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GENERALIZED MONOTONE ITERATIVE METHOD FOR INTEGRO DIFFERENTIAL EQUATIONS WITH PERIODIC BOUNDARY CONDITIONS

IANNA H. WEST AND A. S. VATSALA

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Abstract. In this paper, we will develop a generalized monotone iterative method for first order nonlinear integro differential equations with periodic boundary conditions when the forcing function is the sum of an increasing and decreasing function. We obtain natural monotone sequences or alternating monotone sequences depending on the coupled upper and lower solution used and depending on the iterative scheme used to develop the sequence. These sequences converge to coupled extremal solutions of the integro differential equation.

1. Introduction

It is well known that the method of lower and upper solutions coupled with the monotone iterative technique is used to obtain the existence of extremal solutions for both nonlinear initial value problems and nonlinear boundary value problems. In recent years, the method has been used to obtain existence of solutions to nonlinear integro differential equations of the form

$$u'(t) = f(t, u(t), Tu(t)), \quad u(0) = u(2\pi).$$

See [5] for details. In this paper, we extend this method to nonlinear integro differential equations with nonlinear periodic boundary conditions of the form

$$u' = f(t, u(t), Tu(t)) + g(t, u(t), Tu(t)), \quad u(0) = u(2\pi) \text{ on } J = [0, 2\pi], \tag{1.1}$$

where $f, g \in C[J \times R \times R, R]$, f is increasing in u and Tu, and g is decreasing in u and Tu. Also, $Tu(t) = \int_0^t K(t,s)u(s)ds$ and $K \in C[J \times J, R_+]$.

It is known that equation (1.1) possess four types of lower and upper solutions as in [7, 8]. However, we recall only the two types of coupled upper and lower solutions which we need to develop our results. They are defined as follows:

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DEFINITION 1.1. The functions $v, w \in C^1[J,R]$ are said to be
(i) coupled lower and upper solutions of type I, if

$$v' \leqslant f(t,v(t),Tv(t)) + g(t,w(t),Tw(t)), \quad v(0) \leqslant v(2\pi) \;\; on \;\; J$$
$$w' \geqslant f(t,w(t),Tw(t)) + g(t,v(t),Tv(t)), \quad w(0) \geqslant w(2\pi) \;\; on \;\; J;$$

(ii) coupled lower and upper solutions of type II, if

$$v' \leqslant f(t,w(t),Tw(t)) + g(t,v(t),Tv(t)), \quad v(0) \leqslant v(2\pi) \;\; on \;\; J$$
$$w' \geqslant f(t,v(t),Tv(t)) + g(t,w(t),Tw(t)), \quad w(0) \geqslant w(2\pi) \;\; on \;\; J.$$

We will obtain two results using equation (1.1) when we consider coupled lower and upper solutions of type 1 or type II. Additionally, we will show that type II can be easily constructed when f is nondecreasing in u and Tu, and where g is nonincreasing in u and Tu. Using an appropriate iterative scheme and coupled lower and upper solutions of type I, we obtain natural sequences which converge to coupled extremal solutions of (1.1). Similarly, we also prove the existence of coupled extremal solutions of (1.1) using coupled lower and upper solutions of type II and an appropriate iterative scheme. In the latter case, we obtain intertwined alternating sequences. Our result generalizes the earlier known results on the monotone method for integro differential equations with periodic boundary conditions. See [3] for details. Further, we prove the coupled extremal solutions reduce to the unique solution of (1.1) under suitable uniqueness assumptions of f and g. We provide a numerical example to demonstrate the use of our generalized monotone method.

For each theorem, we will develop monotone sequences using one of two types of iterative schemes. Moreover, we will develop natural monotone sequences or alternating sequences which converge to coupled extremal solutions of (1.1).

2. Preliminaries

In this section, we recall some known results, see [5], relative to the following integro equation

$$\begin{aligned} u' &= f(t,u(t),Tu(t)), \\ u(0) &= u(2\pi) \;\; \text{on} \;\; J = [0,2\pi], \end{aligned} \tag{2.1}$$

where $f \in C[J \times R \times R, R]$, $Tu(t) = \int_0^t K(t,s)u(s)ds$, and $K \in C[J \times J, R_+]$, which we need in our main results.

We will also recall an important comparison result and an existence result using Schauder's fixed point theorem, which are needed in our main results. We merely state the theorems without proof. See [5] for details.

LEMMA 1. *Let $p \in C^1[J,R]$ be such that*

$$\begin{aligned} p' &\leqslant -Mp - NTp, \\ p(0) &\leqslant p(2\pi), \end{aligned} \tag{2.2}$$

where $M > 0, N \geqslant 0$. Then $p(t) \leqslant 0$ for $0 \leqslant t \leqslant 2\pi$ provided the following condition holds $2Nk_0\pi(e^{2M\pi} - 1) \leqslant M$, where $0 \leqslant k_0 = maxK(t,s)$ for $(t,s) \in [0,2\pi] \times [0,2\pi]$ and $K(t,s) \geqslant 0$. If the inequalities are reversed, then $p(t) \geqslant 0$ on J.

THEOREM 2.1. (Schauder) *If E is a closed, bounded, convex subset of a Banach space B and $T : E \to E$ is completely continuous, then T has a fixed point.*

THEOREM 2.2. *Assume that $f \in C[J \times R^n, R^n]$, $K \in C[J \times J \times R^n, R^n]$ and $\int_s^t |K(\sigma, s, u(s))| d\sigma \leqslant N$, for $t_0 \leqslant s \leqslant t \leqslant t_0 + a$, $u \in \Omega = \phi \in C[J, R^n] : \phi(t_0) = u_0$ and $|\phi(t) - u_0| \leqslant b$.*

Then equation (1.1) possesses at least one solution $u(t)$ on $t_0 \leqslant t \leqslant t_0 \leqslant t_0 + \alpha$, for some $0 < \alpha \leqslant a$.

