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HANKEL CONVOLUTION ON THE DUAL OF A SPACE OF ENTIRE FUNCTIONS

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Abstract. In this paper we study the Hankel convolution on a certain space $\mathcal{Q}$ of entire functions rapidly increasing on each horizontal strip in the complex plane. We also describe the #-convolution operators on the dual space $\mathcal{Q}'$ of $\mathcal{Q}$.

1. Introduction and preliminaries

The Hankel transformation is defined by ([12])

$$h_\mu(\phi)(y) = \int_0^\infty (xy)^{-\mu} J_\mu(xy)\phi(x)x^{2\mu+1}dx, \quad y \in (0,\infty),$$

where ϕ is a Lebesgue measurable function on $(0,\infty)$ such that $\int_0^\infty |\phi(x)|x^{2\mu+1}dx < \infty$. Here J_μ represents, as usual, the Bessel function of the first kind and order $\mu > -\frac{1}{2}$.

The study of Hankel transforms on distribution spaces was started by A. H. Zemanian ([17] and [18]) who defined h_μ transforms on distributions of slow growth and of rapid growth. More recently, J. J. Betancor and L. Rodríguez-Mesa ([4] and [6]) have defined Hankel transforms on distributions of exponential growth.

The convolution operation for h_μ-transforms was investigated by I. I. Hirschman [13], D. T. Haimo [10] and F. M. Cholewinski [7].

Suppose that ϕ_i is a Lebesgue measurable function on $(0,\infty)$ such that $\int_0^\infty |\phi_i(x)|x^{2\mu+1}dx < \infty$, for $i = 1,2$. The Hankel convolution of μ, $\phi_1 \#_\mu \phi_2$ of ϕ_1 and ϕ_2, is given by

$$(\phi_1 \#_\mu \phi_2)(x) = \int_0^\infty \phi_2(y)({}_\mu\tau_x\phi_1)(y)\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dy, \quad x \in (0,\infty),$$

where

$$({}_\mu\tau_x\phi_1)(y) = \int_0^\infty D_\mu(x,y,z)\phi_1(z)\frac{z^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dz, \quad y \in (0,\infty), \quad \textit{and} \quad {}_\mu\tau_0\phi_1 = \phi_1.$$

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Here D_μ denotes the Delsarte kernel given by

$$D_\mu(x,y,z) = (2^\mu\Gamma(\mu+1))^2\int_0^\infty (xt)^{-\mu}J_\mu(xt)(yt)^{-\mu}J_\mu(yt)(zt)^{-\mu}J_\mu(zt)t^{2\mu+1}dt,$$
$$x,y,z\in(0,\infty).$$

The Hankel transform h_μ is closely connected with $\#_\mu$-convolution and ${}_\mu\tau_x$-translations, $x\in(0,\infty)$, as the following formulas ([13]) show

$$h_\mu(\phi_1\#_\mu\phi_2) = h_\mu(\phi_1)h_\mu(\phi_2),$$

and

$$h_\mu({}_\mu\tau_x\phi_1)(y) = 2^\mu\Gamma(\mu+1)(xy)^{-\mu}J_\mu(xy)h_\mu(\phi_1)(y),\quad x,y\in(0,\infty).$$

In the sequel, I will write $\#$, τ_x, $x\in[0,\infty)$, and D, instead $\#_\mu$, ${}_\mu\tau_x$, $x\in[0,\infty)$, and D_μ, respectively, to simplify notations. I do not think that this lead to any problem.

The study of Hankel convolutions in distribution spaces was begun by J. Sousa-Pinto [15] who considered distributions having compact support on $(0,\infty)$ and the order $\mu=0$. J. J. Betancor and I. Marrero ([3] and [14]) completed the investigations of J. de Sousa-Pinto. The Hankel convolution was studied on Zemanian's distribution spaces of slow growth in [14] and of rapid growth in [3].

In [4] Hankel transform and Hankel convolution were investigated on distribution spaces of exponential growth.

We now collect some definitions and results stated in [4] which will be very useful in the sequel.

By $\mathcal{X}$ we represent the space constituted by all those C^∞-functions ϕ on $(0,\infty)$ such that, for every $m,n\in\mathbf{N}$,

$$\gamma_{m,n}(\phi) = \sup_{x\in(0,\infty)} e^{mx}\left|\left(\frac{1}{x}\frac{d}{dx}\right)^n\phi(x)\right| < \infty.$$

The space $\mathcal{X}$ is Fréchet when on $\mathcal{X}$ we consider the topology generated by the family $\{\gamma_{m,n}\}_{m,n\in\mathbf{N}}$ of seminorms.

For every $x\in[0,\infty)$, the Hankel translation operator τ_x is continuous from $\mathcal{X}$ into itself ([4, Proposition 3.1]). If $T\in\mathcal{X}'$, the dual space of $\mathcal{X}$, and $\phi\in\mathcal{X}$, the Hankel convolution $T\#\phi$ of T and ϕ is defined by

$$(T\#\phi)(x) = \langle T, \tau_x\phi\rangle,\quad x\in[0,\infty).$$

In [4] the space of Hankel convolution operators on $\mathcal{X}$ was investigated.

The space $\mathcal{Q}$ consists of all those even and entire functions Φ such that, for every $m,k\in\mathbf{N}$,

$$\eta_{m,k}(\Phi) = \sup_{|Im\, z|\leqslant m}(1+|z|^2)^k|\Phi(z)| < \infty.$$

$\mathcal{Q}$ is endowed with the topology associated with the system $\{\eta_{m,k}\}_{m,k\in\mathbf{N}}$ of norms. Thus $\mathcal{Q}$ is a Fréchet space. The Hankel transformation h_μ is an isomorphism from $\mathcal{X}$ onto $\mathcal{Q}$ ([4, Theorem 2.1]). h_μ is defined on the corresponding dual spaces by transposition, that is, if $T \in \mathcal{X}'$ (respectively, $\mathcal{Q}'$, the dual space of $\mathcal{Q}$) the Hankel transform $h_\mu(T)$ is the element of $\mathcal{Q}'$ (respectively, $\mathcal{X}'$) given by

$$\langle h'_\mu(T), \Phi\rangle = \langle T, h_\mu(\Phi)\rangle, \quad \Phi \in \mathcal{Q} \text{ (respectively, } \mathcal{X}).$$

Our objective in this paper is to study the Hankel convolution on the spaces $\mathcal{Q}$ and $\mathcal{Q}'$. We need previously to establish in Section 2 new properties of the spaces $\mathcal{X}$ and $\mathcal{Q}$.

Throughout this paper C always represents a suitable positive constant that can be changed from line to line.

2. Function spaces $\mathcal{X}$ and $\mathcal{Q}$

In this section we establish new properties for the function spaces $\mathcal{X}$ and $\mathcal{Q}$ that were introduced in [4].

As it was mentioned in Section 1 the space $\mathcal{X}$ is constituted by all those C^∞-functions ϕ on $(0,\infty)$ such that, for every $m,n \in \mathbf{N}$,

$$\gamma_{m,n}(\phi) = \sup_{x\in(0,\infty)} e^{mx}\left|\left(\frac{1}{x}\frac{d}{dx}\right)^n \phi(x)\right| < \infty.$$

The space $\mathcal{X}$ is equipped with the topology generated by the family $\{\gamma_{m,n}\}_{m,n\in\mathbf{N}}$ of seminorms. Thus $\mathcal{X}$ is a Fréchet space.

G. Altenburg [1] introduced the space $\mathcal{H}$ that consists of all those C^∞-functions ϕ on $(0,\infty)$ such that, for every $m,n \in \mathbf{N}$,

$$\alpha_{m,n}(\phi) = \sup_{x\in(0,\infty)} (1+x^2)^m\left|\left(\frac{1}{x}\frac{d}{dx}\right)^n \phi(x)\right| < \infty.$$

$\mathcal{H}$ is endowed with the topology associated with the system $\{\alpha_{m,n}\}_{m,n\in\mathbf{N}}$ of seminorms. By virtue of [8, p. 86], $\mathcal{H}$ coincides with the space $\mathcal{S}_e$ of the even functions in the Schwartz class $\mathcal{S}$. The space $\mathcal{X}$ is continuously contained in $\mathcal{H}$.

H. Hasumi [11] considered the space E constituted by the C^∞-functions ϕ on $\mathbf{R}$ such that, for each $m,n \in \mathbf{N}$,

$$\beta_{m,n}(\phi) = \sup_{x\in\mathbf{R}} e^{m|x|}\left|\frac{d^n}{dx^n}\phi(x)\right| < \infty.$$

We denote by E_e the space that consists of the even functions in E.

PROPOSITION 2.1. *$E_e = \mathcal{X}$ where the equality is algebraic and topological.*

Proof. Let $\phi \in \mathcal{X}$. Since $\mathcal{X}$ is contained in $\mathcal{S}_e$, ϕ can be extended to $\mathbf{R}$ as an even function. Moreover, for every $n \in \mathbf{N}$, we can write

$$\frac{d^n}{dx^n} = \sum_{j=0}^{n} a_{j,n} x^{\alpha_{j,n}} \left(\frac{1}{x}\frac{d}{dx}\right)^j,$$

where $a_{j,n} \in \mathbf{R}$ and $\alpha_{j,n} \in \mathbf{N}$, $j = 0, ..., n$. Hence, for every $m, n \in \mathbf{N}$,

$$\beta_{m,n}(\phi) \leqslant C \sum_{j=0}^{n} \gamma_{m+1,j}(\phi). \tag{2.1}$$

Thus we proved that $\phi \in E_e$.

Suppose now $\phi \in E_e$. For every $n \in \mathbf{N}$, we have that

$$\left(\frac{1}{x}\frac{d}{dx}\right)^n = \sum_{j=0}^{n} b_{j,n} x^{-2n+j} \frac{d^j}{dx^j},$$

where $b_{j,n} \in \mathbf{R}$, $j = 0, ..., n$. Then, for each $m, n \in \mathbf{N}$,

$$\sup_{x \geqslant 1} e^{mx} \left| \left(\frac{1}{x}\frac{d}{dx}\right)^n \phi(x) \right| \leqslant C \sum_{j=0}^{n} \beta_{m,j}(\phi). \tag{2.2}$$

On the other hand, since E_e is contained in $\mathcal{S}_e$, we get, for every $m, n \in \mathbf{N}$,

$$\begin{aligned} \sup_{x \in (0,1)} e^{mx} \left| \left(\frac{1}{x}\frac{d}{dx}\right)^n \phi(x) \right| &\leqslant C \sup_{x \in (0,1)} \left| \left(\frac{1}{x}\frac{d}{dx}\right)^n \phi(x) \right| \\ &\leqslant C \max_{0 \leqslant j \leqslant l} \sup_{x \in \mathbf{R}} (1 + |x|^2)^l \left| \frac{d^j}{dx^j} \phi(x) \right| \qquad (2.3) \\ &\leqslant C \max_{0 \leqslant j \leqslant l} \sup_{x \in \mathbf{R}} e^{|x|} \left| \frac{d^j}{dx^j} \phi(x) \right|, \end{aligned}$$

where $l \in \mathbf{N}$ is depending on n.

By combining (2.2) and (2.3) we conclude that $\phi \in \mathcal{X}$.

