i+1,j+1}X_{ij}\right|^k \\
&\leqslant \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left(\sum_{i=0}^{m}\sum_{j=0}^{n}|a_{ijij}||a_{ijij}|^{-1}|\hat{a}_{m-1,n-1,i+1,j+1}\lambda_{i+1,j+1}||X_{ij}|\right)^k \\
&\leqslant \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\sum_{i=0}^{m}\sum_{j=0}^{n}|a_{ijij}|^{1-k}|\hat{a}_{m-1,n-1,i+1,j+1}\lambda_{i+1,j+1}||X_{ij}|^k\times \\
&\qquad \times\left(\sum_{i=0}^{m}\sum_{j=0}^{n}|a_{ijij}||\hat{a}_{m-1,n-1,i+1,j+1}\lambda_{i+1,j+1}|\right)^{k-1} \\
&= O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn|a_{mnmn}|)^{k-1}\sum_{i=0}^{m}\sum_{j=0}^{n}|a_{ijij}|^{1-k}|\hat{a}_{m-1,n-1,i+1,j+1}\lambda_{i+1,j+1}||X_{ij}|^k \\
&= O(1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|a_{ijij}|^{1-k}|X_{ij}|^k\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\hat{a}_{m-1,n-1,i+1,j+1}\lambda_{i+1,j+1}| \\
&= O(1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(ij)^{k-1}|X_{ij}|^k = O(1).
\end{aligned}$$

Using Hölder's inequality, (i), (xi), (xii) and (xxi),

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|I_5|^k &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left|\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\Delta_{0j}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})}{b_{0j0j}}\sum_{r=0}^{i-2}\hat{b}'_{i-1,j-1,r,j}X_{rj}\right|^k \\
&\leqslant \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\sum_{i=0}^{m}\sum_{j=0}^{n}|b_{0j0j}|^{-k}|b_{i0i0}|^{1-k}\left|\sum_{r=0}^{i-2}\hat{b}'_{i-1,j-1,r,j}X_{rj}\right|^k\times \\
&\qquad \times|\Delta_{0j}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij}|\left(\sum_{i=0}^{m}\sum_{j=0}^{n}|b_{i0i0}||\Delta_{0j}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})|\right)^{k-1} \\
&= O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn|a_{mnmn}|)^{k-1}\sum_{i=0}^{m}\sum_{j=0}^{n}|b_{ijij}|^{1-k}|b_{0j0j}|^{-1}\times \\
&\qquad \times|\Delta_{0j}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})|\left|\sum_{r=0}^{i-2}\hat{b}'_{i-1,j-1,r,j}X_{rj}\right|^k \\
&= O(1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{|b_{0j0j}|^{-1}}{|b_{ijij}|^{k-1}}\left|\sum_{r=0}^{i-2}\hat{b}'_{i-1,j-1,r,j}X_{rj}\right|^k\times \\
&\qquad \times\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\Delta_{0j}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})|
\end{aligned}$$

$$= O(1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(ij)^{k-1}\Big|\sum_{r=0}^{i-2}\hat{b}'_{i-1,j-1,r,j}X_{rj}\Big|^k = O(1).$$

Using Hölder's inequality, $(xiii)$, (xiv) and $(xxii)$,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|I_6|^k = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\Big|\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\Delta_{i0}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})}{b_{i0i0}}\sum_{s=0}^{j-2}\hat{b}'_{i-1,j-1,i,s}X_{is}\Big|^k$$

$$\leqslant \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\Big(\sum_{i=0}^{m}\sum_{j=0}^{n}|\Delta_{i0}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})||b_{i0i0}|^{-1}\times$$

$$\times |b_{0j0j}||b_{0j0j}|^{-1}\Big|\sum_{s=0}^{i-1}\hat{b}'_{i-l,j-1,i,s}X_{is}\Big|\Big)^k$$

$$\leqslant \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\sum_{i=0}^{m}\sum_{j=0}^{n}|b_{i0i0}|^{-k}|b_{0j0j}|^{-k}|\Delta_{i0}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij}|\times$$

$$\times\Big|\sum_{s=0}^{j-2}\hat{b}'_{i-1,j-1,i,s}X_{is}\Big|^k\Big(\sum_{i=0}^{m}\sum_{j=0}^{n}|b_{0j0j}||\Delta_{i0}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})|\Big)^{k-1}$$

$$= O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn|a_{mnmn}|)^{k-1}\times$$

$$\times\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{|b_{i0i0}|^{-1}}{|b_{ijij}|^{k-1}}|\Delta_{i0}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})|\Big|\sum_{s=0}^{j-2}\hat{b}'_{i-1,j-1,i,s}X_{is}\Big|^k$$

$$= O(1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{|b_{i0i0}|^{-1}}{|b_{ijij}|^{k-1}}\sum_{s=0}^{j-2}|\hat{b}'_{i-1,j-1,i,s}X_{is}|^k\times$$

$$\times\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\Delta_{i0}(\hat{a}_{m-1,n-1,i,j}\lambda_{ij})|$$

$$= O(1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(ij)^{k-1}\sum_{s=0}^{j-2}|\hat{b}'_{i-1,j-1,i,s}X_{is}|^k = O(1).$$

Using Hölder's inequality, (xv), (xvi) and (xxi),

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|I_7|^k = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\Big|\sum_{i=0}^{m}\sum_{j=0}^{n}\hat{a}_{m-1,n-1,i,j+1}\lambda_{i,j+1}(b_{0j0j})\times$$

$$\times\sum_{r=0}^{i-2}\hat{b}'_{i-1,j-1,r,j}X_{rj}\Big|^k$$

$$\leqslant \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\sum_{i=0}^{m}\sum_{j=0}^{n}|\hat{a}_{m-1,n-1,i,j+1}\lambda_{i,j+1}||b_{0j0j}|^k|a_{ijij}|^{1-k}\times$$

$$\times \left| \sum_{r=0}^{i-2} \hat{b}'_{i-1,j-1,r,j} X_{rj} \right|^k \times \left(\sum_{i=0}^{m} \sum_{j=0}^{n} |a_{ijij}| |\hat{a}_{m-1,n-1,i,j+1} \lambda_{i,j+1}| \right)^{k-1}$$