3. Main results

The usual monotone method developed in literature proves the existence of extremal solutions of equation (2.1) when f is nondecreasing in u and Tu, or when f can be made nondecreasing by adding a linear term to u and Tu respectively. This is precisely the onesided Lipschitz condition of f in u and Tu. In this section, we extend the monotone method to equation (1.1) where f is nondecreasing in u and Tu, and g is nonincreasing in u and Tu.

To develop our theorems, we will consider equation (1.1), and relative to the lower and upper solutions $v, w \in C^1[J, R]$ of (1.1) defined in the introduction section of this paper, we list the following assumptions for convenience.

(A_0) Assume $v_0, w_0 \in C^1[J, R]$ are the coupled lower and upper solutions of (1.1) with $v_0(t) \leqslant u \leqslant w_0$ and $v_0 \leqslant Tu \leqslant w_0$ on J.

(A_1) Assume $f, g \in C[J \times R, R]$, and $M_1 + M_2 > 0$, $N_1 + N_2 \geqslant 0$ satisfying

$$2(N_1 + N_2)k_0\pi(e^{2(M_1+M_2)\pi} - 1) \leqslant M_1 + M_2$$

where k_0 is the $\max K(t, s)$ on $[0, 2\pi] \times [0, 2\pi]$, and either

(i) $f(t, u, Tu) = F(t, u, Tu) - M_1u - N_1Tu$ is nondecreasing in u and Tu, and $g(t, u, Tu) = G(t, u, Tu) - M_2u - N_2Tu$, where G is nonincreasing in u and Tu, or

(ii) $f(t, u, Tu)$ is nondecreasing in u and Tu, and $g(t, u, Tu) = G(t, u, Tu) - M_2u - N_2Tu$, where G is nonincreasing in u and Tu or

(iii) $f(t, u, Tu) = F(t, u, Tu) - M_1u - N_1u$ is nondecreasing in u and Tu, and $g(t, u, Tu)$ is nonincreasing in u and Tu.

THEOREM 3.1. *Assume that (A_0) and $(A_1)(i)$ hold. If $u(t)$ is any solution of equation (1.1) with $v_0(t) \leqslant u \leqslant w_0(t)$ where $v = v_0$, $w = w_0$ are lower and upper solutions of type I, then there exists natural monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ on J such that $v_n(t) \longrightarrow \rho(t)$ and $w_n(t) \longrightarrow r(t)$ uniformly and monotonically and (ρ, r) are coupled minimal and maximal solutions respectively to equation (1.1). That is, (ρ, r) satisfy*

$$\begin{aligned} \rho' &= F(t, \rho, T\rho) - M_1\rho - N_1T\rho + G(t, r, Tr) - M_2\rho - N_2T\rho \\ &= f(t, \rho, T\rho) - M_2\rho - N_2T\rho + G(t, r, Tr), \\ \rho(0) &= \rho(2\pi), \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} r' &= F(t,r,Tr) - M_2 r - N_2 Tr + G(t,\rho,T\rho) - M_2 r - N_2 Tr \\ &= f(t,r,Tr) - M_2 r - N_2 Tr + G(t,\rho,T\rho), \\ r(0) &= r(2\pi). \end{aligned} \tag{3.2}$$

Also $\rho \leqslant u \leqslant r$ *on* J*, where the iterative schemes are given by the following linear integro differential equations with periodic boundary conditions:*

$$\begin{aligned} v'_{n+1} &= F(t,v_n,Tv_n) - M_1 v_{n+1} - N_1 Tv_{n+1} + G(t,w_n,Tw_n) - M_2 v_{n+1} - N_2 Tv_{n+1}, \\ v_{n+1}(0) &= v_{n+1}(2\pi) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} w'_{n+1} &= F(t,w_n,Tw_n) - M_1 w_{n+1} - N_1 Tw_{n+1} + G(t,v_n,Tv_n) - M_2 w_{n+1} - N_2 Tw_{n+1}, \\ w_{n+1}(0) &= w_{n+1}(2\pi) \end{aligned} \tag{3.4}$$

Proof. For any $v_{n+1} \in C[[0,2\pi],R]$ such that $v_0 \leqslant v_{n+1} \leqslant w_0$, the linear integro differential equation (3.3) reduces to a simpler linear integro differential equation given by

$$v'_{n+1} + Mv_{n+1} = -NTv_{n+1} + \sigma_1(t), \quad v_{n+1}(0) = v_{n+1}(2\pi), \tag{3.5}$$

where $M = (M_1 + M_2)$, $N = (N_1 + N_2)$ and $\sigma_1(t) = F(t,v_n,Tv_n) + G(t,w_n,Tw_n)$. Furthermore, for any $w_{n+1} \in C[0,2\pi],R]$ such that $v_0 \leqslant w_{n+1} \leqslant w_0$, the linear integro differential equation (3.4) reduces to

$$w'_{n+1} + \tilde{M}w_n = -\tilde{N}Tw_{n+1} + \sigma_2(t), \quad w_{n+1}(0) = w_{n+1}(2\pi), \tag{3.6}$$

where $\tilde{M} = (M_1 + M_2)$, $\tilde{N} = (N_1 + N_2)$ and $\sigma_2(t) = F(t,w_n,Tw_n) + G(t,v_n,Tv_n)$.