Moreover, (2.1), (2.2) and (2.3) imply that the families $\{\beta_{m,n}\}_{m,n \in \mathbf{N}}$ and $\{\gamma_{m,n}\}_{m,n \in \mathbf{N}}$ are equivalent on $\mathcal{X}$. □

Now, using [2, (2.2) and (2.3)] we can see that for every $\mu > -\frac{1}{2}$, the system $\{\gamma^{\mu}_{m,n}\}_{m,n \in \mathbf{N}}$ of seminorms, where, for each $m, n \in \mathbf{N}$,

$$\gamma^{\mu}_{m,n}(\phi) = \sup_{x \in (0,\infty)} e^{mx} |\Delta^n_{\mu} \phi(x)|, \quad \phi \in \mathcal{X},$$

generates the topology of $\mathcal{X}$. Here Δ_μ denotes the Bessel operator $x^{-2\mu-1} D x^{2\mu+1} D$.

We now introduce new families of seminorms defining the topology of $\mathcal{X}$ which will be useful in the sequel.

We denote by

$$\mathcal{I}_\mu(x) = \sum_{j=0}^{\infty} \frac{x^{2j}}{2^{2j} j! \Gamma(j + \mu + 1)}, \quad x \in \mathbf{R}.$$

Note that $\mathcal{I}_\mu$ is closely connected with the modified Bessel function $\mathbf{I}_\mu$ of the first kind and order μ ([16, p. 77]). According to [19, (5) and (6), Chapter 6] we have that, for a certain $C > 0$,

$$\frac{1}{C} x^{-\mu-1/2} e^x \leqslant \mathcal{I}_\mu(x) \leqslant C e^x, \quad x \in (0, \infty). \tag{2.4}$$

Moreover, for every $k \in \mathbf{N}$,

$$\left(\frac{1}{x}\frac{d}{dx}\right)^k \mathcal{I}_\mu(x) = 2^{-k}\mathcal{I}_{\mu+k}(x), \quad x \in (0,\infty). \tag{2.5}$$

PROPOSITION 2.2. *We define for every $m,n \in \mathbf{N}$ and $\phi \in \mathcal{X}$,*

$$\gamma_{m,n}^{\mu,(1)}(\phi) = \sup_{x\in(0,\infty)} \mathcal{I}_\mu(mx)\left|\left(\frac{1}{x}\frac{d}{dx}\right)^n \phi(x)\right|,$$

$$\gamma_{m,n}^{\mu,(2)}(\phi) = \sup_{x\in(0,\infty)} \left|\left(\frac{1}{x}\frac{d}{dx}\right)^n (\mathcal{I}_\mu(mx)\phi(x))\right|,$$

$$\gamma_{m,n}^{\mu,(3)}(\phi) = \sup_{x\in(0,\infty)} \mathcal{I}_\mu(mx)|\Delta_\mu^n\phi(x)|,$$

$$\gamma_{m,n}^{\mu,(4)}(\phi) = \sup_{x\in(0,\infty)} |\Delta_\mu^n(\mathcal{I}_\mu(mx)\phi(x))|.$$

Then, $\{\gamma_{m,n}^{\mu,(j)}\}_{m,n\in\mathbf{N}}$ is a family of seminorms equivalent to $\{\gamma_{m,n}\}_{m,n\in\mathbf{N}}$ on $\mathcal{X}$, for every $j = 1,2,3,4$.

Proof. We denote by T_j the topology on $\mathcal{X}$ associated with the system $\{\gamma_{m,n}^{\mu,(j)}\}_{m,n\in\mathbf{N}}$ of seminorms, for $j = 1,2,3,4$. We represent by T the topology generated by $\{\gamma_{m,n}\}_{m,n\in\mathbf{N}}$ on $\mathcal{X}$.

Note firstly that by virtue of (2.4) T_1 coincides with T.

1. T_2 *is finer than* T_1. Indeed, let $m,n \in \mathbf{N}$ and $\phi \in \mathcal{X}$. Leibniz formula leads to

$$\left(\frac{1}{x}\frac{d}{dx}\right)^n \phi(x) = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{1}{x}\frac{d}{dx}\right)^j (\mathcal{I}_\mu(2(m+1)x)\phi(x)) \left(\frac{1}{x}\frac{d}{dx}\right)^{n-j} \left(\frac{1}{\mathcal{I}_\mu(2(m+1)x)}\right),$$
$$x \in (0,\infty).$$

Then, from (2.4) and (2.5) we deduce that

$$\begin{aligned}
\gamma_{m,n}^{\mu,(1)}(\phi) &= \sup_{x\in(0,\infty)} \mathcal{I}_\mu(mx)\left|\left(\frac{1}{x}\frac{d}{dx}\right)^n \phi(x)\right| \\
&\leqslant C\sum_{j=0}^{n} \sup_{x\in(0,\infty)} e^{mx}\left|\left(\frac{1}{x}\frac{d}{dx}\right)^{n-j}\left(\frac{1}{\mathcal{I}_\mu(2(m+1)x)}\right)\right| \times \\
&\qquad \times \sup_{x\in(0,\infty)} \left|\left(\frac{1}{x}\frac{d}{dx}\right)^j (\mathcal{I}_\mu(2(m+1)x)\phi(x))\right| \\
&\leqslant C\sum_{j=0}^{n} \gamma_{2(m+1),j}^{\mu,(2)}(\phi).
\end{aligned}$$

Hence T_2 is stronger than T_1.

2. T_1 *is finer than* T_2. To see this we use again (2.4) and (2.5). Indeed, let $m,n \in \mathbf{N}$ and $\phi \in \mathcal{X}$. We can write

$$\begin{aligned} \gamma_{m,n}^{\mu,(2)}(\phi) &\leqslant C\sum_{j=0}^{n} \sup_{x\in(0,\infty)} \left|\left(\frac{1}{x}\frac{d}{dx}\right)^{n-j}\mathcal{I}_\mu(mx)\right|\left|\left(\frac{1}{x}\frac{d}{dx}\right)^{j}\phi(x)\right| \\ &\leqslant C\sum_{j=0}^{n} \sup_{x\in(0,\infty)} \mathcal{I}_{\mu+n-j}(mx)\left|\left(\frac{1}{x}\frac{d}{dx}\right)^{j}\phi(x)\right| \\ &\leqslant C\sum_{j=0}^{n} \sup_{x\in(0,\infty)} e^{mx}\left|\left(\frac{1}{x}\frac{d}{dx}\right)^{j}\phi(x)\right| \\ &\leqslant C\sum_{j=0}^{n} \gamma_{m,j}^{\mu,(1)}(\phi). \end{aligned}$$

3. T_1 *is finer than* T_3. To prove this we note that, for every $n \in \mathbf{N}$, we can find $c_{k,j} \in \mathbf{R}$, $j = n, ..., 2n$, such that

$$\Delta_\mu^n = \sum_{j=n}^{2n} c_{n,j} x^{2(j-n)}\left(\frac{1}{x}\frac{d}{dx}\right)^{j}. \tag{2.6}$$

Hence, according to (2.4), for every $m,n \in \mathbf{N}$, we have

$$\gamma_{m,n}^{\mu,(3)}(\phi) \leqslant C\sum_{j=n}^{2n} \gamma_{m+1,j}^{\mu,(1)}(\phi), \quad \phi \in \mathcal{X}.$$

4. T_1 *is finer than* T_4. Let $m,n \in \mathbf{N}$ and $\phi \in \mathcal{X}$. From (2.5) and (2.6) we infer that

$$\Delta_\mu^n(\mathcal{I}_\mu(mx)\phi(x)) = \sum_{j=n}^{2n} c_{n,j}x^{2(j-n)}\sum_{i=0}^{j}\binom{j}{i} m^{2i}2^{-i}\mathcal{I}_{\mu+i}(mx)\left(\frac{1}{x}\frac{d}{dx}\right)^{j-i}\phi(x), \quad x \in (0,\infty).$$

Then, (2.4) implies that

$$\gamma_{m,n}^{\mu,(4)}(\phi) \leqslant C\sum_{j=0}^{2n} \gamma_{m+1,j}^{\mu,(1)}(\phi).$$

5. T_3 *is finer than* T. By using [2, (2.2) and (2.3)] we can conclude that, for every $m,n \in \mathbf{N}$, there exist $l \in \mathbf{N}$ and $C > 0$ such that

$$\gamma_{m,n}(\phi) \leqslant C\sum_{j=0}^{l} \gamma_{l,j}^{\mu}(\phi), \quad \phi \in \mathcal{X}. \tag{2.7}$$

According to (2.4), (2.7) allows us to see that T_3 is stronger than T.

6. T_4 *is finer than* T. By [2, (2.2)] we have that, for every $\phi \in \mathcal{H}$ and $n \in \mathbf{N}$,

$$\left(\frac{1}{x}\frac{d}{dx}\right)^n \phi(x) = x^{-2(\mu+n)} \int_0^x x_n \int_0^{x_n} x_{n-1} \dots \int_0^{x_2} x_1^{2\mu+1} \Delta_\mu^n \phi(x_1) dx_1 \dots dx_n, \quad x \in (0,\infty). \tag{2.8}$$

On the other hand, from (2.4) and (2.5) we deduce that, for every $m,n \in \mathbf{N}$,

$$\left|\left(\frac{1}{x}\frac{d}{dx}\right)^n \mathcal{I}_\mu(mx)\right| \leqslant Ce^{mx}, \quad x \in (0,\infty).$$

Hence, for every $m \in \mathbf{N}$ and $\phi \in \mathcal{X}$, $\mathcal{I}_\mu(mx)\phi \in \mathcal{X}$. Then, by using (2.8), we can write, for every $m,n \in \mathbf{N}$ and $\phi \in \mathcal{X}$,

$$\begin{aligned}
&\left|\left(\frac{1}{x}\frac{d}{dx}\right)^n (\mathcal{I}_\mu(mx)\phi(x))\right| \\
&\quad \leqslant x^{-2(\mu+n)} \int_0^x x_n \int_0^{x_n} x_{n-1} \dots \int_0^{x_2} x_1^{2\mu+1} |\Delta_\mu^n (\mathcal{I}_\mu(mx_1)\phi(x_1))| dx_1 \dots dx_n \\
&\quad \leqslant \gamma_{m,n}^{\mu,(4)}(\phi) x^{-2(\mu+n)} \int_0^x x_n \int_0^{x_n} x_{n-1} \dots \int_0^{x_2} x_1^{2\mu+1} dx_1 \dots dx_n \\
&\quad \leqslant C\gamma_{m,n}^{\mu,(4)}(\phi), \quad x \in (0,\infty).
\end{aligned}$$

Therefore,

$$\gamma_{m,n}^{\mu,(2)}(\phi) \leqslant C\gamma_{m,n}^{\mu,(4)}(\phi), \quad m,n \in \mathbf{N} \ \textit{and} \ \phi \in \mathcal{X}.$$

Thus the proof of the proposition is finished. □

The space of pointwise multipliers of $\mathcal{X}$ can be characterized as follows ([4]). A smooth function f on $(0,\infty)$ is a pointwise multiplier of $\mathcal{X}$ if, and only if, f can be extended to $\mathbf{R}$ as an even and smooth function and, for every $n \in \mathbf{N}$, there exists $m \in \mathbf{N}$ for which

$$e^{-mx}\left(\frac{1}{x}\frac{d}{dx}\right)^n f(x)$$

is bounded on $(0,\infty)$. We denote by $\mathcal{O}_\mathcal{X}$ the space of pointwise multipliers of $\mathcal{X}$. As it was mentioned in the proof of Proposition 2.2, according to (2.4) and (2.5), $\mathcal{I}_\mu(mx) \in \mathcal{O}_\mathcal{X}$, for every $m \in \mathbf{N}$.