$$= O(1) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn|a_{mnmn}|)^{k-1} \times$$

$$\times \sum_{i=0}^{m} \sum_{j=0}^{n} |a_{ijij}|^{1-k} |\hat{a}_{m-1,n-1,i,j+1} \lambda_{i,j+1}| |b_{0j0j}|^k \left| \sum_{r=0}^{i-2} \hat{b}'_{i-1,j-1,r,j} X_{rj} \right|^k$$

$$= O(1) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |a_{ijij}|^{1-k} |b_{0j0j}|^k \left| \sum_{r=0}^{i-2} \hat{b}'_{i-1,j-1,r,j} X_{rj} \right|^k \times$$

$$\times \sum_{m=i+1}^{\infty} \sum_{n=j+1}^{\infty} (mn|a_{mnmn}|)^{k-1} |\hat{a}_{m-1,n-1,i,j+1} \lambda_{i,j+1}|$$

$$= O(1) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (ij)^{k-1} \left| \sum_{r=0}^{i-2} \hat{b}'_{i-1,j-1,r,j} X_{rj} \right|^k = O(1).$$

Using Hölder's inequality, $(xvii)$, $(xviii)$ and $(xxii)$,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} |I_8|^k = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} \left| \sum_{i=0}^{m} \sum_{j=0}^{n} \hat{a}_{m-1,n-1,i+1,j} \lambda_{i+1,j} (b_{i0i0}) \sum_{s=0}^{j-2} \hat{b}'_{i-1,j-1,i,s} X_{is} \right|^k$$

$$\leqslant \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} \sum_{i=0}^{m} \sum_{j=0}^{n} |\hat{a}_{m-1,n-1,i+1,j} \lambda_{i+1,j}| |b_{i0i0}|^k |a_{ijij}|^{1-k} \times$$

$$\times \left| \sum_{s=0}^{j-2} \hat{b}'_{i-1,j-1,i,s} X_{is} \right|^k \left(\sum_{i=0}^{m} \sum_{j=0}^{n} |a_{ijij}| |\hat{a}_{m-1,n-1,i+1,j} \lambda_{i+1,j}| \right)^{k-1}$$

$$= O(1) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn|a_{mnmn}|)^{k-1} \times$$

$$\times \sum_{i=0}^{m} \sum_{j=0}^{n} |a_{ijij}|^{1-k} |\hat{a}_{m-1,n-1,i+1,j} \lambda_{i+1,j}| |b_{i0i0}|^k \left| \sum_{s=0}^{j-2} \hat{b}'_{i-1,j-1,i,s} X_{is} \right|^k$$

$$= O(1) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |a_{ijij}|^{1-k} |b_{i0i0}|^k \left| \sum_{s=0}^{j-2} \hat{b}'_{i-1,j-1,i,s} X_{is} \right|^k \times$$

$$\times \sum_{m=i+1}^{\infty} \sum_{n=j+1}^{\infty} (mn|a_{mnmn}|)^{k-1} |\hat{a}_{m-1,n-1,i+1,j} \lambda_{i+1,j}|$$

$$= O(1) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (ij)^{k-1} \left| \sum_{s=0}^{j-2} \hat{b}'_{i-1,j-1,i,s} X_{is} \right|^k = O(1).$$

Using Hölder's inequality, (xix), (xx) and $(xxiii)$,

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|I_9|^k &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left|\sum_{i=0}^{m}\sum_{j=0}^{n}\hat{a}_{m-1,n-1,i,j}\lambda_{ij}\sum_{r=0}^{i-2}\sum_{s=0}^{j-2}\hat{b}'_{i-1,j-1,r,s}X_{rs}\right|^k \\
&\leqslant \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\sum_{i=0}^{m}\sum_{j=0}^{n}|a_{ijij}|^{1-k}|\hat{a}_{m-1,n-1,i,j}\lambda_{ij}|\times \\
&\quad\times\left|\sum_{r=0}^{i-2}\sum_{j=0}^{j-2}\hat{b}'_{i-1,j-1,r,s}X_{rs}\right|^k\left(\sum_{i=0}^{m}\sum_{j=0}^{n}|a_{ijij}||\hat{a}_{m-1,n-1,i,j}\lambda_{ij}|\right)^{k-1} \\
&= O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn|a_{mnmn}|)^{k-1}\times \\
&\quad\times\sum_{i=0}^{m}\sum_{j=0}^{n}|a_{ijij}|^{1-k}|\hat{a}_{m-1,n-1,i,j}\lambda_{ij}|\left|\sum_{r=0}^{i-2}\sum_{s=0}^{j-2}\hat{b}'_{i-1,j-1,r,s}X_{rs}\right|^k \\
&= O(1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|a_{ijij}|^{1-k}\left|\sum_{r=0}^{i-2}\sum_{s=0}^{j-2}\hat{b}'_{i-1,j-1,r,s}X_{rs}\right|^k\times \\
&\quad\times\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\hat{a}_{m-1,n-1,i,j}\lambda_{ij}| \\
&= O(1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(ij)^{k-1}\left|\sum_{r=0}^{i-2}\sum_{s=0}^{j-2}\hat{b}'_{i-1,j-1,r,s}X_{rs}\right|^k = O(1).
\end{aligned}$$

THEOREM 2. *Suppose that a double factorable positive sequence* $\{p_{mn}\}$ *and a positive matrix* B *satisfy*

(i) $P_{mn}b_{mnmn} = O(p_{mn})$,

(ii) $\sum_{i=0}^{m}\sum_{j=0}^{n}|\Delta_{ij}(\hat{b}_{m-1,n-1,i,j}\lambda_{ij})| = O(b_{mnmn})$,

(iii) $\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mnb_{mnmn})^{k-1}|\Delta_{ij}(\hat{b}_{m-1,n-1,i,j}\lambda_{ij})| = O\Big((ij)^{k-1}|b_{ijij}|^k\Big)$,

(iv) $\sum_{i=0}^{m}\sum_{j=0}^{n}\left(\frac{p_i}{P_i}\right)|\Delta_{0j}\hat{b}_{m-1,n-1,i+1,j}\lambda_{i+1,j}| = O(b_{mnmn})$,

