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CHORDS HALVING THE AREA OF A PLANAR CONVEX SET

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(*communicated by C. Bandle*)

Abstract. Let $K \subset \mathbb{R}^2$ be a compact convex set in the plane. A halving chord of K is a line segment $p\hat{p}$, $p, \hat{p} \in \partial K$, which divides the area of K into two equal parts. For every direction v there exists exactly one halving chord. Its length $h_A(v)$ is the corresponding (area) halving distance. In this article we give inequalities relating the minimum and maximum (area) halving distance h_A and H_A of a convex closed region $K \subset \mathbb{R}^2$ to other geometric quantities of K, namely the minimal width ω, the diameter D, the perimeter p, the inradius r, the circumradius R, and the area A. We try to find tight inequalities, and characterize their extremal sets (the sets attaining equality).

1. Introduction

The chords of a convex set that divide the perimeter of the set into two parts of equal length are called *halving chords*. Recently several results concerning the geometry of these chords have been obtained ([8], [9], [10], [11]). The motivation of these articles comes from graph theory; in particular from the notion of graph dilation and the authors improve the lower bounds of the dilation. Their proofs rely on a transformation defined by halving chords and they obtain different geometric inequalities relating the minimum and the maximum length of the halving chords of a convex set with other geometric quantities. Additionally they analyze curves of constant halving distance.

The aim of this paper is to analyze the chords halving the area of a bounded convex set instead of the ones halving the perimeter, and to describe their properties by means of geometric inequalities. We summarize the results known in the past, and contribute with new inequalities.

Chords halving either the area or the perimeter of a planar bounded convex set have been known for a long time:

1) In particular there is a remarkable result by Zindler [21] who in 1921 showed that there exist non circular planar curves with the property that all chords bisecting the area have equal length and also bisect the perimeter; they are called Zindler curves. He also observed that a convex set has constant area halving distance if and only if it has constant perimeter halving distance.

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2) Later Auerbach [1] extended Zindler's results and related Zindler curves to a classical problem of S. Ulam about floating bodies and also to curves of constant width.

3) Santaló [7, section A26] asked if there is for every planar convex set with area A always a chord of length at most $(4/\pi)^{1/2}A^{1/2}$ that bisects the area. The answer to this question turns out to be negative as every simple convex Zindler curve besides the circle is a counter-example (cf. [9, Lemma 8]).

4) Radziszewski [18] found a lower bound for the maximum length of area bisecting chords in terms of the diameter: $\frac{3}{4}D$.

5) Eggleston [12] obtained also a lower bound in terms of the circumradius: $\frac{3}{2}R$.

6) Hammer and Smith [15] characterized the disks as the only sets of constant width such that each of its diametral chords bisect the perimeter (area).

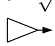
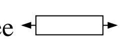
7) Chakerian and Goodey [5] using techniques from Integral Geometry obtained several inequalities relating the area of a convex set to the integral of the squared halving distances.

The chords halving the area of a planar convex set are also involved in the so called *fencing problems* which consider the best way to divide by a "fence" a convex bounded set into two subsets of equal area. See for instance [7], [17].

For the purpose of this article we define the maximum halving distance $H_A = H_A(K)$, the length of the longest chord bisecting the area of a planar convex set K, and the minimum halving distance $h_A = h_A(K)$, the length of the shortest halving chord.

We first prove some basic properties of H_A and h_A, and then we obtain upper and lower bounds of the ratio between either the maximum or the minimum halving area distance with the six classical geometric magnitudes: area A, perimeter p, diameter D, minimal width ω, inradius r and circumradius R. In many cases we determine the extremal sets which attain the bounds or at least give examples of these sets. In the end we state open problems.

The following table gives an overview of the results. Below the inequality you find symbols representing the corresponding extremal sets.

	ω	D	p	r	R	A
$H_A \leqslant$	none see –	$H_A \leqslant D$ $\supsetneq CS$	$H_A \leqslant \frac{1}{2}p$ –	none see –	$H_A \leqslant 2R$ $\supsetneq CS$	none see –
$H_A \geqslant$	$H_A \geqslant \omega$ $\supsetneq W$, $\triangle_E$	$H_A \geqslant \frac{3}{4}D$ 	$H_A \geqslant \frac{3}{4\pi}p$ not tight	$H_A \geqslant 2r$ ○	$H_A \geqslant \frac{3}{2}R$ $\triangle_E$	$H_A^2 \geqslant \frac{4}{\pi}A$ ○
$h_A \leqslant$	$h_A \leqslant \omega$ $\supsetneq CS$	$h_A \leqslant D$ ○	$h_A \leqslant \frac{1}{\pi}p$ ○	$h_A \leqslant 3r$ not tight	$h_A \leqslant 2R$ ○	$h_A^2 \leqslant \sqrt{3}A$ not tight
$h_A \geqslant$	$h_A \geqslant \frac{1}{\sqrt{2}}\omega$ 	none see –	none see –	$h_A \geqslant \sqrt{2}r$ 	none see –	none see 