Then, using the method of variation of parameters and the boundary condition $v_{n+1}(0) = v_{n+1}2\pi)$, we get

$$\begin{aligned} v_{n+1}(t) = e^{-Mt}\{&\frac{1}{e^{2M\pi}-1}\int_0^{2\pi}[\sigma(s) - N\int_0^s K(s,\xi)v_n(\xi)d\xi]e^{Ms}ds\} \\ &+ e^{-Mt}\int_0^t[\sigma(s) - N\int_0^s K(s,\xi)v_n(\xi)d\xi]e^{Ms}ds. \end{aligned} \tag{3.7}$$

In view of the condition $2Nk_0\pi(e^{2M\pi}-1) \leqslant M$, applying Theorem 2.2 to (3.7), we can show that there exists a solution $v_{n+1}(t)$ for equation (3.5). Similarly, we can show that there is a solution $w_{n+1}(t)$ to equation (3.6).

We claim that the solutions of equations (3.5) and (3.6) are unique. For this purpose, let u,v be two distinct solutions of either (3.5) or (3.6), and let $p(t) = u(t) - v(t)$, then we get

$$\begin{aligned} p'(t) &= u'(t) - v'(t) \\ &= -Mu(t) - NTu(t) + \sigma(t) + Mv(t) + NTv(t) - \sigma(t) \\ &= -Mp - NTp, \end{aligned}$$

where $p(0) = p(2\pi)$. Hence by Lemma 1 it follows $u \equiv v$.

Now our aim is to show that

$$v_0 \leqslant v_1 \leqslant v_2 \leqslant ... \leqslant v_k \leqslant w_k \leqslant ... \leqslant w_2 \leqslant w_1 \leqslant w_0 \ \ on \ \ J. \tag{3.8}$$

Since v_0 is a lower solution and $v_0 \leqslant w_0$, we obtain

$$\begin{aligned} v_0' &\leqslant F(t, v_0, Tv_0) - M_1 v_0 - N_1 T v_0 + G(t, w_0, Tw_0) - M_2 w_0 - N_2 T w_0 \\ &\leqslant F(t, v_0, Tv_0) - M_1 v_0 - N_1 T v_0 + G(t, w_0, Tw_0) - M_2 v_0 - N_2 T v_0. \end{aligned}$$

and since w_0 is an upper solution, we obtain

$$\begin{aligned} w_0' &\geqslant F(t, w_0, Tw_0) - M_1 w_0 - N_1 T w_0 + G(t, v_0, Tv_0) - M_2 v_0 - N_2 T v_0 \\ &\geqslant F(t, w_0, Tw_0) - M_1 w_0 - N_1 T w_0 + G(t, v_0, Tv_0) - M_2 w_0 - N_2 T w_0. \end{aligned}$$

Then, our claim is to show that $v_0 \leqslant v_1$. For this purpose let $p(t) = v_0 - v_1$, then

$$\begin{aligned} p'(t) = v_0' - v_1' &\leqslant F(t, v_0, Tv_0) - M_1 v_0 - N_1 T v_0 + G(t, w_0, Tw_0) - M_2 v_0 - N_2 v_0 \\ &\quad - F(t, v_0, Tv_0) + M_1 v_1 + N_1 T v_1 - G(t, w_0, Tw_0) + M_2 v_1 + N_2 T v_1 \\ &= -(M_1 + M_2)(v_0 - v_1) - (N_1 + N_2) T (v_0 - v_1) \\ &= -Mp - NTp, \end{aligned}$$

where $(M_1 + M_2) = M > 0$ and $(N_1 + N_2) = N \geqslant 0$. Also $p(0) = v_0(0) - v_1(0) \leqslant v_0(2\pi) - v_1(2\pi) = p(2\pi)$. Hence by Lemma 1, $p(t) \leqslant 0$, which proves that $v_0 \leqslant v_1$ on J. Similarly, we can show that $w_0 \geqslant w_1$. Next, we will show that $v_1 \leqslant w_1$

$$\begin{aligned} p'(t) = v_1' - w_1' &= F(t, v_0, Tv_0) - M_1 v_1 - N_1 T v_1 + G(t, w_0, Tw_0) - M_2 v_1 - N_2 v_1 \\ &\quad - F(t, w_0, Tw_0) + M_1 w_1 + N_1 T w_1 - G(t, v_0, Tv_0) + M_2 w_1 + N_2 T w_1 \\ &\leqslant -(M_1 + M_2)(v_1 - v_2) - (N_1 + N_2) T (v_1 - v_2) \\ &= -Mp - NTp, \end{aligned}$$

by the monotone nature of f and g,where $(M_1+M_2) = M>0$ and $(N_1+N_2) = N\geqslant 0$. Also $p(0) = p(2\pi)$, hence by Lemma 1, $p(t) \leqslant 0$, which proves that $v_1 \leqslant w_1$ Thus, giving us $v_0(t) \leqslant v_1(t) \leqslant w_1(t) \leqslant w_0(t)$ holds on J. Hence (3.8) is true for $k = 1$.