(v) $\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mnb_{mnmn})^{k-1}|\Delta_{0j}(\hat{b}_{m-1,n-1,i+1,j}\lambda_{i+1,j})| = \left((ij|b_{ijij}|)^{k-1}\frac{q_j}{Q_j}\right)$,

(vi) $\sum_{i=0}^{m}\sum_{j=0}^{n}\left(\frac{q_j}{Q_j}\right)|\Delta_{i0}(\hat{b}_{m-1,n-1,i,j+1}\lambda_{i,j+1}| = O(b_{mnmn})$,

(vii) $\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mnb_{mnmn})^{k-1}|\Delta_{i0}(\hat{b}_{m-1,n-1,i,j+1}\lambda_{i,j+1})| = O\Big((ijb_{ijij})^{k-1}\frac{p_i}{P_i}\Big)$,

$$(viii)\quad \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ijij}|\hat{b}_{m-1,n-1,i+1,j+1}\lambda_{i+1,j+1}| = O(b_{mnmn}),$$

$$(ix)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty} (mnb_{mnmn})^{k-1}|\hat{b}_{m-1,n-1,i+1,j+1}\lambda_{i+1,j+1}| = O\Big((ijb_{ijij})^{k-1}\Big),$$

$$(x)\quad \sum_{i=0}^{m}\sum_{j=0}^{n} \frac{p_i}{P_i}|\Delta_{0j}\hat{b}_{m-1,n-1,i,j}\lambda_{ij}| = O(b_{mnmn}),$$

$$(xi)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty} (mnb_{mnmn})^{k-1}|\Delta_{0j}\hat{b}_{m-1,n-1,i,j}\lambda_{ij}| = O\Big((ijb_{ijij})^{k-1}\frac{q_j}{Q_j}\Big),$$

$$(xii)\quad \sum_{i=0}^{m}\sum_{j=0}^{n} \frac{q_j}{Q_j}|\Delta_{i0}(\hat{b}_{m-1,n-1,i,j}\lambda_{ij}| = O(b_{mnmn}),$$

$$(xiii)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty} (mnb_{mnmn})^{k-1}|\Delta_{i0}(\hat{b}_{m-1,n-1,i,j}\lambda_{ij})| = O\Big((ijb_{ijij})^{k-1}\frac{p_i}{P_i}\Big),$$

$$(xiv)\quad \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ijij}|(\hat{b}_{m-1,n-1,i,j+1}\lambda_{i,j+1})| = O(|b_{mnmn}|),$$

$$(xv)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty} (mnb_{mnmn})^{k-1}|\hat{b}_{m-1,n-1,i,j+1}\lambda_{i,j+1}| = O((ijb_{ijij})^{k-1}(\frac{q_j}{Q_j})^k,$$

$$(xvi)\quad \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ijij}|\hat{b}_{m-1,n-1,i+1,j}\lambda_{i+1,j}| = O(|b_{mnmn}|),$$

$$(xvii)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty} (mnb_{mnmn})^{k-1}|\hat{b}_{m-1,n-1,i+1,j}\lambda_{i+1,j}| = O((ijb_{ijij}|)^{k-1}(\frac{p_i}{P_i})^k),$$

$$(xviii)\quad \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ijij}|\hat{b}_{m-1,n-1,i,j}\lambda_{ij})| = O(b_{mnmn}), \quad \text{and}$$

$$(xix)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty} (mnb_{mnmn})^{k-1}|\hat{b}_{m-1,n-1,i,j}\lambda_{ij}| = O((ijb_{ijij})^{k-1}),$$

If $\sum\sum a_{ij}$ *is summable* $|\overline{N},p_{mn}|_k$, *then* $\sum\sum a_{ij}\lambda_{ij}$ *is also summable* $|B|_k, k \geqslant 1$.

Proof. With A replaced by B and B replaced by $(\overline{N},p_{mn})$, condition (i) of Theorem 1 becomes condition (i) of Theorem 2.

$$b_{ijij} - b_{i+1,j,i,j} = \frac{p_iq_j}{P_iQ_j} - \frac{p_iq_j}{P_{i+1}Q_j} = \frac{p_iq_j}{Q_j}\Big(\frac{1}{P_i} - \frac{1}{P_{i+1}}\Big)$$
$$= \frac{p_ip_{i+1}q_j}{P_iP_{i+1}Q_j} = b_{i0i0}b_{i+1,j,i+1,j}.$$

Similarly,

$$b_{ijij} - b_{i,j+1,i,j} = b_{0j0j}b_{i,j+1,i,j+1},$$

and condition (ii) of Theorem 1 is automatically satisfied.

Conditions (iii) - (xx) of Theorem 1 reduce to conditions (ii) - (xix) of Theorem 2, respectivelly. Since the inverse of $(\hat{N},p_{mn})$ is bidiagonal, conditions (xxi) - $(xxiii)$ of Theorem 1 are trivially satisfied.

COROLLARY 1. *Suppose that* $B=(\overline{N},r_ms_n), A=(\overline{N},p_mq_n)$. *Then, if*

(i) $P_mQ_nr_ms_n = O(p_mq_nR_mS_m)$,

(ii) $$\sum_{i=0}^{m}\sum_{j=0}^{n}|\Delta_{ij}(R_{i-1}S_{j-1}\lambda_{ij})| = O(R_{m-1}S_{n-1}),$$

(iii) $$\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnr_ms_n}{R_mS_n}\right)^{k-1}\frac{r_ms_n}{R_mR_{m-1}S_mS_{n-1}}|\Delta_{ij}(R_{i-1}S_{j-1}\lambda_{ij}| = O\Big((ij)^{k-1}\Big(\frac{r_is_j}{R_iS_j}\Big)^k\Big),$$

(iv) $$\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{p_i}{P_i}|\Delta_{0j}(R_{i-1}S_{j-1}\lambda_{ij}| = O(R_{m-1}S_{n-1}),$$

(v) $$\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnr_ms_n}{R_mS_n}\right)^{k-1}\frac{r_ms_n}{R_mR_{m-1}S_mS_{n-1}}|\Delta_{0j}(R_{i-1}S_{j-1}\lambda_{ij}| = O\Big(\Big(\frac{ijr_is_j}{R_iS_j}\Big)^{k-1}\frac{q_j}{Q_j}\Big),$$