On $\mathcal{O}_\mathcal{X}$ we consider the topology defined by the family $\{\gamma_{m,n,\phi}\}_{m,n\in\mathbf{N},\phi\in\mathcal{X}}$ of seminorms, where

$$\gamma_{m,n,\phi}(f) = \gamma_{m,n}(f\phi), \quad f \in \mathcal{O}_\mathcal{X},$$

for every $m,n \in \mathbf{N}$ and $\phi \in \mathcal{X}$.

PROPOSITION 2.3. *Let* $m \in \mathbf{N}$. *Then*

$$\lim_{k\to\infty} \sum_{j=0}^{k} \frac{(mx)^{2j}}{2^{2j} j! \Gamma(\mu+j+1)} = \mathcal{I}_\mu(mx),$$

in the sense of convergence in $\mathcal{O}_\mathcal{X}$.

Proof. We are going to see that, for every $\phi \in \mathcal{X}$,

$$\lim_{k\to\infty} \phi(x) \sum_{j=0}^{k} \frac{(mx)^{2j}}{2^{2j} j!\Gamma(\mu+j+1)} = \phi(x)\mathcal{I}_\mu(mx),$$

in the sense of convergence in $\mathcal{X}$.

Let $\phi \in \mathcal{X}$, $\varepsilon > 0$ and $l, n \in \mathbf{N}$. According to (2.4), we can write

$$\begin{aligned} e^{lx}&\left|\left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\phi(x) \sum_{j=k+1}^{\infty} \frac{(mx)^{2j}}{2^{2j} j!\Gamma(\mu+j+1)}\right)\right| \\ &\leqslant C \sum_{\alpha=0}^{n} e^{lx} \left|\left(\frac{1}{x}\frac{d}{dx}\right)^\alpha \phi(x)\right| \left|\left(\frac{1}{x}\frac{d}{dx}\right)^{n-\alpha} \left(\sum_{j=k+1}^{\infty} \frac{(mx)^{2j}}{2^{2j} j!\Gamma(\mu+j+1)}\right)\right| \\ &\leqslant C \sum_{\alpha=0}^{n} \gamma_{l+m+1,\alpha}(\phi) e^{-(m+1)x} \sum_{j=k+1}^{\infty} \frac{(mx)^{2(j-n+\alpha)}}{2^{2(j-n+\alpha)}(j-n+\alpha)!\Gamma(\mu+j+1)} \\ &\leqslant C \sum_{\alpha=0}^{n} \gamma_{l+m+1,\alpha}(\phi) e^{-(m+1)x} \mathcal{I}_{\mu+n-\alpha}(mx) \\ &\leqslant C e^{-x} \sum_{\alpha=0}^{n} \gamma_{l+m+1,\alpha}(\phi), \qquad x \in (0,\infty) \text{ and } k \in \mathbf{N}. \end{aligned}$$

Then, there exists $x_0 \in (0,\infty)$ such that, for every $x \in (x_0,\infty)$ and $k \in \mathbf{N}$,

$$e^{lx}\left|\left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\phi(x) \sum_{j=k+1}^{\infty} \frac{(mx)^{2j}}{2^{2j} j!\Gamma(\mu+j+1)}\right)\right| < \varepsilon. \tag{2.9}$$

Moreover, we have

$$\begin{aligned} e^{lx}&\left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\phi(x) \sum_{j=k+1}^{\infty} \frac{(mx)^{2j}}{2^{2j} j!\Gamma(\mu+j+1)}\right) \\ &\leqslant \sum_{\alpha=0}^{n} \binom{n}{\alpha} e^{lx} \left(\frac{1}{x}\frac{d}{dx}\right)^\alpha (\phi(x)) \left(\frac{1}{x}\frac{d}{dx}\right)^{n-\alpha} \left(\sum_{j=k+1}^{\infty} \frac{(mx)^{2j}}{2^{2j} j!\Gamma(\mu+j+1)}\right) \to 0, \end{aligned} \tag{2.10}$$

as $k \to \infty$, uniformly in $x \in [0, x_0]$.

By combining (2.9) and (2.10) we conclude that

$$\gamma_{l,n}\left(\phi(x) \sum_{j=k+1}^{\infty} \frac{(mx)^{2j}}{2^{2j} j!\Gamma(\mu+j+1)}\right) \to 0, \quad \text{as } k \to \infty.$$

Thus we have proved that

$$\lim_{k\to\infty} \phi(x) \sum_{j=0}^{k} \frac{(mx)^{2j}}{2^{2j} j!\Gamma(\mu+j+1)} = \phi(x)\mathcal{I}_\mu(mx),$$

in the sense of convergence in $\mathcal{X}$. □

The dual space of $\mathcal{X}$ is denoted by $\mathcal{X}'$. If f is a Lebesgue measurable function on $(0,\infty)$ such that, for some $k \in \mathbf{N}$, $e^{-kx}f(x)x^{2\mu+1}$ is absolutely integrable on $(0,\infty)$, then f defines an element of $\mathcal{X}'$, that we continue denoting by f, through

$$\langle f,\phi\rangle = \int_0^\infty f(x)\phi(x)\frac{x^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dx, \quad \phi \in \mathcal{X}.$$

Thus $\mathcal{X}$ and $\mathcal{O}_{\mathcal{X}}$ can be seen as subspaces of $\mathcal{X}'$.

The space $\mathcal{Q}$ was introduced in [4] as follows. An even and entire function Φ is in $\mathcal{Q}$ if, and only if, for every $m,k \in \mathbf{N}$,

$$\eta_{m,k}(\Phi) = \sup_{|Im\, z|\leqslant m} (1+|z|^2)^k|\Phi(z)| < \infty.$$

$\mathcal{Q}$ is a Fréchet space when $\mathcal{Q}$ is endowed with the topology associated with the family $\{\eta_{m,k}\}_{m,k\in\mathbf{N}}$ of seminorms.

The Hankel transformation h_μ is an isomorphism from $\mathcal{X}$ onto $\mathcal{Q}$ ([4, Theorem 2.1]).

We now give a new family of seminorms on $\mathcal{Q}$ that generates the same topology as $\{\eta_{m,k}\}_{m,k\in\mathbf{N}}$ on $\mathcal{Q}$.

If $m \in \mathbf{N}$ and Φ is an smooth function on $(0,\infty)$ we define

$$\mathcal{J}_{\mu,m}(\Phi)(x) = \sum_{k=0}^\infty \frac{(-1)^k m^{2k}\Delta_\mu^k\Phi(x)}{2^{2k}k!\Gamma(\mu+k+1)}, \quad x \in (0,\infty),$$

provided that the series converges for every $x \in (0,\infty)$. Note that $\mathcal{J}_{\mu,m}$ is closely related to the Bessel function J_μ.

For every $m,k \in \mathbf{N}$ and $\Phi \in \mathcal{Q}$ we define $\eta^\mu_{m,k}(\Phi)$ as follows

$$\eta^\mu_{m,k}(\Phi) = \sup_{x\in(0,\infty)} (1+x^2)^k|\mathcal{J}_{\mu,m}(\Phi)(x)|.$$

Note that in the definition of $\eta^\mu_{m,k}(\Phi)$ we consider only the restriction of Φ to $(0,\infty)$ in contrast to the definition of $\eta_{m,k}(\Phi)$ where Φ is considered on the whole complex plane.

PROPOSITION 2.4. *The family* $\{\eta^\mu_{m,k}\}_{m,k\in\mathbf{N}}$ *of seminorms generates the topology of* $\mathcal{Q}$*, that is, the topology associated with* $\{\eta_{m,k}\}_{m,k\in\mathbf{N}}$.

Proof. Let $m,k \in \mathbf{N}$. According to Proposition 2.3, [4, Theorem 2.1] and [1, (6)] we can write

$$\begin{aligned} h_\mu(\Delta_\mu^k(\mathcal{I}_\mu(mx)h_\mu(\Phi)))(z) &= (-1)^k z^{2k} h_\mu\left(\lim_{n\to\infty}\sum_{j=0}^n \frac{(mx)^{2j}}{2^{2j}j!\Gamma(\mu+j+1)}h_\mu(\Phi)\right)(z) \\ &= (-1)^k z^{2k}\lim_{n\to\infty}\sum_{j=0}^n \frac{(-1)^j m^{2j}\Delta_\mu^j\Phi(z)}{2^{2j}j!\Gamma(\mu+j+1)} \\ &= (-1)^k z^{2k}\mathcal{J}_{\mu,m}(\Phi)(z), \quad z \in \mathbf{C}, \qquad (2.11)\end{aligned}$$

for every $\Phi \in \mathcal{Q}$.

Equality (2.11) and again [4, Theorem 2.1] and Proposition 2.3 allow us to find $l \in \mathbf{N}$ for which

$$\begin{aligned}\eta^{\mu}_{m,k}(\Phi) &\leqslant C\sum_{j=0}^{k}\sup_{x\in(0,\infty)}|h_\mu(\Delta^j_\mu(\mathcal{I}_\mu(mt)h_\mu(\Phi)(t)))(x)|\\ &\leqslant \sup_{|Im\, x|\leqslant 1}|h_\mu(\Delta^k_\mu(\mathcal{I}_\mu(mt)h_\mu(\Phi)(t)))(x)|\\ &\leqslant C\eta_{l,l}(\Phi),\quad \Phi\in\mathcal{Q}.\end{aligned}$$

We have also taken into account that $\mathcal{I}_\mu(mx) \in \mathcal{O}_{\mathcal{X}}$ and that Δ_μ defines a continuous linear mapping from $\mathcal{X}$ into itself.

On the other hand we have, since $z^{-\mu}J_\mu(z)$ is bounded on $(0,\infty)$,

$$\begin{aligned}\eta_{m,k}(\Phi) &= \eta_{m,k}(h_\mu(h_\mu(\Phi)))\\ &\leqslant C\max_{0\leqslant i,j\leqslant l}\gamma^{\mu,(4)}_{i,j}(h_\mu(\Phi))\\ &\leqslant C\max_{0\leqslant i,j\leqslant l}\sup_{x\in(0,\infty)}|h_\mu(z^{2i}\mathcal{J}_{\mu,j}(\Phi)(z))(x)|\\ &\leqslant C\max_{0\leqslant i,j\leqslant l}\sup_{x\in(0,\infty)}\int_0^\infty|(xz)^{-\mu}J_\mu(xz)|z^{2\mu+1}|z^{2i}\mathcal{J}_{\mu,j}(\Phi)(z)|dz\\ &\leqslant C\max_{0\leqslant j\leqslant s}\sup_{z\in(0,\infty)}|(1+z^2)^s\mathcal{J}_{\mu,j}(\Phi)(z)|\\ &\leqslant C\max_{0\leqslant j\leqslant s}\eta^{\mu}_{s,j}(\Phi),\quad \Phi\in\mathcal{Q},\end{aligned}$$

for certain $l,s\in\mathbf{N}$.