Now assume that (3.8) holds for some $k > 1$, such that,

$$v_{k-1} \leqslant v_k \leqslant w_k \leqslant w_{k-1} \ \ on \ \ J. \tag{3.9}$$

Thus, our aim is to show that (3.8) holds for $k+1$ by proving that

$$v_k \leqslant v_{k+1} \leqslant w_{k+1} \leqslant w_k \tag{3.10}$$

holds on J. For this purpose, let $p(t) = v_k(t) - v_{k+1}(t)$, and note that $p(0) = v_k(0) - v_{k+1}(0) = v_k(2\pi) - v_{k+1}(2\pi) = p(2\pi)$. We get

$$\begin{aligned} p'(t) &= v_k'(t) - v_{k+1}'(t) \\ &= F(t, v_{k-1}, Tv_{k-1}) - M_1 v_k - N_1 T v_k + G(t, w_{k-1}, Tw_{k-1}) - M_2 v_k - N_2 T v_k \\ &\quad - F(t, v_k, Tv_k) + M_1 v_{k+1} + N_1 T v_{k+1} - G(t, w_k, Tw_k) + M_2 v_{k+1} + N_2 T v_{k+1} \\ &\leqslant -(M_1 + M_2)(v_k - v_{k+1}) - (N_1 + N_2) T (v_k - v_{k+1}) \\ &= -Mp - NTp \end{aligned}$$

using (3.9) and the monotone nature of f and g. By Lemma 1 $p(t) \leqslant 0$ which proves $v_k(t) \leqslant v_{k+1}(t)$. Similarly, we can prove that $w_{k+1}(t) \leqslant w_k(t)$ and $v_{k+1}(t) \leqslant w_{k+1}(t)$. This proves that (3.8) holds for $k+1$. Hence (3.8) is valid for all $k = 1, 2 \ldots$.

Also, the sequences $\{v_k(t)\}$ and $\{w_k(t)\}$ can be shown to be equicontinuous and uniformly bounded using (3.7) and a similar form for w_n. Thus by Ascoli-Arzela's Theorem, subsequences $\{v_{n_k}(t)\}, \{w_{n_k}(t)\}$ converge to $\rho(t)$ and $r(t)$ respectively on J. Since both the sequences $\{v_k(t)\}$ and $\{w_k(t)\}$ are monotone, the entire sequences converge uniformly and monotonically to $\rho(t)$ and $r(t)$ respectively on J. Therefore, $\rho(t)$ and $r(t)$ satisfy the integro period boundary value problems (3.1) and (3.2).

Finally, we claim that ρ and r are coupled minimal and maximal solutions of (1.1). Thus we need to show that

$$v_k(t) \leqslant \rho(t) \leqslant u(t) \leqslant r(t) \leqslant w_k(t) \text{ on } J.$$

Suppose that u is any solution of (1.1), such that $v_0(t) \leqslant u(t) \leqslant w_0(t)$ on J. It is easy to show as before using induction that $v_k(t) \leqslant u(t) \leqslant w_k(t)$ on J for all $k \geqslant 1$. Then, taking the limit, we get $\lim_{k\to\infty} v_k(t) = \rho(t)$ and $\lim_{k\to\infty} w_k(t) = r(t)$. This completes the proof.

LEMMA 2. *In addition to the assumptions of Theorem* 3.1*, if for* $u \geqslant \bar{u}$*,* f *and* G *satisfy*

$$f(t,u,Tu) - f(t,\bar{u},T\bar{u}) \leqslant K_1(u-\bar{u}) + L_1T(u-\bar{u}), \quad t \in J;$$
$$G(t,u,Tu) - G(t,\bar{u},T\bar{u}) \geqslant -K_2(u-\bar{u}) - L_2T(u-\bar{u}), \quad t \in J,$$

where $K_\mu > 0, L_\mu \geqslant 0, \ \mu = 0,1$*, then* $\rho = u = r$ *is the unique solution of* (1.1)*, provided* $M_2 - (K_1+K_2) > 0$ *and* $N_2 - (L_1+L_2) \geqslant 0$.

Proof. Since we have $\rho \leqslant r$, it is enough to show that $r \leqslant \rho$. For this purpose, set $p(t) = r - \rho$, we have $p(0) = p(2\pi)$, and

$$\begin{aligned} p'(t) &= r'(t) - \rho'(t) \\ &= f(t,r,Tr) - M_2r - N_2Tr + G(t,\rho,T\rho) - f(t,\rho,T\rho) + M_2\rho + N_2T\rho - G(t,r,Tr) \\ &\leqslant K_1(r-\rho) + L_1T(r-\rho) + K_2(r-\rho) + L_2T(r-\rho) - M_2(r-\rho) - N_2T(r-\rho) \\ &= (K_1+K_2-M_2)(r-\rho) + (L_1+L_2-N_2)T(r-\rho). \end{aligned}$$

This implies that $p'(t) \leqslant -Mp - NTp$, where $(K_1+K_2-M_2) = M$ and $(L_1+L_2-N_2) = N$. Thus $p(t) \leqslant 0$ which proves $r \leqslant \rho$. Therefore, $\rho = u = r$ is the unique solution. This completes the proof.

REMARK 3.1. When f is nondecreasing and g is non-increasing, we can always construct coupled upper and lower solutions of type II. We state this result below as a lemma. Then we will use the constructed upper and lower solutions to develop the next result.

LEMMA 3. *Suppose that* $f(t,u,Tu)$*,* $g(t,u,Tu)$ *are monotone nondecreasing and monotone nonincreasing in* u *respectively on* J*, then there exists coupled lower and upper solutions of type II for equation* (1.1).