(vi) $$\sum_{i=0}^{m}\sum_{j=0}^{n}\left(\frac{q_j}{Q_j}\right)|\Delta_{i0}(R_{i-1}S_{j-1}\lambda_{ij}| = O(R_{m-1}S_{n-1}),$$

(vii) $$\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnr_ms_n}{R_mS_n}\right)^{k-1}\frac{r_ms_n}{R_mR_{n-1}S_nS_{n-1}}|\Delta_{i0}(R_{i-1}S_{j-1}\lambda_{i,j+1})| = O\Big(\Big(\frac{ijr_is_j}{R_iS_j}\Big)^{k-1}\frac{p_i}{P_i}\Big),$$

$(viii)$ $$\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{r_is_j}{R_iS_j}|R_iS_j\lambda_{i+1,j+1}| = O(R_{m-1}S_{n-1}),$$

(ix) $$\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnr_ms_n}{R_mS_n}\right)^{k-1}\frac{r_ms_n}{R_mR_{m-1}S_nS_{n-1}}|R_iS_j\lambda_{i+1,j+1}|$$

(x) $$\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{p_i}{P_i}|\Delta_{0j}(R_{i-1}S_{j-1}\lambda_{ij})| = O(R_{m-1}S_{n-1}),$$

(xi) $$\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnr_ms_n}{R_mS_n}\right)^{k-1}\frac{r_ms_n}{R_mR_{m-1}S_nS_{n-1}}|\Delta_{0j}(R_{i-1}S_j\lambda_{i,j})| = \Big(\Big(\frac{ijr_is_j}{R_iS_j}\Big)^{k-1}\frac{q_j}{Q_j}\Big),$$

(xii) $$\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{q_j}{Q_j}|\Delta_{i0}(R_{i-1}S_{j-1}\lambda_{ij})| = O(R_{m-1}S_{n-1}),$$

$(xiii)$ $$\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnr_ms_n}{R_mS_n}\right)^{k-1}\frac{r_ms_n}{R_mR_{m-1}S_nS_{n-1}}|\Delta_{i0}(R_{i-1}S_{j-1}\lambda_{ij})| = O\Big(\Big(\frac{ijr_is_j}{R_iS_j}\Big)^{k-1}\frac{p_i}{P_i}\Big),$$

(xiv) $$\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{r_is_j}{R_iS_j}|R_{i-1}S_j\lambda_{i,j+1}| = O(R_{m-1}S_{n-1}),$$

(xv) $$\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnr_ms_n}{R_mS_n}\right)^{k-1}\frac{r_ms_n}{R_mR_{m-1}S_nS_{n-1}}|(R_{i-1}S_j\lambda_{i,j+1}| = O\Big(\Big(\frac{ijr_is_j}{R_iS_j}\Big)^{k-1}\Big(\frac{q_j}{Q_j}\Big)^k\Big).$$

$$(xvi)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{r_is_j}{R_iS_j}|R_iS_{j-1}\lambda_{i+1,j}| = O(R_{m-1}S_{n-1}),$$

$$(xvii)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnr_ms_n}{R_mS_n}\right)^{k-1}\frac{r_ms_n}{R_mR_{m-1}S_nS_{n-1}}|(R_iS_{j-1}\lambda_{i+1,j}| = O\Big(\left(\frac{ijr_is_j}{R_iS_j}\right)^{k-1}\left(\frac{p_i}{P_i}\right)^{k}\Big).$$

$$(xviii)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{r_is_j}{R_iS_j}|R_{i-1}S_{j-1}\lambda_{ij}| = O(R_{m-1}S_{n-1}),\quad and$$

$$(xix)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnr_ms_n}{R_mS_n}\right)^{k-1}\frac{r_ms_n}{R_mR_{m-1}S_nS_{n-1}}|(R_{i-1}S_{j-1}\lambda_{ij}|O\Big(\left(\frac{ijr_is_j}{R_iS_j}\right)^{k-1}\Big).$$

Then $\sum\sum a_{mn}$ *summable* $|\overline{N},p_mq_n|_k$ *implies that* $\sum\sum\lambda_{ij}a_{ij}$ *is summable* $|\overline{N},r_ms_n|_k, k\geqslant 1$.

Proof. From (7) it follows that

$$\hat{b}_{m-1,n-1,i,j} = \hat{c}_{mi}\hat{d}_{nj} = \frac{r_ms_nR_{i-1}S_{j-1}}{R_mR_{m-1}S_mS_{m-1}}.$$

Conditions (i) - (xix) of Corollary 1 follow from Theorem 2 by setting $B = (\overline{N},r_ms_n)$.

THEOREM 3. *Let A be a factorable double weighted mean method, B a factorable double triangular summability method satisfying conditions*

$$(i)\quad \frac{P_mQ_nb_{mnmn}}{p_mq_n} = O(1),$$

$$(ii)\quad (b_{ijij}-b_{i+1,j,i,j}) = O(b_{i0i0}b_{i+1,j,i+1,j}),(b_{ijij}-b_{i,j+1,i,j}) = O(b_{0j0j}b_{i,j+1,i,j+1}),$$

$$(iii)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}|\Delta_{ij}(P_{i-1}Q_{j-1}\lambda_{ij})| = O(P_{m-1}Q_{n-1}),$$

$$(iv)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{k-1}\frac{p_mq_n}{P_mP_{m-1}Q_nQ_{n-1}}|\Delta_{ij}(P_{i-1}Q_{j-1}\lambda_{ij})| = O\Big((ij)^{k-1}\left(\frac{p_iq_j}{P_iQ_j}\right)^{k}\Big),$$

$$(v)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}b_{i0i0}|\Delta_{0j}(P_{i-1}Q_{j-1}\lambda_{i+1,j})| = O(P_{m-1}Q_{n-1}),$$

$$(vi)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{k-1}\frac{p_mq_n}{P_mP_{m-1}Q_nQ_{n-1}}|\Delta_{0j}(P_{i-1}Q_{j-1}\lambda_{i+1,j})| = O\Big(\left(\frac{ijp_iq_j}{P_iQ_j}\right)^{k-1}b_{0j0j}\Big),$$