Thus we conclude that $\{\eta^{\mu}_{m,n}\}_{m,n\in\mathbf{N}}$ and $\{\eta_{m,n}\}_{m,n\in\mathbf{N}}$ are equivalent on $\mathcal{Q}$. □

According to Proposition 2.4, for every $m\in\mathbf{N}$, the operator $\mathcal{J}_{\mu,m}$ defines a continuous mapping from $\mathcal{Q}$ into itself. The operator $\mathcal{J}_{\mu,m}$, $m\in\mathbf{N}$, is defined on $\mathcal{Q}'$ by transposition.

By $\mathcal{O}_{\mathcal{Q}}$ we denote the space of pointwise multipliers of $\mathcal{Q}$. We now obtain a new characterization of the elements of $\mathcal{O}_{\mathcal{Q}}$.

PROPOSITION 2.5. *A function F is in $\mathcal{O}_{\mathcal{Q}}$ if, and only if, F is even and entire and, for every $m\in\mathbf{N}$, there exists $k\in\mathbf{N}$ such that*

$$\sup_{|Im\, z|\leqslant m}(1+|z|^2)^{-k}|F(z)|<\infty.$$

Proof. Suppose that F is in $\mathcal{O}_{\mathcal{Q}}$. We consider the function $\Phi(z)=e^{-z^2}$, $z\in\mathbf{C}$. Thus Φ is even and entire and it is in $\mathcal{Q}$. Indeed, if $m,k\in\mathbf{N}$ we can write

$$\begin{aligned}(1+|z|^2)^k|\Phi(z)| &\leqslant (1+|Re\, z|^2+m^2)^k e^{-(Re\, z)^2+m^2}\\ &\leqslant C(1+|Re\, z|^2)^k e^{-(Re\, z)^2},\qquad |Im\, z|\leqslant m.\end{aligned}$$

Hence, $\eta_{m,k}(\Phi)<\infty$, for every $m,k\in\mathbf{N}$.

Since $F\in\mathcal{O}_{\mathcal{Q}}$, $F\Phi=\Psi\in\mathcal{Q}$. Then $F=\Psi/\Phi$ is an even and entire function.

Assume that there exists $m \in \mathbf{N}$ such that $\sup_{|Im\, z| \leqslant m}(1+|z|^2)^{-k}|F(z)| = \infty$, for every $k \in \mathbf{N}$.

Hence, for every $k \in \mathbf{N}$ there exists z_k such that $|Im\, z_k| \leqslant m$ and

$$(1+|z_k|^2)^{-k}|F(z_k)| \geqslant k.$$

Moreover we can suppose that $|Re\, z_k| > k+1$, $k \in \mathbf{N}$.

We define, for every $k \in \mathbf{N}$, the function Φ_k by

$$\Phi_k(z) = (\Phi(z-z_k) + \Phi(z+z_k))(1+|z_k|^2)^{-k}, \;\; z \in \mathbf{C}.$$

The sequence $\{\Phi_k\}_{k\in\mathbf{N}}$ converges to zero in $\mathcal{Q}$. Indeed, let $l, n \in \mathbf{N}$. We can write, for every $k \in \mathbf{N}$ and $z \in \mathbf{C}$,

$$(1+|z|^2)^n|\Phi_k(z)| \leqslant C(1+|z_k|^2)^{n-k}((1+|z-z_k|^2)^n|\Phi(z-z_k)|+(1+|z+z_k|^2)^n|\Phi(z+z_k)|).$$

Hence, there exists $C>0$ such that

$$\eta_{l,n}(\Phi_k) \leqslant C(1+|z_k|^2)^{n-k}\eta_{l+m,n}(\Phi), \;\; k \in \mathbf{N}.$$

Then $\Phi_k \to 0$, as $k \to \infty$, in $\mathcal{Q}$.

As a consequence of the closed graph theorem, since F is a pointwise multiplier of $\mathcal{Q}$, the mapping $\Phi \longrightarrow F\Phi$ is continuous from $\mathcal{Q}$ into itself. Hence $F\Phi_k \to 0$, as $k \to \infty$, in $\mathcal{Q}$.

However, we have, since $\Phi(2z_k) \to 0$, as $k \to \infty$,

$$\begin{aligned} |F(z_k)\Phi_k(z_k)| &= |F(z_k)(1+|z_k|^2)^{-k}(\Phi(0)+\Phi(2z_k))| \\ &\geqslant |F(z_k)|(1+|z_k|^2)^{-k}(1-|\Phi(2z_k)|) \\ &\geqslant k(1-|\Phi(2z_k)|) \\ &\geqslant \frac{k}{2}, \end{aligned}$$

provided that k is large enough.

Hence $\eta_{m,0}(F\Phi_k) \geqslant \frac{k}{2}$, when k is large enough. Thus, we conclude that $\{F\Phi_k\}_{k\in\mathbf{N}}$ does not converge to zero in Q in contradiction to the fact that F is a multiplier of $\mathcal{Q}$. Therefore, if F is a multiplier of $\mathcal{Q}$, for every $m \in \mathbf{N}$, there exist $k \in \mathbf{N}$ and $C>0$ for which

$$\sup_{|Im\, z| \leqslant m}(1+|z|^2)^{-k}|F(z)| \leqslant C.$$

Suppose now that F is an even and entire function and that, for every $m \in \mathbf{N}$, we can find $k \in \mathbf{N}$ such that $(1+|z|^2)^{-k}F$ is a bounded function on the strip $\{z \in \mathbf{C} : |Im\, z| \leqslant m\}$. Let $\Phi \in \mathcal{Q}$ and $m, n \in \mathbf{N}$. We can write

$$\begin{aligned} \eta_{m,n}(\Phi F) &= \sup_{|Im\, z| \leqslant m}(1+|z|^2)^n|\Phi(z)F(z)| \\ &\leqslant C\eta_{m,n+k}(\Phi), \end{aligned}$$

for a certain $k \in \mathbf{N}$. Thus we have proved that $\Phi F \in \mathcal{Q}$.

On $\mathcal{O}_{\mathcal{Q}}$ we consider the topology associated with the family $\{\eta_{m,n,\Phi}\}_{m,n\in\mathbf{N},\Phi\in\mathcal{Q}}$ of seminorms, where, for every $m,n\in\mathbf{N}$ and $\Phi\in\mathcal{Q}$,

$$\eta_{m,n,\Phi}(F)=\eta_{m,n}(\Phi F),\ \ F\in\mathcal{O}_{\mathcal{Q}}.$$

By $\mathcal{Q}'$ we represent the dual space of $\mathcal{Q}$. We now obtain a representation of the elements of $\mathcal{Q}'$ that will be very useful in the sequel. □

PROPOSITION 2.6. *Let $T\in\mathcal{Q}'$. There exist $m\in\mathbf{N}$ and $m+1$ complex Borel measures $\gamma_0,\gamma_1,...,\gamma_m$ on the strip $I_m=\{z\in\mathbf{C}:|Im\,z|\leqslant m\}$ such that*

$$\langle T,\Phi\rangle=\sum_{j=0}^{m}\int_{I_m}\Phi(w)w^{2j}d\gamma_j(w),\ \ \Phi\in\mathcal{Q}.$$

Proof. Since $T\in\mathcal{Q}'$ there exist $C>0$ and $m\in\mathbf{N}$ such that

$$\begin{aligned}|\langle T,\Phi\rangle|&\leqslant C\sup_{|Im\,z|\leqslant m}(1+|z|^2)^m|\Phi(z)|\\&\leqslant C\max_{0\leqslant j\leqslant m}\sup_{|Im\,z|\leqslant m}|z^{2j}\Phi(z)|,\ \ \Phi\in\mathcal{Q}.\end{aligned}$$

We now define the mappings J and H as follows

$$\begin{aligned}J:\ &\mathcal{Q}\longrightarrow\mathcal{Q}\times\overset{m+1}{\ldots}\times\mathcal{Q}\\&\Phi\longrightarrow(\Phi,z^2\Phi,...,z^{2m}\Phi)\end{aligned}$$

and

$$\begin{aligned}H:\ &J(\mathcal{Q})\longrightarrow\mathbf{C}\\&(\Phi,z^2\Phi,...,z^{2m}\Phi)\longrightarrow\langle T,\Phi\rangle.\end{aligned}$$

Thus H is a continuous mapping from $J(\mathcal{Q})$ into $\mathbf{C}$ when on $J(\mathcal{Q})$ we consider the topology induced by the product topology $L_m^\infty\times\overset{m+1}{\ldots}\times L_m^\infty$, where L_m^∞ denotes the space of essentially (with respect to the Lebesgue measure on the strip $I_m=\{z\in\mathbf{C}:|Im\,z|\leqslant m\}$) bounded functions on I_m. Note also that $J(\mathcal{Q})$ is contained in $C_m^0\times\overset{m+1}{\ldots}\times C_m^0$, where C_m^0 represents the subspace L_m^∞ constituted by all those functions $\Psi\in L_m^\infty$ such that $\lim_{|z|\to\infty,z\in I_m}\Psi(z)=0$.

By invoking now the Hahn-Banach theorem and the Riesz representation theorem we can conclude that there exist $m+1$ complex Borel measures $\gamma_0,\gamma_1,...,\gamma_m$ on I_m such that

$$\langle T,\Phi\rangle=\sum_{j=0}^{m}\int_{I_m}\Phi(w)w^{2j}d\gamma_j(w),\ \ \ \phi\in\mathcal{Q}.\ \ \square$$

If f is a Lebesgue measurable function on $(0,\infty)$ such that, for some $k\in\mathbf{N}$, $(1+x^2)^{-k}f(x)x^{2\mu+1}$ is absolutely integrable on $(0,\infty)$, then f defines an element on $\mathcal{Q}'$, that we denote again by f, through

$$\langle f,\Phi\rangle=\int_0^\infty f(x)\Phi(x)\frac{x^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dx,\ \ \Phi\in\mathcal{Q}.$$

Thus $\mathcal{Q}$ and $\mathcal{O}_{\mathcal{Q}}$ can be identified with subspaces of $\mathcal{Q}'$.

3. Hankel convolution on $\mathcal{Q}$ and $\mathcal{Q}'$

In this section we study Hankel translation and Hankel convolutions on the space $\mathcal{Q}$ and $\mathcal{Q}'$.

The space $\mathcal{X}$ is contained in $\mathcal{H}$. Hence, according to [1, Satz 5], $\mathcal{Q}$ is also a subspace of $\mathcal{H}$. Then, by using [3, (3.2)], we have that

$$(\tau_x\Phi)(y) = 2^\mu\Gamma(\mu+1)h_\mu((xt)^{-\mu}J_\mu(xt)h_\mu(\Phi)(t))(y), \quad x,y \in (0,\infty), \tag{3.1}$$

for every $\Phi \in \mathcal{Q}$.