Proof. Choose a constant R_0 large enough such that

$$v(t) = z(t) - R_0 \leqslant 0, \;\; v(0) \leqslant v(2\pi) \;\; \text{and} \;\; w(t) = z(t) + R_0 \geqslant 0, \;\; w(0) \geqslant w(2\pi),$$

where $z(t)$ is the solution of

$$z'(t) = f(t,0,0) + g(t,0,0), \;\; z(0) = z(2\pi) \;\; \text{on} \;\; J.$$

Then

$$\begin{aligned} v'(t) &= z'(t) - 0 = f(t,0,0) + g(t,0,0) \leqslant f(t,w,Tw) + g(t,v,Tv), \;\; v(0) \leqslant (2\pi), \\ w'(t) &= z'(t) + 0 = f(t,0,0) + g(t,0,0) \geqslant f(t,v,Tv) + g(t,w,Tw), \;\; w(0) \geqslant w(2\pi). \end{aligned}$$

THEOREM 3.2. *Assume the hypothesis of Lemma 3.2 holds, and let v_0, w_0 be constructed coupled lower and upper solutions respectively of type II with $v_0(t) \leqslant w_0(t)$ on J, and let (A_1)(i) hold. Then for any solution $u(t)$ of equation (1.1) with $v_0(t) \leqslant u \leqslant w_0(t)$ on J, we get the intertwining alternating sequences $\{ v_{2n}, w_{2n+1} \}$ and $\{ w_{2n}, v_{2n+1} \}$ satisfying*

$$v_0 \leqslant w_1 \leqslant \ldots \leqslant v_{2n} \leqslant w_{2n+1} \leqslant u \leqslant v_{2n+1} \leqslant w_{2n} \leqslant \ldots \leqslant v_1 \leqslant w_0 \tag{3.11}$$

for every $n \geqslant 1$, provided $v_0 \leqslant w_1$ and $w_0 \geqslant v_1$, where the iterative schemes are given by

$$v'_{n+1} = F(t,w_n,Tw_n) - M_1 v_{n+1} - N_1 T v_{n+1} + G(t,v_n,Tv_n) - M_2 v_{n+1} - N_2 T v_{n+1}, \tag{3.12}$$

where $v_{n+1}(0) = v_{n+1}(2\pi)$ on J,

$$w'_{n+1} = F(t,v_n,Tv_n) - M_1 w_{n+1} - N_1 T w_{n+1} + G(t,w_n,Tw_n) - M_2 w_{n+1} - N_2 T w_{n+1}, \tag{3.13}$$

where $w_{n+1}(0) = w_{n+1}(2\pi)$ on J.

Moreover, the monotone sequence $\{v_{2n}, w_{2n+1}\}$ converges to ρ and $\{w_{2n}, v_{2n+1}\}$ converges to r on J, where (ρ, r) are coupled minimal and maximal solutions of equation (1.1) respectively, satisfying the coupled system

$$\begin{aligned} \rho' &= F(t,\rho,T\rho) - M_1\rho - N_1 T\rho + G(t,r,Tr) - M_2\rho - N_2 T\rho \\ &= f(t,\rho) - M_2\rho - N_2 T\rho + G(t,r), \end{aligned} \tag{3.14}$$

where $\rho(0) = \rho(2\pi)$ on J,

$$\begin{aligned} r' &= F(t,r,Tr) - M_2 r - N_2 Tr + G(t,\rho,T\rho) - M_2 r - N_2 Tr \\ &= f(t,r) - M_2 r - N_2 Tr + G(t,\rho), \end{aligned} \tag{3.15}$$

where $r(0) = r(2\pi)$ on J.

Proof. Similar to the results of Theorem 3.1, we can show the existence and uniqueness of the solution of the linear integro differential equations (3.12) and (3.13).

Now our aim is to show that equation (3.11) holds for every $n \geqslant 1$. First, we need to show that (3.11) is true for $k = 1$. Thus we need to show that

$$v_0 \leqslant w_1 \leqslant v_2 \leqslant w_3 \leqslant u \leqslant v_3 \leqslant w_2 \leqslant v_1 \leqslant w_0$$

holds on J. Now, since v_0, w_0 are lower and upper solutions respectively, and $v_0 \leqslant w_0$, we obtain

$$\begin{aligned} v_0' &\leqslant F(t, w_0, Tw_0) - M_1 w_0 - N_1 Tw_0 + G(t, v_0, Tv_0) - M_2 v_0 - N_2 Tv_0 \\ &\leqslant F(t, w_0, Tw_0) - M_1 v_0 - N_1 Tv_0 + G(t, v_0, Tv_0) - M_2 v_0 - N_2 Tv_0. \end{aligned}$$

and

$$\begin{aligned} w_0' &\geqslant F(t, v_0, Tv_0) - M_1 v_0 - N_1 Tv_0 + G(t, w_0, Tw_0) - M_2 w_0 - N_2 Tw_0 \\ &\geqslant F(t, v_0, Tv_0) - M_1 w_0 - N_1 Tw_0 + G(t, w_0, Tw_0) - M_2 w_0 - N_2 Tw_0. \end{aligned}$$

We can now show that $v_0 \leqslant v_1$. For this purpose let $p(t) = v_0(t) - v_1(t)$, then

$$\begin{aligned} p'(t) = v_0' - v_1' &\leqslant F(t, w_0, Tw_0) - M_1 v_0 - N_1 Tv_0 + G(t, v_0, Tv_0) - M_2 v_0 - N_2 v_0 \\ &\quad - F(t, w_0, Tw_0) + M_1 v_1 + N_1 Tv_1 - G(t, v_0, Tv_0) + M_2 v_1 + N_2 Tv_1 \\ &= -(M_1 + M_2)(v_0 - v_1) - (N_1 + N_2)T(v_0 - v_1) \\ &= -Mp - NTp, \end{aligned}$$

where $(M_1 + M_2) = M > 0$ and $(N_1 + N_2) = N \geqslant 0$. Also $p(0) = p(2\pi)$. Thus, from Lemma 1, we get $p(t) \leqslant 0$, which proves $v_0(t) \leqslant v_1(t)$ on J. Similarly, we can show $w_0 \geqslant w_1$.