$$(vii)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}b_{0j0j}|\Delta_{i0}(P_{i-1}Q_{j-1})\lambda_{i,j+1}| = O(P_{m-1}Q_{n-1}),$$

$$(viii)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{k-1}\frac{p_mq_n}{P_mP_{m-1}Q_nQ_{n-1}}|\Delta_{i0}(P_{i-1}Q_{j-1}\lambda_{i,j+1})| = O\Big(\left(\frac{ijp_iq_j}{P_iQ_j}\right)^{k-1}b_{i0i0}\Big),$$

$$(ix)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{p_iq_j}{P_iQ_j}|P_iQ_j\lambda_{i+1,j+1}| = O(P_{m-1}Q_{n-1}),$$

$$(x)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{k-1}\frac{p_mq_n}{P_mP_{m-1}Q_nQ_{n-1}}|(P_iQ_j\lambda_{i+1,j+1})| = O\Big(\Big(\frac{ijp_iq_j}{P_iQ_j}\Big)^{k-1}\Big),$$

$$(xi)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}|b_{i0i0}||\Delta_{0j}(P_{i-1}Q_{j-1}\lambda_{ij})| = O(P_{m-1}Q_{n-1}),$$

$$(xii)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{k-1}\frac{p_mq_n}{P_mP_{m-1}Q_nQ_{n-1}}|\Delta_{0j}(P_{i-1}Q_{j-1}\lambda_{ij})|O\Big(\Big(\frac{ijp_iq_j}{P_iQ_j}\Big)^{k-1}b_{0j0j}\Big),$$

$$(xiii)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}|b_{0j0j}||\Delta_{i0}(P_{i-1}Q_{j-1}\lambda_{ij})| = O(P_{m-1}Q_{n-1}),$$

$$(xiv)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{k-1}\frac{p_mq_n}{P_mP_{m-1}Q_nQ_{n-1}}|\Delta_{i0}(P_{i-1}Q_{j-1}\lambda_{ij})|O\Big(\Big(\frac{ijp_iq_j}{P_iQ_j}\Big)^{k-1}|b_{i0i0}|\Big),$$

$$(xv)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{p_iq_j}{P_iQ_j}P_{i-1}Q_j|\lambda_{i,j+1}| = O(P_{m-1}Q_{n-1}),$$

$$(xvi)\quad \sum_{n=j+1}^{\infty}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{k-1}\frac{p_mq_n}{P_mP_{m-1}Q_nQ_{n-1}}|(P_{i-1}Q_j\lambda_{i,j+1})| = O\Big(\Big(\frac{ijp_iq_j}{P_iQ_j}\Big)^{k-1}|b_{0j0j}|^{-k}\Big),$$

$$(xvii)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{p_iq_j}{P_iQ_j}|P_iQ_{j-1}\lambda_{i+1,j}| = O(P_{m-1}Q_{n-1}),$$

$$(xviii)\quad \sum_{n=j+1}^{\infty}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{k-1}\frac{p_mq_n}{P_mP_{m-1}Q_nQ_{n-1}}|(P_iQ_{j-1}\lambda_{i+1,j})| = O\Big(\Big(\frac{ijp_iq_j}{P_iQ_j}\Big)^{k-1}|b_{i0i0}|^{-k}\Big),$$

$$(xix)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{p_iq_j}{P_iQ_j}|P_{i-1}Q_{j-1}\lambda_{ij}| = O(P_{m-1}Q_{n-1}),$$

$$(xx)\quad \sum_{n=j+1}^{\infty}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{k-1}\frac{p_mq_n}{P_mP_{m-1}Q_nQ_{n-1}}|(P_{i-1}Q_{j-1}\lambda_{ij})| = O\Big(\Big(\frac{ijp_iq_j}{P_iQ_j}\Big)^{k-1}\Big),$$

$$(xxi)\quad \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(ij)^{k-1}\Big|\sum_{r=0}^{i-2}\hat{b}_{i-1,j-1,r,j}X_{rj}\Big|^{k} = O(1),$$

$$(xxii)\quad \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(ij)^{k-1}\Big|\sum_{s=0}^{j-1}\hat{b}_{i-1,j-1,i,s}X_{is}\Big|^{k} = O(1), \quad \textit{and}$$

$$(xxiii)\quad \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(ij)^{k-1}\Big|\sum_{r=0}^{i-1}\sum_{s=0}^{j-2}\hat{b}_{i-1,j-1,r,s}X_{rs}\Big|^{k} = O(1).$$

Then $\sum\sum a_{ij}$ *summable* $|\overline{N},p_mq_n|_k$ *implies that* $\sum\sum a_{ij}\lambda_{ij}$ *is summable* $|B|_k, k\geqslant 1$.

Proof. From (7), and from the proof of Corollary 2 of [1],

$$\hat{a}_{m-1,n-1,i,j} = \frac{p_mP_{i-1}}{P_mP_{m-1}}\frac{q_nQ_{j-1}}{Q_nQ_{n-1}}.$$

Making these substitutions into the conditions of Theorem 1 yields the conditions of Theorem 3.

Every double summability factor theorem of the type in this paper leads to a double inclusion theorem by setting each $\lambda_{mn} = 1$.

THEOREM 4. *Let A and B be doubly infinite triangles, B factorable, satisfying*

(i) $|b_{mnmn}| = O(|a_{mnmn}|)$,

(ii) $(b_{ijij} - b_{i+1,j,i,j}) = O(b_{i0i0}b_{i+1,j,i+1,j}), (b_{ijij} - b_{i.,j+1,i,j}) = O(b_{0j0j}b_{i,j+1,i,j+1})$,

$$(iii)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}|\Delta_{ij}(\hat{a}_{m-1,n-1,i,j})| = O(|a_{mnmn}|),$$

$$(iv)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\Delta_{ij}(\hat{a}_{m-1,n-1,i,j})| = O((ij)^{k-1}|a_{ijij}|^{k}),$$

$$(v)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}|b_{i0i0}||\Delta_{0j}(\hat{a}_{m-1,n-1,i+1,j})| = O(|a_{mnmn}|),$$