By invoking [19, (7), Chapter 5], we can write

$$\left(\frac{1}{t}\frac{d}{dt}\right)^k((xt)^{-\mu}J_\mu(xt)) = (-1)^k x^{2k}(xt)^{-\mu-k}J_{\mu+k}(xt), \quad x,t \in \mathbf{C} \ \textit{and} \ k \in \mathbf{N}.$$

From [9, 7.12, (6)] we deduce

$$\left|\left(\frac{1}{t}\frac{d}{dt}\right)^k((xt)^{-\mu}J_\mu(xt))\right| \leqslant C|x|^{2k}e^{t|Im\,x|}, \quad x \in \mathbf{C}, \ \ t \in (0,\infty) \ \textit{and} \ k \in \mathbf{N}.$$

Hence $(xt)^{-\mu}J_\mu(xt) \in \mathcal{O}_{\mathcal{X}}$, for every $x \in \mathbf{C}$. Then, for each $x \in \mathbf{C}$, [4, Theorem 1] implies that $h_\mu((xt)^{-\mu}J_\mu(xt)h_\mu(\Phi)(t)) \in \mathcal{Q}$, provided that $\Phi \in \mathcal{Q}$. In the sequel we will adopt as definition of the Hankel translated $(\tau_x\Phi)(y)$, $x,y \in \mathbf{C}$, and $\Phi \in \mathcal{Q}$, the right hand side of (3.1), that is,

$$(\tau_x\Phi)(y) = 2^\mu\Gamma(\mu+1)h_\mu((xt)^{-\mu}J_\mu(xt)h_\mu(\Phi)(t))(y), \quad x,y \in \mathbf{C} \ \textit{and} \ \Phi \in \mathcal{Q}.$$

PROPOSITION 3.1.

(a) Let $x \in \mathbf{C}$. The Hankel translated τ_x defines a continuous linear mapping from $\mathcal{Q}$ into itself.

(b) Let $\Phi \in \mathcal{Q}$. The mapping M_Φ defined by

$$M_\Phi(z) = \tau_z\Phi, \quad \Phi \in \mathcal{Q},$$

is holomorphic from $\mathbf{C}$ into $\mathcal{Q}$.

Proof. (a) It is a consequence of [4, Theorem 2.1].

(b) According to again [4, Theorem 2.1], the assertion in (b) will be proved when we establish that, for every $\phi \in \mathcal{X}$, the mapping $m_\phi(z) = (zt)^{-\mu}J_\mu(zt)\phi$ is holomorphic from $\mathbf{C}$ into $\mathcal{X}$.

Let $\phi \in \mathcal{X}$ and $z_0 \in \mathbf{C}$. We are going to see that m_ϕ is holomorphic in z_0. We can write, for each $t \in (0,\infty)$ and $h \in \mathbf{C}\setminus\{0\}$,

$$\frac{m_\phi(z_0+h)-m_\phi(z_0)}{h} - \frac{\partial}{\partial z}((zt)^{-\mu}J_\mu(zt))_{|z=z_0}\phi(t) = \phi(t)\frac{h}{2\pi i}\int_{\mathcal{C}}\frac{(wt)^{-\mu}J_\mu(wt)}{(w-z_0-h)(w-z_0)^2}dw,$$

where $\mathcal{C}$ represents a circle having as a parametric representation $w(\theta) = z_0 + e^{i\theta}$, $\theta \in [0,2\pi)$.

Let $m,n\in\mathbf{N}$. By using [9, 7.12, (6)] and [19, (7), Chapter 5] we get

$$\begin{aligned}
&\left|e^{mt}\left(\frac{1}{t}\frac{d}{dt}\right)^{n}\left(\phi(t)\int_{\mathcal{C}}\frac{(wt)^{-\mu}J_{\mu}(wt)}{(w-z_0-h)(w-z_0)^2}dw\right)\right|\\
&\quad\leqslant Ce^{mt}\sum_{j=0}^{n}\left|\left(\frac{1}{t}\frac{d}{dt}\right)^{n-j}\phi(t)\right|\left|\int_{\mathcal{C}}\frac{w^{2j}(wt)^{-\mu-j}J_{\mu+j}(wt)}{(w-z_0-h)(w-z_0)^2}dw\right|\\
&\quad\leqslant Ce^{mt}\sum_{j=0}^{n}\left|\left(\frac{1}{t}\frac{d}{dt}\right)^{n-j}\phi(t)\right|e^{(|Im\, z_0|+1)t}\\
&\quad\leqslant C\sum_{j=0}^{n}\gamma_{l,j}(\phi),\;\; t\in(0,\infty)\;\; and\;\; |h|<\frac{1}{2},
\end{aligned}$$

for a certain $l\in\mathbf{N}$.

Hence, for $0<|h|<\frac{1}{2}$,

$$\gamma_{m,n}\left(\frac{m_\phi(z_0+h)-m_\phi(z_0)}{h}-\frac{\partial}{\partial z}((zt)^{-\mu}J_\mu(zt))_{|z=z_0}\phi(t)\right)\leqslant C|h|\sum_{j=0}^{n}\gamma_{l,j}(\phi).$$

Then

$$\gamma_{m,n}\left(\frac{m_\phi(z_0+h)-m_\phi(z_0)}{h}-\frac{\partial}{\partial z}((zt)^{-\mu}J_\mu(zt))_{|z=z_0}\phi(t)\right)\to 0,\;\; as\;\; h\to 0.$$

Thus we have proved that

$$\lim_{h\to 0}\frac{m_\phi(z_0+h)-m_\phi(z_0)}{h}=\frac{\partial}{\partial z}((zt)^{-\mu}J_\mu(zt))_{|z=z_0}\phi(t),$$

in the sense of convergence in $\mathcal{X}$. $\square$

According to [3, (3.2)] we can write, for each $\Phi,\Psi\in\mathcal{Q}$,

$$\Phi\#\Psi=h_\mu(h_\mu(\Phi)h_\mu(\Psi)).$$

Since $\mathcal{X}$ is contained in $\mathcal{O}_{\mathcal{X}}$, [4, Theorem 2.1] allows us to see that $\Phi\#\Psi\in\mathcal{Q}$, for every $\Phi,\Psi\in\mathcal{Q}$. Moreover the bilinear mapping $(\Phi,\Psi)\longrightarrow\Phi\#\Psi$ is continuous from $\mathcal{Q}\times\mathcal{Q}$ into $\mathcal{Q}$.

By taking into account Proposition 3.1, (a), we define the Hankel convolution $T\#\Phi$ of $T\in\mathcal{Q}'$ and $\Phi\in\mathcal{Q}$ as follows

$$(T\#\Phi)(z)=\langle T,\tau_z\Phi\rangle,\;\; z\in\mathbf{C}.$$

According to Proposition 3.1, (b), $T\#\Phi$ is an entire function, for every $T\in\mathcal{Q}'$ and $\Phi\in\mathcal{Q}$. Moreover, since the function $z^{-\mu}J_\mu(z)$ is even, $T\#\Phi$ is even, for each $T\in\mathcal{Q}'$ and $\Phi\in\mathcal{Q}$.

PROPOSITION 3.2. *Let* $T\in\mathcal{Q}'$ *and* $\Phi\in\mathcal{Q}$. *Then* $T\#\Phi\in\mathcal{O}_{\mathcal{Q}}$.

Proof. By invoking Proposition 2.6 it is suffficient to see that if $m,n \in \mathbf{N}$ and γ is a complex Borel measure on the strip $I_m = \{z \in \mathbf{C} : |Im\, z| \leqslant m\}$ that the function F defined by

$$F(z) = \int_{I_m} (\tau_z \Phi)(w) w^{2n} d\gamma(w), \;\; z \in \mathbf{C},$$

is in $\mathcal{O}_{\mathcal{Q}}$.

Let $k \in \mathbf{N}$. We are going to prove that there exists $l \in \mathbf{N}$ for which

$$\sup_{|Im\, z| \leqslant k} (1+|z|^2)^{-l} |F(z)| < \infty.$$

From [1, Lemma 8, (b), (6)] we infer

$$F(z) = 2^\mu \Gamma(\mu+1)(-1)^n \int_{I_m} h_\mu(\Delta^n_{\mu,t}((zt)^{-\mu} J_\mu(zt) h_\mu(\Phi)(t)))(w) d\gamma(w), \;\; z \in \mathbf{C}.$$

According to (2.6), it get

$$\begin{aligned}
&\Delta^n_{\mu,t}((zt)^{-\mu} J_\mu(zt) h_\mu(\Phi)(t)) \\
&\quad = \sum_{j=n}^{2n} c_{n,j} t^{2(j-n)} \left(\frac{1}{t}\frac{d}{dt}\right)^j ((zt)^{-\mu} J_\mu(zt) h_\mu(\Phi)(t)) \\
&\quad = \sum_{j=n}^{2n} c_{n,j} t^{2(j-n)} \sum_{\alpha=0}^{j} \binom{j}{\alpha} \left(\frac{1}{t}\frac{d}{dt}\right)^{j-\alpha} (h_\mu(\Phi)(t))(-1)^\alpha z^{2\alpha} (zt)^{-\mu-\alpha} J_{\mu+\alpha}(zt),
\end{aligned}$$

for every $t \in (0,\infty)$ and $z \in \mathbf{C}$.

Hence [9, 7.12, (6)] leads to

$$\begin{aligned}
&|h_\mu(\Delta^n_{\mu,t}((zt)^{-\mu} J_\mu(zt) h_\mu(\Phi)(t)))(w)| \\
&\quad \leqslant C \int_0^\infty |(wt)^{-\mu} J_\mu(wt)| \sum_{j=n}^{2n} t^{2(j-n)} \times \\
&\qquad \times \sum_{\alpha=0}^{j} \left| \left(\frac{1}{t}\frac{d}{dt}\right)^{j-\alpha} (h_\mu(\Phi)(t)) \right| |z|^{2\alpha} |(zt)^{-\mu-\alpha} J_{\mu+\alpha}(zt)| t^{2\mu+1} dt \\
&\quad \leqslant C \sum_{j=n}^{2n} \sum_{\alpha=0}^{j} |z|^{2\alpha} \int_0^\infty e^{t(|Im\, w|+|Im\, z|)} t^{2(j-n)} \left| \left(\frac{1}{t}\frac{d}{dt}\right)^{j-\alpha} (h_\mu(\Phi)(t)) \right| t^{2\mu+1} dt \\
&\quad \leqslant C \sum_{j=n}^{2n} \sum_{\alpha=0}^{j} |z|^{2\alpha} \int_0^\infty e^{t(k+m)} t^{2(j-n)} \left| \left(\frac{1}{t}\frac{d}{dt}\right)^{j-\alpha} (h_\mu(\Phi)(t)) \right| t^{2\mu+1} dt \\
&\quad \leqslant C(1+|z|^2)^{2n} \sum_{j=0}^{2n} \gamma_{k+m+1,j}(h_\mu(\Phi)), \;\; |Im\, z| \leqslant k \;\; and \;\; |Im\, w| \leqslant m.
\end{aligned}$$

Since $h_\mu(\Phi) \in \mathcal{X}$ we conclude that

$$|F(z)| \leqslant C(1+|z|^2)^{2n} |\gamma|(I_m), \;\; |Im\, z| \leqslant k.$$

Here $|\gamma|$ represents the total variation measure of γ. Hence $|\gamma|(I_m) < \infty$.

Thus we prove that $F \in \mathcal{O}_{\mathcal{Q}}$ and the proof is finished. □

For every $n \in \mathbf{N}$, we represent by $\mathcal{P}_n$ the space constituted by all those even and entire functions F such that, for every $m \in \mathbf{N}$,

$$\varepsilon_m^n(F) = \sup_{|Im\, z| \leqslant m} (1+|z|^2)^{-n}|F(z)| < \infty.$$

$\mathcal{P}_n$ is endowed with the topology associated with the family $\{\varepsilon_m^n\}_{m\in\mathbf{N}}$ of norms. Thus $\mathcal{P}_n$ is a Fréchet space. $\mathcal{Q}$ is continuously contained in $\mathcal{P}_n$. If $n \geqslant m$ then $\mathcal{P}_m$ is continuously contained in $\mathcal{P}_n$. By $\mathcal{P}$ we denote the space $\cup_{n\in\mathbf{N}}\mathcal{P}_n$ that is endowed with the locally convex inductive limit topology, that is $\mathcal{P} = ind_n\mathcal{P}_n$.