Also, let u be any solution of equation (1.1) such that $v_0(t) \leqslant u \leqslant w_0(t)$ on J and set $p(t) = u - v_1$, we get

$$\begin{aligned} p'(t) = u' - v_1' &= F(t, u, Tu) - M_1 u - N_1 Tu + G(t, u, Tu) - M_2 u - N_2 u \\ &\quad - F(t, w_0, Tw_0) + M_1 v_1 + N_1 Tv_1 - G(t, v_0, Tv_0) + M_2 v_1 + N_2 Tv_1 \\ &\leqslant -(M_1 + M_2)(u - v_1) - (N_1 + N_2)T(u - v_1) \\ &= -Mp - NTp, \end{aligned}$$

where $(M_1 + M_2) = M > 0$ and $(N_1 + N_2) = N \geqslant 0$ using the monotone nature of f and g and the fact that $v_0 \leqslant u \leqslant w_0$. Also $p(0) = u(0) - v_1(0) = u(2\pi) - v_1(2\pi) = p(2\pi)$. Thus $p(t) \leqslant 0$. From Lemma 1 we get $u \leqslant v_1$. A similar argument yields $u \geqslant w_1$. To avoid repetition, we can prove $u \geqslant v_2$, $u \leqslant w_2$, $u \leqslant v_3$, and $u \geqslant w_3$ in a similar fashion. Now we want to show that $v_0 \leqslant w_1 \leqslant v_2 \leqslant w_3$, and $v_3 \leqslant w_2 \leqslant v_1 \leqslant w_0$. We have by hypothesis that $v_0 \leqslant w_1$ and $w_0 \geqslant v_1$. For this purpose let $p(t) = w_1 - v_2$, then

$$\begin{aligned} p'(t) = w_1' - v_2' &= F(t, v_0, Tv_0) - M_1 w_1 - N_1 Tw_1 + G(t, w_0, Tw_0) - M_2 w_1 - N_2 w_1 \\ &\quad - F(t, w_1, Tw_1) + M_1 v_2 + N_1 Tv_2 - G(t, v_1, Tv_1) + M_2 v_2 + N_2 Tv_2 \\ &\leqslant -(M_1 + M_2)(w_1 - v_2) - (N_1 + N_2)T(w_1 - v_2) \\ &= -Mp - NTp, \end{aligned}$$

by the monotone nature of f and G. Also $p(0) = p(2\pi)$. Hence by Lemma 1, $p(t) \leqslant 0$, giving us $w_1 \leqslant v_2$. Similarly, $v_1 \geqslant w_2$, $v_2 \leqslant w_3$ and $v_3 \leqslant w_2$. Hence, it is clear that (3.11) is true for $k = 1$.

Now assume that (3.11) holds for some $k > 1$, such that,

$$w_{k-2} \leqslant v_{k-1} \leqslant w_k \leqslant u \leqslant v_k \leqslant w_{k-1} \leqslant v_{k-2} \text{ on } J.$$

We will use this and prove that (3.11) holds for $k+1$. Thus we need to show that the inequality

$$v_{k-1} \leqslant w_k \leqslant v_{k+1} \leqslant u \leqslant w_{k+1} \leqslant v_k \leqslant w_{k-1}$$

holds on J. For this purpose, let $p(t) = v_{k-1}(t) - w_k(t)$, then we have

$$\begin{aligned} p'(t) &= v'_{k-1}(t) - w'_k(t) \\ &= F(t, w_{k-2}, Tw_{k-2}) - M_1 v_{k-1} - N_1 T v_{k-1} + G(t, v_{k-2}, Tv_{k-2}) - M_2 v_{k-1} - N_2 v_{k-1} \\ &\quad - F(t, v_{k-1}, Tv_{k-1}) + M_1 w_k + N_1 T w_k - G(t, w_{k-1}, Tw_{k-1}) + M_2 w_k + N_2 T w_k \\ &\leqslant -(M_1 + M_2)(v_{k-1} - w_k) - (N_1 + N_2) T (v_{k-1} - w_k) \\ &= -Mp - NTp, \end{aligned}$$

using (3.14) and by the monotone nature of f and G. We also have that $p(0) = v_{k-1}(0) - w_k(0) = v_{k-1}(2\pi) - w_k(2\pi) = p(2\pi)$, thus by Lemma 1, $p(t) \leqslant 0$, giving us $v_{k-1}(t) \leqslant w_k(t)$. Similarly, we can show $w_k(t) \leqslant v_{k+1}(t)$, $w_{k+1}(t) \leqslant v_k(t)$, and $v_k(t) \leqslant w_{k-1}(t)$. We can also show in a similar fashion as before that $u \geqslant v_{k+1}$ and $u \leqslant w_{k+1}$ on J. Thus (3.11) holds for $k+1$. Hence, by induction, (3.11) is valid for all $k = 0, 1, 2, \ldots$.

Also the sequences $\{v_{2k}, w_{2k+1}\}$ and $\{w_{2k}, v_{2k+1}\}$ can be shown to be equicontinuous and uniformly bounded. So by Ascoli-Arzela's Theorem, the subsequences $\{w_{2n_k}, w_{(2n+1)_k}\}$ and $\{w_{2n_k}, v_{(2n+1)_k}\}$ converge to $\rho(t)$ and $r(t)$ respectively on J. Since the sequences $\{v_{2k}, w_{2k+1}\}$ and $\{w_{2k}, v_{2k+1}\}$ are monotone, the entire sequences converge uniformly and monotonically to $\rho(t)$ and $r(t)$ respectively on J. Thus the coupled system (3.12) and (3.13) are satisfied.

Finally, we can show that $\rho(t)$ and $r(t)$ are coupled minimal and maximal solutions of equation (1.1) in a fashion similar to the one in the proof of Theorem 3.1. This completes the proof.

REMARK 3.2. We can prove uniqueness of the solution of equation (1.1) on the same lines as in Theorem 3.1.