$$(vi)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\Delta_{0j}(\hat{a}_{m-1,n-1,i+1,j})| = O((ij|a_{ijij}|)^{k-1}|b_{0j0j}|),$$

$$(vii)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}|b_{0j0j}||\Delta_{i0}(\hat{a}_{m-1,n-1,i,j+1}| = O(|a_{mnmn}|),$$

$$(viii)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\Delta_{i0}(\hat{a}_{m-1,n-1,k,j+1})| = O((ij|a_{ijij}|)^{k-1}|b_{i0i0}|),$$

$$(ix)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}|a_{ijij}||\hat{a}_{m-1,n-1,i+1,j+1}| = O(|a_{mnmn}|),$$

$$(x)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\hat{a}_{m-1,n-1,i+1,j+1}| = O((ij|a_{ijij}|)^{k-1}),$$

$$(xi)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}|b_{i0i0}||\Delta_{0j}(\hat{a}_{m-1,n-1,i,j})| = O(|a_{mnmn}|),$$

$$(xii)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\Delta_{0j}(\hat{a}_{m-1,n-1,i,j})| = O((ij)|a_{ijij}|)^{k-1}|b_{ojoj}|),$$

$$(xiii)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}|b_{0j0j}||\Delta_{i0}(\hat{a}_{m-1,n-1,i,j})| = O(|a_{mnmn}|),$$

$$(xiv)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\Delta_{i0}(\hat{a}_{m-1.n-1,i,j})| = O((ij|a_{ijij}|)^{k-1}|b_{i0i0}|),$$

$$(xv)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}|a_{ijij}||(\hat{a}_{m-1,n-1,i,j+1})| = O(|a_{mnmn}|),$$

$$(xvi)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\hat{a}_{m-1,n-1,i,j+1}| = O((ij|a_{ijij}|)^{k-1}|b_{0j0j}|^{-k}),$$

(xvii) $\sum_{i=0}^{m}\sum_{j=0}^{n}|a_{ijij}||\hat{a}_{m-1,n-1,i+1,j}| = O(|a_{mnmn}|)$,

(xviii) $\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\hat{a}_{m-1,n-1,i+1,j}| = O((ij|a_{ijij}|)^{k-1}|b_{ioio}|^{-k})$,

(xix) $\sum_{i=0}^{m}\sum_{j=0}^{n}|a_{ijij}||\hat{a}_{m-1,n-1,i,j}| = O(|a_{mnmn}|)$,

(xx) $\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mn|a_{mnmn}|)^{k-1}|\hat{a}_{m-1,n-1,i,j}| = O(ij|a_{ijij}|)^{k-1})$,

(xxi) $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(ij)^{k-1}\left|\sum_{r=0}^{i-2}\hat{b}_{i-1,j-1,r,j}X_{rj}\right|^{k} = O(1)$,

(xxii) $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(ij)^{k-1}\left|\sum_{s=0}^{j-1}\hat{b}_{i-1,j-1,i,s}X_{is}\right|^{k} = O(1)$,

(xxiii) $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(ij)^{k-1}\left|\sum_{r=0}^{i-1}\sum_{s=0}^{j-2}\hat{b}_{i-1,j-1,r,s}X_{rs}\right|^{k} = O(1)$.

Then $\sum\sum a_{ij}$ *summable* $|A|_k$ *implies that* $\sum\sum a_{ij}$ *is summable* $|B|_k$, *where* X_{mn} *is as defined in* (5).

THEOREM 5. *Suppose that a double factorable positive sequence* $\{p_{mn}\}$ *and a positive matrix* B *satisfy*

(i) $P_{mn}b_{mnmn} = O(p_{mn})$,

(ii) $\sum_{i=0}^{m}\sum_{j=0}^{n}|\Delta_{ij}(\hat{b}_{m-1,n-1,i,j})| = O(b_{mnmn})$,

(iii) $\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mnb_{mnmn})^{k-1}|\Delta_{ij}(\hat{b}_{m-1,n-1,i,j})| = O\Big((ij)^{k-1}|b_{ijij}|^{k}\Big)$,

(iv) $\sum_{i=0}^{m}\sum_{j=0}^{n}\left(\frac{p_i}{P_i}\right)|\Delta_{0j}\hat{b}_{m-1,n-1,i+1,j}| = O(b_{mnmn})$,

(v) $\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mnb_{mnmn})^{k-1}|\Delta_{0j}(\hat{b}_{m-1,n-1,i+1,j})| = \Big((ij|b_{ijij}|)^{k-1}\frac{q_j}{Q_j}\Big)$,

(vi) $\sum_{i=0}^{m}\sum_{j=0}^{n}\left(\frac{q_j}{Q_j}\right)|\Delta_{i0}(\hat{b}_{m-1,n-1,i,j+1}| = O(b_{mnmn})$,

(vii) $\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mnb_{mnmn})^{k-1}|\Delta_{i0}(\hat{b}_{m-1,n-1,i,j+1})| = O\Big((ijb_{ijij})^{k-1}\frac{p_i}{P_i}\Big)$,

(viii) $\sum_{i=0}^{m}\sum_{j=0}^{n}b_{ijij}|\hat{b}_{m-1,n-1,i+1,j+1}| = O(b_{mnmn})$,

$$(ix)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mnb_{mnmn})^{k-1}|\hat{b}_{m-1,n-1,i+1,j+1}| = O\Big((ijb_{ijij})^{k-1}\Big),$$

$$(x)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{p_i}{P_i}|\Delta_{0j}\hat{b}_{m-1,n-1,i,j}| = O(b_{mnmn}),$$

$$(xi)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mnb_{mnmn})^{k-1}|\Delta_{0j}\hat{b}_{m-1,n-1,i,j}| = O\Big((ijb_{ijij})^{k-1}\frac{q_j}{Q_j}\Big),$$

$$(xii)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{q_j}{Q_j}|\Delta_{i0}(\hat{b}_{m-1,n-1,i,j}| = O(b_{mnmn}),$$

$$(xiii)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mnb_{mnmn})^{k-1}|\Delta_{i0}(\hat{b}_{m-1,n-1,i,j})| = O\Big((ijb_{ijij})^{k-1}\frac{p_i}{P_i}\Big),$$