Along the proof of Proposition 3.2 we established the following.

PROPOSITION 3.3. *Let $T \in \mathcal{Q}'$. There exists $n \in \mathbf{N}$ such that $T\#\Phi \in \mathcal{P}_n$, for every $\Phi \in \mathcal{Q}$.*

We now prove an associative property for the distributional #-convolution.

PROPOSITION 3.4. *Let $T \in \mathcal{Q}'$ and $\Phi, \Psi \in \mathcal{Q}$. Then $T\#\Phi \in \mathcal{Q}'$ and*

$$(T\#\Phi)\#\Psi = T\#(\Phi\#\Psi).$$

Proof. According to Proposition 3.2, $T\#\Phi \in \mathcal{O}_{\mathcal{Q}}$. Hence $T\#\Phi$ defines an element of $\mathcal{Q}'$ by

$$\langle T\#\Phi, \Lambda\rangle = \int_0^\infty (T\#\Phi)(x)\Lambda(x)x^{2\mu+1}\frac{dx}{2^\mu\Gamma(\mu+1)}, \quad \Lambda \in \mathcal{Q}.$$

By Proposition 2.6 there exists $m \in \mathbf{N}$ and $m+1$ complex Borel measures $\gamma_0, \gamma_1, \ldots, \gamma_m$ on the strip $I_m = \{z \in \mathbf{C} : |Im\, z| \leqslant m\}$ such that

$$\langle T, \Lambda\rangle = \sum_{j=0}^m \int_{I_m} z^{2j}\Lambda(z)d\gamma_j(z), \quad \Lambda \in \mathcal{Q}.$$

Then, we can write

$$\begin{aligned}
\langle T\#\Phi, \Lambda\rangle &= \int_0^\infty \langle T, \tau_x\Phi\rangle\, \Lambda(x)x^{2\mu+1}\frac{dx}{2^\mu\Gamma(\mu+1)} \\
&= \sum_{j=0}^m \int_0^\infty \Lambda(x)x^{2\mu+1}\int_{I_m} z^{2j}h_\mu((xt)^{-\mu}J_\mu(xt)h_\mu(\Phi)(t))(z)d\gamma_j(z)dx \\
&= \sum_{j=0}^m \int_{I_m} z^{2j}\int_0^\infty \Lambda(x)h_\mu((xt)^{-\mu}J_\mu(xt)h_\mu(\Phi)(t))(z)x^{2\mu+1}dxd\gamma_j(z) \\
&= \sum_{j=0}^m \int_{I_m} z^{2j}\int_0^\infty \Lambda(x)h_\mu((zt)^{-\mu}J_\mu(zt)h_\mu(\Phi)(t))(x)x^{2\mu+1}dxd\gamma_j(z) \\
&= \sum_{j=0}^m \int_{I_m} z^{2j}(\Phi\#\Lambda)(z)d\gamma_j(z) \\
&= \langle T, \Phi\#\Lambda\rangle, \quad \Lambda \in \mathcal{Q}.
\end{aligned}$$

Hence, by virtue of Proposition 3.1,

$$\begin{aligned}((T\#\Phi)\#\Psi)(z) &= \langle T\#\Phi, \tau_z\Psi\rangle \\ &= \langle T, \Phi\#\tau_z\Psi\rangle \\ &= \langle T, h_\mu(h_\mu(\Phi)h_\mu(\tau_z\Psi))\rangle \\ &= \langle T, 2^\mu\Gamma(\mu+1)h_\mu((z.)^{-\mu}J_\mu(z.)h_\mu(\Phi)h_\mu(\psi))\rangle \\ &= \langle T, \tau_z(\Phi\#\Psi)\rangle \\ &= (T\#(\Phi\#\Psi))(z), \quad z\in\mathbf{C}. \qquad \square\end{aligned}$$

In the following proposition we establish a distributional interchange formula.

PROPOSITION 3.5. *Let $T\in\mathcal{Q}'$ and $\Phi\in\mathcal{Q}$. Then*

$$h'_\mu(T\#\Phi) = h'_\mu(T)h_\mu(\Phi). \tag{3.2}$$

Proof. For every $\phi\in\mathcal{X}$ Proposition 3.4 leads to

$$\begin{aligned}\langle h'_\mu(T\#\Phi), \phi\rangle &= \langle T\#\Phi, h_\mu(\phi)\rangle \\ &= ((T\#\Phi)\#h_\mu(\phi))(0) \\ &= (T\#(\Phi\#h_\mu(\phi)))(0) \\ &= \langle T, \Phi\#h_\mu(\phi)\rangle \\ &= \langle T, h_\mu(h_\mu(\Phi)\phi)\rangle \\ &= \langle h'_\mu(T)h_\mu(\Phi), \phi\rangle . \qquad \square\end{aligned}$$

If $T\in\mathcal{Q}'$ and $\Phi\in\mathcal{Q}$ it is not always true that $T\#\phi\in\mathcal{Q}$. Indeed, we define the functional T on $\mathcal{Q}$ by

$$\langle T, \Phi\rangle = \int_0^\infty \Phi(x)x^{2\mu+1}dx, \quad \Phi\in\mathcal{Q}.$$

Thus T is in $\mathcal{Q}'$. Moreover, by [13, (2), Section 2] we obtain, for every $\Phi\in\mathcal{Q}$,

$$\begin{aligned}(T\#\Phi)(x) &= \int_0^\infty (\tau_x\Phi)(y)y^{2\mu+1}dy \\ &= \int_0^\infty \Phi(z)z^{2\mu+1}\int_0^\infty D(x,y,z)\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dydz \\ &= \int_0^\infty \Phi(z)z^{2\mu+1}dz, \quad x\in(0,\infty).\end{aligned}$$

Then, since $T\#\Phi$ is an entire function (Proposition 3.2), for every $\Phi\in\mathcal{Q}$,

$$(T\#\Phi)(x) = \int_0^\infty \Phi(z)z^{2\mu+1}dz, \quad x\in\mathbf{C}.$$

Hence, if $\Phi\in\mathcal{Q}$ is such that $\int_0^\infty \Phi(z)z^{2\mu+1}dz\neq 0$ then $T\#\Phi\notin\mathcal{Q}$.

Our next objetive is to determine the elements T of $\mathcal{Q}'$ such that $T\#\Phi\in\mathcal{Q}$, for each $\Phi\in\mathcal{Q}$.

PROPOSITION 3.6. *Let $T\in\mathcal{Q}'$. Then $T\#\Phi\in\mathcal{Q}$, for every $\Phi\in\mathcal{Q}$, if, and only if, $h'_\mu(T)\in\mathcal{O}_{\mathcal{X}}$.*

Proof. Suppose that $T\#\Phi \in \mathcal{Q}$, for every $\Phi \in \mathcal{Q}$. Then, the interchange formula (3.2) implies that, for every $\Phi \in \mathcal{Q}$, $h'_\mu(T)h_\mu(\Phi) \in \mathcal{X}$. By invoking [4, Theorem 2.1] we have that, for every $\phi \in \mathcal{X}$, $S\phi \in \mathcal{X}$, where $S = h'_\mu(T) \in \mathcal{X}'$.

Let $n \in \mathbf{N}$. We choose $\phi_n \in \mathcal{X}$ such that $\phi_n(x) = 1$, $x \in (0,n)$, and $\phi_n(x) = 0$, $x > n+1$. As in [1] (see also [18]) we denote by $\mathcal{B}_n$ the subspace of $\mathcal{H}$ constituted by all those $\phi \in \mathcal{H}$ such that $\phi(x) = 0$, $x \geqslant n$. If $\varphi \in \mathcal{B}_n$ we have that

$$\langle S\phi_n, \varphi\rangle = \int_0^n (S\phi_n)(x)\varphi(x)\frac{x^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dx = \langle S, \phi_n\varphi\rangle = \langle S, \varphi\rangle .$$

Hence $S = S\phi_n$ on $\mathcal{B}_n$. We define $f_n = S\phi_n$, $n \in \mathbf{N}$. For every $\varphi \in \mathcal{B}_n$, we get

$$\langle S, \varphi\rangle = \int_0^n f_n(x)\varphi(x)\frac{x^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dx = \int_0^n f_{n+k}(x)\varphi(x)\frac{x^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dx, \quad k \in \mathbf{N}.$$

Then, $f_n(x) = f_{n+k}(x)$, $0 \leqslant x \leqslant n$ and $k \in \mathbf{N}$.

We denote by f the function defined on $[0,\infty)$ by

$$f(x) = f_n(x), \quad 0 \leqslant x \leqslant n \ \text{ and } \ n \in \mathbf{N}.$$

Thus f is a smooth function on $(0,\infty)$. Moreover if $\varphi \in \mathcal{B} = \cup_{n\in\mathbf{N}}\mathcal{B}_n$, we have

$$\langle S, \varphi\rangle = \int_0^\infty f(x)\varphi(x)\frac{x^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dx.$$

We can also write, for every $\phi \in \mathcal{X}$ and $\varphi \in \mathcal{B}$,

$$\langle S\phi, \varphi\rangle = \langle S, \phi\varphi\rangle = \int_0^\infty f(x)\phi(x)\varphi(x)\frac{x^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dx.$$

Then $S\phi = f\phi \in \mathcal{X}$, for each $\phi \in \mathcal{X}$. Hence, since $\mathcal{B}$ is a dense subspace of $\mathcal{X}$, $S = f \in \mathcal{O}_\mathcal{X}$.