REMARK 3.3. If we would consider the assumptions $(A_1)(ii)$ or $(A_1)(iii)$, we would get the same results as we did using assumption $(A_1)(i)$ in Theorems 3.1 and 3.2.

REMARK 3.4. If $g \equiv 0$ and f is not nondecreasing in u and Tu, but

$$\widetilde{f}(t, u, Tu) = f(t, u, Tu) + Mu + NTu$$

is nondecreasing in u and Tu for $M > 0$, $N \geqslant 0$. Then we can write

$$\begin{aligned} u' &= f(t, u, Tu) + Mu + NTu - Mu - NTu \\ &= \widetilde{f}(t, u, Tu) + \widetilde{g}(t, u, Tu), \end{aligned}$$

where $\widetilde{g} = -Mu - NTu$ is nonincreasing in u and Tu. Then, if v_0, w_0 are lower and upper solutions of the original problem with $v_0 \leqslant w_0$, then they are lower and upper solutions of the integro differential equation with boundary conditions

$$\begin{aligned} u' &= \widetilde{f}(t,u,Tu) - Mu - NTu, \\ u(0) &= u(2\pi). \end{aligned} \tag{3.18}$$

Then we can construct upper and lower solutions of type II, given by

$$\begin{aligned} v_0' &\leqslant f(t,v_0,Tv_0) + Mv_0 + NTv_0 - Mv_0 - NTv_0 \\ &\leqslant f(t,w_0,Tw_0) + Mw_0 + NTw_0 - Mv_0 - NTv_0 \\ &= \widetilde{f}(t,w_0,Tw_0) - Mv_0 - NTv_0, \\ &= \widetilde{f}(t,w_0,Tw_0) + \widetilde{g}(t,v_0,Tv_0) \\ v_0(0) &\leqslant v_0(2\pi) \end{aligned}$$

and

$$\begin{aligned} w_0' &\geqslant f(t,w_0,Tw_0) + Mw_0 + Nw_0 - Mw_0 - Nw_0 \\ &\geqslant f(t,v_0,Tv_0) + Mv_0 + NTv_0 - Mw_0 - NTw_0 \\ &= \widetilde{f}(t,v_0,Tv_0) - Mw_0 - NTw_0, \\ &= \widetilde{f}(t,v_0,Tv_0) + \widetilde{g}(t,w_0,Tw_0) \\ w_0(0) &\geqslant w_0(2\pi). \end{aligned}$$

Note that we get the same results as in Theorem 3.2, using an appropriate iterative scheme.

REMARK 3.5 If $f \equiv 0$ and g is not nonincreasing in u and Tu, but

$$\widetilde{g}(t,u,Tu) = g(t,u,Tu) - Mu - NTu$$

is nonincreaing in u and Tu for $M > 0$, $N \geqslant 0$. Then we can write

$$\begin{aligned} u' &= g(t,u,Tu) - Mu - NTu + Mu + NTu \\ &= \widetilde{g}(t,u,Tu) + \widetilde{f}(t,u,Tu), \end{aligned}$$

where $\widetilde{f} = Mu + NTu$ is nondecreasing in u and Tu. Then, if v_0, w_0 are lower and upper solutions of the original problem with $v_0 \leqslant w_0$, then they are lower and upper solutions of the integro differential equation with boundary conditions

$$\begin{aligned} u' &= \widetilde{g}(t,u,Tu) + Mu + NTu, \\ u(0) &= u(2\pi). \end{aligned} \tag{3.19}$$

Then we can construct upper and lower solutions of type II, given by

$$\begin{aligned} v_0' &\leqslant g(t,v_0,Tv_0) - Mv_0 - NTv_0 + Mv_0 + NTv_0 \\ &\leqslant g(t,v_0,Tv_0) - Mv_0 - NTv_0 + Mw_0 + NTw_0 \\ &= \widetilde{g}(t,v_0,Tv_0) + Mw_0 + NTw_0 \\ &= \widetilde{g}(t,v_0,Tv_0) + \widetilde{f}(t,w_0,Tw_0), \\ v_0(0) &\leqslant v_0(2\pi) \end{aligned}$$

and

$$\begin{aligned} w_0' &\geqslant g(t,w_0,Tw_0) - Mw_0 - NTw_0 + Mw_0 + NTw_0 \\ &\geqslant g(t,w_0,Tw_0) - Mw_0 - NTw_0 + Mv_0 + NTv_0 \\ &= \widetilde{g}(t,w_0,Tw_0) + Mv_0 + NTv_0 \\ &= \widetilde{g}(t,w_0,Tw_0) + \widetilde{f}(t,v_0,Tv_0), \\ w_0(0) &\geqslant w_0(2\pi). \end{aligned}$$

Note that we get the same results as in Theorem 3.2, using an appropriate iterative scheme.

4. Numerical results

In this section we give an example of a nonlinear integro-differential equation with periodic boundary conditions to demonstrate a special case of Theorem 3.2 by using Mathematica to compute the iterates of the sequences which converge to the solution of the nonlinear problem. For this purpose we will consider the nonlinear integro-differential equation with periodic boundary conditions given by

$$\begin{aligned} u' &= \frac{1}{\pi^4}u^4 - \frac{1}{24\pi^2} - \frac{1}{16\pi^2}\int_0^t uds - \frac{5}{8\pi}u, \\ u(0) &= u(2\pi) \quad \text{on} \quad J=[0,2\pi]. \end{aligned} \tag{4.1}$$

Note that

$$\begin{aligned} f(t,u,Tu) &= F(t,u,Tu) - M_1u - N_1Tu \\ &= \frac{1}{\pi^4}u^4 - \frac{1}{24\pi^2} \end{aligned}$$

is nondecreasing in u, where $-M_1 = 0$ and $-N_1 = 0$.