$$(xiv)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}b_{ijij}|(\hat{b}_{m-1,n-1,i,j+1})| = O(|b_{mnmn}|),$$

$$(xv)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mnb_{mnmn})^{k-1}|\hat{b}_{m-1,n-1,i,j+1}| = O((ijb_{ijij})^{k-1}(\frac{q_j}{Q_j})^k,$$

$$(xvi)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}b_{ijij}|\hat{b}_{m-1,n-1,i+1,j}| = O(|b_{mnmn}|),$$

$$(xvii)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mnb_{mnmn})^{k-1}|\hat{b}_{m-1,n-1,i+1,j}| = O((ijb_{ijij}|)^{k-1}(\frac{p_i}{P_i})^k),$$

$$(xvii)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}b_{ijij}|\hat{b}_{m-1,n-1,i,j})| = O(b_{mnmn}),\quad and$$

$$(xix)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}(mnb_{mnmn})^{k-1}|\hat{b}_{m-1,n-1,i,j}| = O((ijb_{ijij})^{k-1}),$$

If $\sum\sum a_{ij}$ *is summable* $|\overline{N},p_{mn}|_k$ *, then* $\sum\sum a_{ij}$ *is also summable* $|B|_k, k \geqslant 1$.

COROLLARY 2. *Suppose that* $B = (\overline{N}, r_m s_n), A = (\overline{N}, p_m q_n)$. *Then, if*

$$(i)\quad P_m Q_n r_m s_n = O(p_m q_n R_m S_m),$$

$$(ii)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}|\Delta_{ij}(R_{i-1}S_{j-1})| = O(R_{m-1}S_{n-1}),$$

$$(iii)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\Big(\frac{mnr_m s_n}{R_m S_n}\Big)^{k-1}\frac{r_m s_n}{R_m R_{m-1} S_m S_{n-1}}|\Delta_{ij}(R_{i-1}S_{j-1}| = O\Big((ij)^{k-1}\Big(\frac{r_i s_j}{R_i S_j}\Big)^k\Big),$$

$$(iv)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{p_i}{P_i}|\Delta_{0j}(R_{i-1}S_{j-1}| = O(R_{m-1}S_{n-1}),$$

$$(v)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\Big(\frac{mnr_m s_n}{R_m S_n}\Big)^{k-1}\frac{r_m s_n}{R_m R_{m-1} S_m S_{n-1}}|\Delta_{0j}(R_{i-1}S_{j-1}| = O\Big(\Big(\frac{ijr_i s_j}{R_i S_j}\Big)^{k-1}\frac{q_j}{Q_j}\Big),$$

$$(vi)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\left(\frac{q_j}{Q_j}\right)|\Delta_{i0}(R_{i-1}S_{j-1}| = O(R_{m-1}S_{n-1}),$$

$$(vii)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnr_ms_n}{R_mS_n}\right)^{k-1}\frac{r_ms_n}{R_mR_{n-1}S_nS_{n-1}}|\Delta_{i0}(R_{i-1}S_{j-1})| = O\left(\left(\frac{ijr_is_j}{R_iS_j}\right)^{k-1}\frac{p_i}{P_i}\right),$$

$$(viii)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{r_is_j}{R_iS_j}|R_iS_j| = O(R_{m-1}S_{n-1}),$$

$$(ix)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnr_ms_n}{R_mS_n}\right)^{k-1}\frac{r_ms_n}{R_mR_{m-1}S_nS_{n-1}}|R_iS_j| = O\left(\left(\frac{ijr_is_j}{R_iS_j}\right)^{k-1}\right),$$

$$(x)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{p_i}{P_i}|\Delta_{0j}(R_{i-1}S_{j-1})| = O(R_{m-1}S_{n-1}),$$

$$(xi)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnr_ms_n}{R_mS_n}\right)^{k-1}\frac{r_ms_n}{R_mR_{m-1}S_nS_{n-1}}|\Delta_{0j}(R_{i-1}S_{j-1})|\left(\left(\frac{ijr_is_j}{R_iS_j}\right)^{k-1}\frac{q_j}{Q_j}\right),$$

$$(xii)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{q_j}{Q_j}|\Delta_{i0}(R_{i-1}S_{j-1})| = O(R_{m-1}S_{n-1}),$$

$$(xiii)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnr_ms_n}{R_mS_n}\right)^{k-1}\frac{r_ms_n}{R_mR_{m-1}S_nS_{n-1}}|\Delta_{i0}(R_{i-1}S_{j-1}|O\left(\left(\frac{ijr_is_j}{R_iS_j}\right)^{k-1}\frac{p_i}{P_i}\right),$$

$$(xiv)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{r_is_j}{R_iS_{j-1}}|R_{i-1}S_j| = O(R_{m-1}S_{n-1}),$$

$$(xv)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnr_ms_n}{R_mS_n}\right)^{k-1}\frac{r_ms_n}{R_mR_{m-1}S_nS_{n-1}}|(R_{i-1}S_j|O\left(\left(\frac{ijr_is_j}{R_iS_j}\right)^{k-1}\left(\frac{q_j}{Q_j}\right)^{k}\right).$$

$$(xvi)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{r_is_j}{R_iS_j}|R_iS_{j-1}| = O(R_{m-1}S_{n-1}),$$

$$(xvii)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnr_ms_n}{R_mS_n}\right)^{k-1}\frac{r_ms_n}{R_mR_{m-1}S_nS_{n-1}}|(R_iS_{j-1}|O\left(\left(\frac{ijr_is_j}{R_iS_j}\right)^{k-1}\left(\frac{p_i}{P_i}\right)^{k}\right).$$

$$(xviii)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{r_is_j}{R_iS_j}|R_{i-1}S_{j-1}| = O(R_{m-1}S_{n-1}),\quad and$$

$$(xix)\quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnr_ms_n}{R_mS_n}\right)^{k-1}\frac{r_ms_n}{R_mR_{m-1}S_nS_{n-1}}|(R_{i-1}S_{j-1}|O\left(\left(\frac{ijr_is_j}{R_iS_j}\right)^{k-1}\right).$$

Then $\sum\sum a_{mn}$ *summable* $|\overline{N}, p_mq_n|_k$ *implies that* $\sum\sum a_{ij}$ *is summable* $|\overline{N}, r_ms_n|_k, k \geqslant 1$.