Assume now that $h'_\mu(T) \in \mathcal{O}_\mathcal{X}$. By using the interchange formula (3.2) we have

$$h'_\mu(T\#\Phi) = h'_\mu(T)h_\mu(\phi), \quad \in \mathcal{X}.$$

According to [4, Theorem 2.1], $h'_\mu(T\#\Phi) \in \mathcal{X}$, $\Phi \in \mathcal{X}$. Hence $T\#\Phi = h_\mu(h'_\mu(T\#\Phi))$ is in $\mathcal{Q}$, for every $\Phi \in \mathcal{Q}$. $\square$

PROPOSITION 3.7. *Let $T \in \mathcal{Q}'$ such that $h'_\mu(T) \in \mathcal{O}_\mathcal{X}$. Then, for every $m, s \in \mathbf{N}$ there exists a function $G_{m,s}$ that is continuous and even in the strip $I_s = \{z \in \mathbf{C} : |Im\, z| \leqslant s\}$ and holomorphic in the interior of I_s and such that*

$$\sup_{|Im\, z|\leqslant s} (1+|z|^2)^m |G_{m,s}(z)| < \infty,$$

and $T = \mathcal{J}_{\mu,k_{m,s}}G_{m,s}$, for certain $k_{m,s} \in \mathbf{N}$, that is,

$$\langle T, \Phi\rangle = \lim_{n\to\infty}\sum_{j=0}^n \frac{(-1)^j k_{m,s}^{2j}}{2^j j!\Gamma(\mu+j+1)}\int_0^\infty G_{m,s}(x)\Delta_\mu^j(\Phi(x))x^{2\mu+1}dx, \quad \Phi \in \mathcal{Q}.$$

Proof. Since $h'_\mu(T) \in \mathcal{O}_{\mathcal{X}}$, for every $m \in \mathbf{N}$ there exist $C > 0$ and $n \in \mathbf{N}$ such that

$$\left|\left(\frac{1}{x}\frac{d}{dx}\right)^k h'_\mu(T)(x)\right| \leqslant Ce^{nx}, \ \ x \in (0,\infty) \ \ \textit{and} \ \ k = 0,1,...,2m. \tag{3.3}$$

To simplify we write $g_k(x) = \mathcal{I}_\mu(kx)$, $x \in (0,\infty)$ and $k \in \mathbf{N}$. From (2.4) and (2.5) we deduce that for every $m,k \in \mathbf{N}$, there exists $C > 0$ for which

$$\left|\left(\frac{1}{x}\frac{d}{dx}\right)^m \frac{1}{g_k(x)}\right| \leqslant Ce^{-kx/2}, \ \ x \in (0,\infty). \tag{3.4}$$

Let $s,m \in \mathbf{N}$. Assume that $n \in \mathbf{N}$ is associated with m satisfying (3.3). According to (3.4) we have

$$\left|\left(\frac{1}{x}\frac{d}{dx}\right)^k \frac{h'_\mu(T)(x)}{g_{2(n+s+1)}(x)}\right| \leqslant Ce^{-(s+1)x}, \ \ x \in (0,\infty) \ \ \textit{and} \ \ k = 0,1,...,2m. \tag{3.5}$$

We write $F_{m,s} = h'_\mu(T)/g_{2(n+s+1)}$. Then $T = h'_\mu(F_{m,s}g_{2(n+s+1)})$, that is,

$$\langle T,\Phi\rangle = \langle F_{m,s}, g_{2(n+s+1)}h_\mu(\Phi)\rangle, \ \ \Phi \in \mathcal{Q}.$$

Note that $g_k \in \mathcal{O}_{\mathcal{X}}$ and $\frac{1}{g_k} \in \mathcal{O}_{\mathcal{X}}$, $k \in \mathbf{N}$.

By using now Proposition 2.3 and [4, Theorem 2.1], it follows

$$\begin{aligned}
\langle T,\Phi\rangle &= \langle F_{m,s}(x), \lim_{j\to\infty}\sum_{k=0}^{j}\frac{(2(n+s+1)x)^{2k}}{2^{2k}k!\Gamma(\mu+k+1)}h_\mu(\Phi)(x)\rangle \\
&= \lim_{j\to\infty}\sum_{k=0}^{j}\langle F_{m,s}(x), h_\mu\left(\frac{(-1)^k(2(n+s+1))^{2k}}{2^{2k}k!\Gamma(\mu+k+1)}\Delta_\mu^k\Phi\right)(x)\rangle \\
&= \lim_{j\to\infty}\sum_{k=0}^{j}\langle h'_\mu(F_{m,s}), \frac{(-1)^k(2(n+s+1))^{2k}}{2^{2k}k!\Gamma(\mu+k+1)}\Delta_\mu^k\Phi\rangle \\
&= \langle \sum_{k=0}^{\infty}\frac{(-1)^k(2(n+s+1))^{2k}}{2^{2k}k!\Gamma(\mu+k+1)}\Delta_\mu^k h'_\mu(F_{m,s}),\Phi\rangle, \ \ \Phi \in \mathcal{Q}.
\end{aligned}$$

Hence

$$T = \sum_{k=0}^{\infty}\frac{(-1)^k(2(n+s+1))^{2k}}{2^{2k}k!\Gamma(\mu+k+1)}\Delta_\mu^k G_{m,s} = \mathcal{J}_{\mu,2(n+s+1)}(G_{m,s}),$$

where $G_{m,s} = h'_\mu(F_{m,s}) = h_\mu(F_{m,s})$.

On the other hand, by interchanging the order of integration and by (3.5) we obtain

$$\begin{aligned}
\langle G_{m,s},\Phi\rangle &= \langle \frac{h'_\mu(T)}{g_{2(n+s+1)}}, h_\mu(\Phi)\rangle \\
&= \int_0^\infty \frac{h'_\mu(T)(x)}{g_{2(n+s+1)}(x)}h_\mu(\Phi)(x)\frac{x^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dx \\
&= \int_0^\infty h_\mu\left(\frac{h'_\mu(T)(x)}{g_{2(n+s+1)}(x)}\right)(y)\Phi(y)\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dy, \ \ \Phi \in \mathcal{Q}.
\end{aligned}$$

Therefore

$$G_{m,s}(z) = \int_0^\infty (zt)^{-\mu} J_\mu(zt) F_{m,s}(t) t^{2\mu+1} dt, \ \ z \in (0,\infty).$$

Moreover, since $|z^{-\mu}J_\mu(z)| \leqslant Ce^{|Im\, z|}$, $z \in \mathbf{C}$ ([9, 7.12, (6)]), from (3.5) we infer that $G_{m,s}$ can be extended to the strip I_s as a continuous function that is holomorphic in the interior of I_s. According to [1, Lemma 8, (b), (6)] and by using (2.6) and (3.5) we obtain

$$\begin{aligned}
(1+|z|^2)^m |G_{m,s}(z)| &= \sum_{j=0}^m \binom{m}{j} |z^{2j} G_{m,s}(z)| \\
&= \sum_{j=0}^m \binom{m}{j} \left| \int_0^\infty \Delta_\mu^j (F_{m,s}(x))(xz)^{-\mu} J_\mu(xz) x^{2\mu+1} dx \right| \\
&\leqslant C \sum_{j=0}^m \sum_{i=j}^{2j} \int_0^\infty \left| x^{2(i-j)} \left(\frac{1}{x}\frac{d}{dx}\right)^i (F_{m,s}(x)) \right| |(xz)^{-\mu} J_\mu(xz)| x^{2\mu+1} dx \\
&\leqslant C \sum_{j=0}^m \sum_{i=j}^{2j} \int_0^\infty x^{2(i-j)} e^{-(s+1)x+|Im\, z|x} x^{2\mu+1} dx \\
&\leqslant C \sum_{j=0}^m \sum_{i=j}^{2j} \int_0^\infty x^{2(i-j)+2\mu+1} e^{-x} dx, \ \ |Im\, z| \leqslant s.
\end{aligned}$$

Thus the proof is finished. □

Let $k \in \mathbf{N}$. We define the space A_k as follows. An even and smooth function Φ on $\mathbf{R}$ is in A_k if, and only if, for every $m \in \mathbf{N}$,

$$\delta_m^k(\Phi) = \sup_{x \in (0,\infty)} (1+x^2)^{-k} |\mathcal{J}_{\mu,m}(\Phi)(x)| < \infty.$$

A_k is endowed with the topology generated by the family $\{\delta_m^k\}_{m \in \mathbf{N}}$ of seminorms. Thus A_k is continuously contained in A_{k+1}. Moreover, according to Proposition 2.4, $\mathcal{Q}$ is continuously contained in A_k. We represent by $\mathcal{A}_k$ the closure of $\mathcal{Q}$ in A_k. By $\mathcal{A}$ we denote the inductive space $\mathcal{A} = \cup_{k \in \mathbf{N}} \mathcal{A}_k$ and by $\mathcal{A}'$, as usual, we represent the dual space of $\mathcal{A}$.

PROPOSITION 3.8. *Let* $T \in \mathcal{A}'$. *Then* $h'_\mu(T) \in \mathcal{O}_\mathcal{X}$.

Proof. Let $k \in \mathbf{N}$. Since $T \in \mathcal{A}'_k$ there exists $C > 0$ and $m \in \mathbf{N}$ such that

$$|\langle T, \Phi \rangle| \leqslant C \max_{0 \leqslant l \leqslant m} \delta_l^k(\Phi), \ \ \Phi \in \mathcal{A}_k.$$

In particular, since $\mathcal{Q}$ is contained in $\mathcal{A}_k$, we have

$$|\langle T, \Phi \rangle| \leqslant C \max_{0 \leqslant l \leqslant m} \sup_{x \in (0,\infty)} (1+x^2)^{-k} |\mathcal{J}_{\mu,l}(\Phi)(x)|, \ \ \Phi \in \mathcal{Q}.$$

Since, for every $\Phi \in \mathcal{Q}$ and $l \in \mathbf{N}$,

$$\lim_{k\to\infty}(1+x^2)^{-k}\mathcal{J}_{\mu,l}(\Phi)(x)=0,$$

by employing a procedure similar to the one used in Proposition 2.6, we can prove that

$$\langle T,\Phi\rangle = \sum_{l=0}^{m}\int_0^\infty (1+x^2)^{-k}\mathcal{J}_{\mu,l}(\Phi)(x)d\gamma_l(x), \quad \Phi\in\mathcal{Q},$$

for certain complex Borel measures $\gamma_0,\gamma_1,...,\gamma_m$ on $[0,\infty)$.

Then, Proposition 2.3 implies that, for every $\phi \in \mathcal{X}$,

$$\begin{aligned}\langle h'_\mu(T),\phi\rangle &= \langle T, h_\mu(\phi)\rangle\\ &= \sum_{l=0}^{m}\int_0^\infty (1+x^2)^{-k}\mathcal{J}_{\mu,l}(h_\mu(\phi))(x)d\gamma_l(x)\\ &= \sum_{l=0}^{m}\int_0^\infty (1+x^2)^{-k}h_\mu(\mathcal{I}_\mu(lt)\phi(t))(x)d\gamma_l(x)\\ &= \sum_{l=0}^{m}\int_0^\infty \mathcal{I}_\mu(lt)\phi(t)\left(\int_0^\infty (1+x^2)^{-k}(xt)^{-\mu}J_\mu(xt)d\gamma_l(x)\right)t^{2\mu+1}dt.\end{aligned}$$

Hence,

$$h'_\mu(T)(t)=2^\mu\Gamma(\mu+1)\sum_{l=0}^{m}\mathcal{I}_\mu(lt)\int_0^\infty(1+x^2)^{-k}(xt)^{-\mu}J_\mu(xt)d\gamma_l(x), \quad t\in(0,\infty). \quad (3.6)$$

Let $n\in\mathbf{N}$. We write (3.6) for $k=n+1$. According to (2.4), (2.5) and [9, 7.12, (6)] we obtain

$$\begin{aligned}&\left|\left(\frac{1}{t}\frac{d}{dt}\right)^n h'_\mu(T)(t)\right|\\ &\leqslant C\sum_{l=0}^{m}\sum_{j=0}^{n} l^{2j}\mathcal{I}_{\mu+j}(lt)\int_0^\infty(1+x^2)^{-n-1}x^{2(n-j)}|(xt)^{-\mu-n+j}J_{\mu+n-j}(xt)|d|\gamma_l|(x)\\ &\leqslant C\sum_{l=0}^{m}\sum_{j=0}^{n} e^{lt}\int_0^\infty \frac{x^{2(n-j)}}{(1+x^2)^{n+1}}d|\gamma_l|(x)\\ &\leqslant Ce^{mt}, \quad t\in(0,\infty).\end{aligned}$$

Here $|\gamma_l|$ represents the total variation measure of γ_l, $l=0,1,...,m$.