Also

$$\begin{aligned} g(t,u,Tu) &= G(t,u,Tu) - M_2u - N_2Tu \\ &= -\frac{5}{8\pi}u - \frac{1}{16\pi^2}\int_0^t uds \end{aligned}$$

is nonincreasing in u and Tu, where $G(t,u,Tu) = 0, -M_2u = \frac{5}{8\pi}u$ and $-N_2Tu(t) = -\frac{1}{16\pi^2}\int_0^t u(s)ds$. Furthermore, the equation (4.1) satisfies the condition

$$2(N_1+N_2)k_0\pi(e^{2(M_1+M_2)\pi} - 1) \leqslant M_1 + M_2,$$

since

$$(\frac{1}{8\pi})(e^{\frac{5}{4}} - 1) \leqslant \frac{5}{8\pi}.$$

Also, we have $v_0 = -\frac{1}{3}$ and $w_0 = \frac{3}{4}$ are lower and upper solutions of type II of (4.1). Using v_0, w_0 and a special case of the iterative scheme given by equations (3.12) and (3.13), we have used Laplace Transform and Mathematica to compute v_1 and w_1 analytically as follows:

$$v_1 = \frac{1}{36\pi^2}\{e^{\frac{-t}{8\pi}}[211 - 96\pi^3(-\frac{211(-1+e^{\frac{1}{4}})(1+2e^{\frac{1}{4}}+3\sqrt{e})}{384(1+e^{\frac{1}{4}}+\sqrt{e})\pi^3})]\} + \frac{1}{144\pi^2}\{e^{\frac{-t}{2\pi}}[-211 + 384\pi^3(-\frac{211(-1+e^{\frac{1}{4}})(1+2e^{\frac{1}{4}}+3\sqrt{e})}{384(1+e^{\frac{1}{4}}+\sqrt{e})\pi^3})]\}$$

and

$$w_1 = \frac{2}{243\pi^2}e^{\frac{-t}{2\pi}}\{[19 + 324\pi^3(\frac{19(-1+e^{\frac{1}{4}})(1+2e^{\frac{1}{4}}+3\sqrt{e})}{324(1+e^{\frac{1}{4}}+\sqrt{e})\pi^3})]\} - \frac{8}{243\pi^2}e^{\frac{-t}{8\pi}}[19 + 81\pi^3(\frac{19(-1+e^{\frac{1}{4}})(1+2e^{\frac{1}{4}}+3\sqrt{e})}{324(1+e^{\frac{1}{4}}+\sqrt{e})\pi^3})]\}$$

We illustrate in Figure 4.1 the lower and upper solutions v_0, w_0 and the two iterates v_1, w_1, and we show clearly that $v_0 \leqslant w_1$ and $w_0 \geqslant v_1$, therefore the conditions of Theorem 3.2 are satisfied. Also, it is clear that the periodic boundary conditions are met. Furthermore, the graph illustrates intertwined alternating sequences, which are converging to the solution of equation (4.1). Note, we have chosen appropriate scales to demonstrate these conditions. Therefore the graphs of v_1 and w_1 appear to be linear.

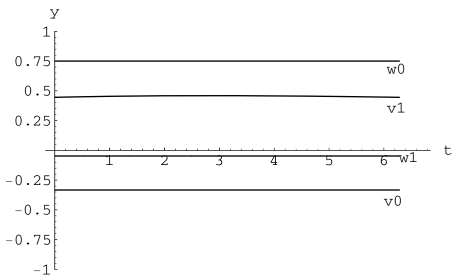

Figure 4.1.

We illustrate v_1 and w_1 in Figures 4.2 and 4.3. Note, we have chosen appropriate small scales to clearly see the curves of the graphs.

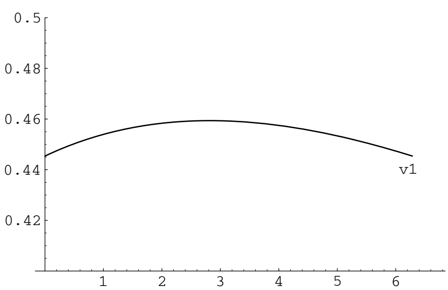

Figure 4.2

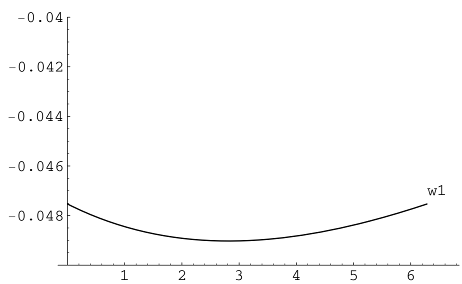

Figure 4.3

Conclusion. It was known that when using the monotone iterative technique coupled with lower and upper solutions for nonlinear integro differential equations with periodic boundary conditions, we obtained natural sequences which converge to minimal and maximal solutions of the nonlinear problems. Our results prove that we can obtain natural sequences as well as alternating intertwining sequences when the forcing function is the sum of a nondecreasing and nonincreasing function. We have shown, that it depends entirely on the construction of the sequences. We have also shown that the solutions to the nonlinear problem is unique. Our result generalizes the monotone method for periodic boundary value problems for the nonlinear integro equation 1.1, which includes some earlier known results.

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Ianna H. West
Department of Mathematics and Computer Science
Nicholls State University
P. O. Box 2031
Thibodaux, LA 70310
USA

A. S. Vatsala
Department of Mathematics
University of Louisiana at Lafayette
Lafayette, LA 70504
USA