THEOREM 6. *Let* A *be a factorable double weighted mean method,* B *a factorable double triangular summability method satisfying conditions*

$$(i)\quad \frac{P_mQ_nb_{mnmn}}{p_mq_n} = O(1),$$

$$(ii) \quad (b_{ijij} - b_{i+1,j,i,j}) = O(b_{i0i0}b_{i+1,j,i+1,j}),$$
$$(b_{ijij} - b_{i,j+1,i,j}) = O(b_{0j0j}b_{i,j+1,i,j+1}),$$

$$(iii) \quad \sum_{i=0}^{m}\sum_{j=0}^{n}|\Delta_{ij}(P_{i-1}Q_{j-1})| = O(P_{m-1}Q_{n-1}),$$

$$(iv) \quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{k-1}\frac{p_mq_n}{P_mP_{m-1}Q_nQ_{n-1}}|\Delta_{ij}(P_{i-1}Q_{j-1})| = O\Big((ij)^{k-1}\Big(\frac{p_iq_j}{P_iQ_j}\Big)^k\Big),$$

$$(v) \quad \sum_{i=0}^{m}\sum_{j=0}^{n}b_{i0i0}|\Delta_{0j}(P_{i-1}Q_{j-1})| = O(P_{m-1}Q_{n-1}),$$

$$(vi) \quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{k-1}\frac{p_mq_n}{P_mP_{m-1}Q_nQ_{n-1}}|\Delta_{0j}(P_{i-1}Q_{j-1})| = O\Big(\Big(\frac{ijp_iq_j}{P_iQ_j}\Big)^{k-1}b_{0j0j}\Big),$$

$$(vii) \quad \sum_{i=0}^{m}\sum_{j=0}^{n}b_{0j0j}|\Delta_{i0}(P_{i-1}Q_{j-1})| = O(P_{m-1}Q_{n-1}),$$

$$(viii) \quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{k-1}\frac{p_mq_n}{P_mP_{m-1}Q_nQ_{n-1}}|\Delta_{i0}(P_{i-1}Q_{j-1})| = O\Big(\Big(\frac{ijp_iq_j}{P_iQ_j}\Big)^{k-1}b_{i0i0}\Big),$$

$$(ix) \quad \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{p_iq_j}{P_iQ_j}|P_iQ_j| = O(P_{m-1}Q_{n-1}),$$

$$(x) \quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{k-1}\frac{p_mq_n}{P_mP_{m-1}Q_nQ_{n-1}}|(P_iQ_j)| = O\Big(\Big(\frac{ijp_iq_j}{P_iQ_j}\Big)^{k-1}\Big),$$

$$(xi) \quad \sum_{i=0}^{m}\sum_{j=0}^{n}|\Delta_{0j}(P_{i-1}Q_{j-1})| = O(P_{m-1}Q_{n-1}),$$

$$(xii) \quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{k-1}\frac{p_mq_n}{P_mP_{m-1}Q_nQ_{n-1}}|\Delta_{0j}(P_{i-1}Q_{j-1})|O\Big(\Big(\frac{ijp_iq_j}{P_iQ_j}\Big)^{k-1}b_{0j0j}\Big),$$

$$(xiii) \quad \sum_{i=0}^{m}\sum_{j=0}^{n}b_{0j0j}|\Delta_{i0}(P_{i-1}Q_{j-1})| = O(P_{m-1}Q_{n-1}),$$

$$(xiv) \quad \sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{k-1}\frac{p_mq_n}{P_mP_{m-1}Q_nQ_{n-1}}|\Delta_{0j}(P_{i-1}Q_{j-1})|O\Big(\Big(\frac{ijp_iq_j}{P_iQ_j}\Big)^{k-1}b_{i0i0}\Big),$$

$$(xv) \quad \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{p_iq_j}{P_iQ_j}P_{i-1}Q_j|| = O(P_{m-1}Q_{n-1}),$$

$$(xvi) \quad \sum_{n=j+1}^{\infty}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{k-1}\frac{p_mq_n}{P_mP_{m-1}Q_nQ_{n-1}}|(P_{i-1}Q_j)| = O\Big(\Big(\frac{ijp_iq_j}{P_iQ_j}\Big)^{k-1}|b_{0j0j}|^{-k}\Big),$$

$$(xvii) \quad \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{p_iq_j}{P_iQ_j}P_iQ_{j-1} = O(P_{m-1}Q_{n-1}),$$

$$(xviii) \quad \sum_{n=j+1}^{\infty}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{k-1}\frac{p_mq_n}{P_mP_{m-1}Q_nQ_{n-1}}|(P_iQ_{j-1} = O\Big(\Big(\frac{ijp_iq_j}{P_iQ_j}\Big)^{k-1}|b_{i0i0}|^{-k}\Big),$$

$$(xix)\quad \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{p_iq_j}{P_iQ_j}P_{i-1}Q_{j-1}=O(P_{m-1}Q_{n-1}),$$

$$(xx)\quad \sum_{n=j+1}^{\infty}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{k-1}\frac{p_mq_n}{P_mP_{m-1}Q_nQ_{n-1}}|P_{i-1}Q_{j-1}|=O\Big(\Big(\frac{ijp_iq_j}{P_iQ_j}\Big)^{k-1}\Big),$$

$$(xxi)\quad \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(ij)^{k-1}\Big|\sum_{r=0}^{i-2}\hat{b}_{i-1,j-1,r,j}X_{rj}\Big|^k=O(1),$$

$$(xxii)\quad \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(ij)^{k-1}\Big|\sum_{s=0}^{j-1}\hat{b}_{i-1,j-1,i,s}X_{is}\Big|^k=O(1),\quad \textit{and}$$

$$(xxiii)\quad \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(ij)^{k-1}\Big|\sum_{r=0}^{i-1}\sum_{s=0}^{j-2}\hat{b}_{i-1,j-1,r,s}X_{rs}\Big|^k=O(1).$$

Then $\sum\sum a_{ij}$ *summable* $|\overline{N},p_mq_n|_k$ *implies that* $\sum\sum a_{ij}$ *is summable* $|B|_k, k\geqslant 1$.

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