Thus we conclude that $h'_\mu(T)\in\mathcal{O}_\mathcal{X}$. □

PROPOSITION 3.9. *Let $T\in\mathcal{Q}'$. Suppose that for every $m\in\mathbf{N}$ there exists an even and continuous function G on $\mathbf{R}$ and $k\in\mathbf{N}$ for which*

$$\sup_{z\in\mathbf{R}}(1+z^2)^m|G(z)|<\infty,$$

and $T=\mathcal{J}_{\mu,k}(G)$. Then $T\in\mathcal{A}'$.

Proof. We are going to see that $T \in \mathcal{A}'_m$, for every $m \in \mathbf{N}$.

Let $m \in \mathbf{N}$. We choose $l \in \mathbf{N}$ such that that $m + l > \mu + 1$ and then we take an even and continuous function G on $\mathbf{R}$ and $k \in \mathbf{N}$ satisfying

$$\sup_{z \in \mathbf{R}} (1 + z^2)^{m+l} |G(z)| < \infty,$$

and $T = \mathcal{J}_{\mu,k}(G)$.

Hence, for every $\Phi \in \mathcal{Q}$, we have

$$\begin{aligned} \langle T, \Phi \rangle &= \langle \mathcal{J}_{\mu,k}(G), \Phi \rangle \\ &= \langle G, \mathcal{J}_{\mu,k}(\Phi) \rangle \\ &= \int_0^\infty G(x) \mathcal{J}_{\mu,k}(\Phi)(x) \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dx, \end{aligned} \tag{3.7}$$

Then

$$\begin{aligned} |\langle T, \Phi \rangle| &\leqslant C \int_0^\infty (1 + x^2)^{-l-m} |\mathcal{J}_{\mu,k}(\Phi)(x)| x^{2\mu+1} dx \\ &\leqslant C \sup_{x \in (0,\infty)} (1 + x^2)^{-m} |\mathcal{J}_{\mu,k}(\Phi)(x)|, \quad \Phi \in \mathcal{Q}. \end{aligned}$$

Thus we have proved that T can be extended to $\mathcal{A}_m$ by (3.7) as an element of $\mathcal{A}'_m$. □

The properties established in above propositions allow us to obtain the following theorem that presents the main result of this section.

THEOREM 3.10. *Let $T \in \mathcal{Q}'$. The following assertions are equivalent.*

(i) $T \in \mathcal{A}'$.

(ii) $h'_\mu(T) \in \mathcal{O}_\mathcal{X}$.

(iii) *For every $m \in \mathbf{N}$ there exists an even and continuous function G on $\mathbf{R}$ and $k \in \mathbf{N}$ such that*

$$\sup_{x \in \mathbf{R}} (1 + x^2)^m |G(x)| < \infty,$$

and $T = \mathcal{J}_{\mu,k}(G)$.

(iv) $T \# \Phi \in \mathcal{Q}$, *for every* $\Phi \in \mathcal{Q}$.

Proof. Proposition 3.6 establishes that (ii) is equivalent to (iv). Property $(i) \Rightarrow (ii)$ was proved in Proposition 3.8. From Proposition 3.7 we deduce that $(ii) \Rightarrow (iii)$. Finally, Proposition 3.9 shows that $(iii) \Rightarrow (i)$. □

An interesting consequence of Theorem 3.10 is the following.

PROPOSITION 3.11. *Suppose that $T \in \mathcal{P}'$, where $\mathcal{P}'$ denotes the dual space of $\mathcal{P}$. Then $T \# \Phi \in \mathcal{Q}$, for every $\Phi \in \mathcal{Q}$.*

Proof. Note that, since $\mathcal{Q}$ is continuously contained in $\mathcal{P}_n$, for each $n \in \mathbf{N}$, $\mathcal{P}'$ is contained in $\mathcal{Q}'$. Hence $T \in \mathcal{Q}'$.

According to Theorem 3.10 to prove that $T \# \Phi \in \mathcal{Q}$, $\Phi \in \mathcal{Q}$, it is sufficient to see that $h'_\mu(T) \in \mathcal{O}_\mathcal{Q}$.

Let $m \in \mathbf{N}$. Since $T \in \mathcal{P}'$, T is also in $\mathcal{P}'_m$, the dual space of $\mathcal{P}_m$. Hence, there exist $C > 0$ and $l \in \mathbf{N}$ such that

$$|\langle T, \Phi \rangle| \leqslant C \sup_{|Im\, z| \leqslant l} (1 + |z|^2)^{-m} |\Phi(z)|, \quad \Phi \in \mathcal{P}_m.$$

By taking into account that

$$\lim_{|z| \to \infty,\, |Im\, z| \leqslant l} (1 + |z|^2)^{-m} \Phi(z) = 0,$$

for every $\Phi \in \mathcal{Q}$, the procedure developed in the proof of Propostion 2.6 allows us to show that

$$\langle T, \Phi \rangle = \int_{I_l} (1 + |z|^2)^{-m} \Phi(z) d\gamma(z), \quad \Phi \in \mathcal{Q},$$

for a certain complex Borel measure γ on $I_l = \{z \in \mathbf{C} : |Im\, z| \leqslant l\}$.

Then, for every $\phi \in \mathcal{X}$, we have

$$\begin{aligned} \langle h'_\mu(T), \phi \rangle &= \langle T, h_\mu(\phi) \rangle \\ &= \int_{I_l} (1 + |z|^2)^{-m} h_\mu(\phi)(z) d\gamma(z) \\ &= \int_{I_l} (1 + |z|^2)^{-m} \int_0^\infty (xz)^{-\mu} J_\mu(xz) \phi(x) x^{2\mu+1} dx d\gamma(z) \\ &= \int_0^\infty \phi(x) x^{2\mu+1} \int_{I_l} (1 + |z|^2)^{-m} (xz)^{-\mu} J_\mu(xz) d\gamma(z) dx. \end{aligned}$$

Hence,

$$h'_\mu(T)(x) = 2^\mu \Gamma(\mu + 1) \int_{I_l} (1 + |z|^2)^{-m} (xz)^{-\mu} J_\mu(xz) d\gamma(z), \quad x \in (0, \infty).$$

Moreover, from [19, (7), Chapter 5] and [9, 7.12, (6)] we infer that

$$\begin{aligned} &\left(\frac{1}{x}\frac{d}{dx}\right)^j h'_\mu(T)(x) \\ &\qquad = 2^\mu \Gamma(\mu + 1) \int_{I_l} (1+|z|^2)^{-m} z^{2j} (-1)^j (xz)^{-\mu-j} J_{\mu+j}(xz) d\gamma(z), \quad x \in (0, \infty), \end{aligned}$$

and

$$\left|\left(\frac{1}{x}\frac{d}{dx}\right)^j h'_\mu(T)(x)\right| \leqslant C \int_{I_l} (1 + |z|^2)^{-m} z^{2j} e^{x|Im\, z|} d|\gamma|(z) \leqslant C e^{lx}, \quad x \in (0, \infty),$$

provided that $j \in \mathbf{N}$, $0 \leqslant j \leqslant m$.

The arbitrariness of $m \in \mathbf{N}$ allows us to conclude that $h'_\mu(T) \in \mathcal{O}_{\mathcal{X}}$. □

According to Theorem 3.10 the Hankel convolution $T \# S$ of $T \in \mathcal{Q}'$ and $S \in \mathcal{A}'$ is the functional on $\mathcal{Q}$ given by

$$\langle T \# S, \Phi \rangle = \langle T, S \# \Phi \rangle, \quad \Phi \in \mathcal{Q}.$$

Note that, since $h'_\mu(S) \in \mathcal{O}_\mathcal{X}$ (Theorem 3.10), the mapping $\Phi \longrightarrow S\#\Phi$ is continuous from $\mathcal{Q}$ into itself. Hence $T\#S \in \mathcal{Q}'$.

A distributional interchange formula for the Hankel transform that extends the one proved in Proposition 3.5 is the following.

PROPOSITION 3.12. *Let* $T \in \mathcal{Q}'$ *and* $S \in \mathcal{A}'$. *Then*

$$h'_\mu(T\#S) = h'_\mu(T)h'_\mu(S).$$

Proof. For every $\phi \in \mathcal{X}$ we have

$$\begin{aligned}\langle h'_\mu(T\#S), \phi\rangle &= \langle T\#S, h_\mu(\phi)\rangle \\ &= \langle T, S\#h_\mu(\phi)\rangle \\ &= \langle T, h_\mu(h'_\mu(S)\phi)\rangle \\ &= \langle h'_\mu(T)h'_\mu(S), \phi\rangle . \qquad \square\end{aligned}$$

We now present some algebraic properties of the distributional #-convolution.

PROPOSITION 3.13. *Let* $T \in \mathcal{Q}'$ *and* $S, R \in \mathcal{A}'$. *Then*
(i) $S\#R \in \mathcal{A}'$ *and* $S\#R = R\#S$.
(ii) *(T#S)#R=T#(S#R).*
(iii) *For every* $m \in \mathbf{N}$, $\mathcal{J}_{\mu,m}S \in \mathcal{A}'$ *and* $\mathcal{J}_{\mu,m}(T\#S) = T\#\mathcal{J}_{\mu,m}S$.
(iv) *The Dirac functional* δ *defined, as usual, by* $\langle \delta, \Phi\rangle = \Phi(0)$, $\Phi \in \mathcal{Q}$, *is in* $\mathcal{A}'$ *and* $T\#\delta = T$.

Proof. To see (i) and (ii) it is sufficient to use the interchange formula established in Proposition 3.12 and Theorem 3.10.

According to Proposition 2.3 we can write, for each $m \in \mathbf{N}$,

$$\begin{aligned}\langle h'_\mu(\mathcal{J}_{\mu,m}S), \phi\rangle &= \langle \mathcal{J}_{\mu,m}S, h_\mu(\phi)\rangle \\ &= \langle S, \mathcal{J}_{\mu,m}h_\mu(\phi)\rangle \\ &= \langle S, h_\mu(\mathcal{I}_\mu(mx)\phi(x))\rangle \\ &= \langle h'_\mu(S)\mathcal{I}_\mu(mx), \phi\rangle , \quad \phi \in \mathcal{X}.\end{aligned}$$

Hence $h'_\mu(\mathcal{J}_{\mu,m}S) \in \mathcal{O}_\mathcal{X}$ and then $\mathcal{J}_{\mu,m}S \in \mathcal{A}'$, $m \in \mathbf{N}$.

In a similar way we can see

$$h'_\mu(\mathcal{J}_{\mu,m}(T\#S)) = \mathcal{I}_\mu(mx)h'_\mu(T)h'_\mu(S) = h'_\mu(T\#\mathcal{J}_{\mu,m}S).$$

Thus (iii) is shown.

To prove that $\delta \in \mathcal{A}'$ it is sufficient to note that $h'_\mu(\delta) = 1 \in \mathcal{O}_\mathcal{X}$. By again using Proposition 3.12 we can show that $T\#\delta = T$. $\square$

REMARK. H. Hasumi [11] investigated the Fourier transform of distributions with exponential growth. He characterized the Fourier transform of the space E as the space F constituted by all those entire functions Φ such that, for every $k, m \in \mathbf{N}$, $\eta_{m,k}(\Phi) < \infty$. By using the ideas developed in this paper we can investigate the usual convolution in F', the dual space of F, when F is endowed with the topology generated by the family $\{\eta_{m,k}\}_{m,k\in\mathbf{N}}$ of norms.

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