The extremal sets

– line segment
○ circle
$\triangle_E$ equilateral triangle

W sets of constant width
CS centrally symmetric convex sets

The symbols "$\supsetneq CS$" mean, for instance, that all the centrally symmetric sets are extremal sets, but there are more. For example all the sets symmetric with respect to their diameter satisfy $H_A = D$ but are not necessarily centrally symmetric. And all the sets being symmetric with respect to their shortest chord satisfy $h_A = \omega$.

We would like to thank Paul Goodey for pointing out some of the known results.

2. Definitions and basic properties

Let K be a planar convex bounded set. Let $C = \partial K$ be the boundary of K. Let $p \in C$ be a point on C. Then the unique *halving partner* $\hat{p} \in C$ of p is characterized by the fact that the straight line segment $p\hat{p}$ divides K into two subsets of equal area. We say that $(p, \hat{p})$ is a *halving pair* of C.

Following [11] we give the corresponding definitions of *breadth measures*:

Let K be a bounded, planar, convex set, and let $v \in \mathbb{S}^1$ be an arbitrary direction, an element of the unit circle $\mathbb{S}^1$.

(1) The v*-length* of K is the length of the longest chord defined by a pair of points in ∂K with direction v, i.e.:

$$l(K, v) := \max\{|pq| : p, q \in C, q - p = |q - p|v\}.$$

The line segment connecting the corresponding pair of points is also called a *diametral chord*.

(2) The v*-breadth* *(*v*-width)* of K is the distance of the two supporting lines of K perpendicular to v, i.e.:

$$b(K, v) := \max_{p \in \partial K} < p, v > - \min < p, v >$$

where $< p, v >$ denotes the scalar product.

(3) The v*-halving distance*, $h_A(K, v)$, of K is the distance of the halving pair with direction v, i.e. the length of the unique chord $p\hat{p}$ halving the area of K in direction v.

(4) The *diameter*

$$D(K) := \max_{v \in \mathbb{S}^1} l(K, v)$$

of K is the maximum v-length or equivalently the maximum v-breadth. The *minimal width*

$$\omega(K) := \min_{v \in \mathbb{S}^1} l(K, v)$$

of K is the minimum v-length or equivalently the minimum v-breadth.

(5) The *maximum halving distance* is denoted by

$$H_A(K) := \max_{v \in \mathbb{S}^1} h_A(K, v).$$

Analogously the *minimum halving distance* is

$$h_A(K) := \min_{v \in \mathbb{S}^1} h_A(K, v).$$

Let $u = (\cos\theta, \sin\theta)$ be a unit vector in the plane. There is a diametral chord l_u of K parallel to u. Sometimes this chord is not unique (consider for instance a rectangle and u parallel to one of its edges), but it is possible always to select such a chord so that the function $A_{K^+}(u)$ representing the area of the subset of K at the right of l_u depends continuously on u.

LEMMA 1. *With the above notation, the function* $A_{K^+} : \mathbb{S}^1 \mapsto \mathbb{R}$ *is continuous.*

Proof. *i*) The statement is true if K is a convex polygon: the diametral chord AB in a particular direction u has one of its endpoints A in a vertex of the polygon K, and if we rotate u slightly, then A is fixed and the other endpoint B moves along an edge of the polygon till it reaches a second vertex of the polygon (so the area $A_{K^+}(u)$ changes continuously); if we continue rotating u, one of the vertices (it can be again A but it can be also B) is still fixed and the other endpoint slides along an edge of the polygon till it reaches another vertex, and still the area $A_{K^+}(u)$ changes continuously.

ii) Any convex set K can be approximated by a sequence of convex polygons inscribed in K, $P_1, P_2, ..., P_i, ...$, in such a way that the Hausdorff distance $\delta(K, P_i)$ is as small as required [19]; hence the statement is also true for any convex set. □

PROPOSITION 1. $D \geqslant H_A \geqslant \omega \geqslant h_A$.

In general these inequalities are strict as we can check in an isosceles right-angled triangle:

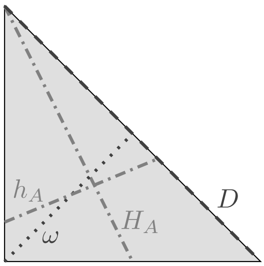


Figure 1. H_A, h_A, D *and* ω *of an isosceles right-angled triangle.*

For some sets (for instance for all centrally symmetric convex sets) $D = H_A$ and $\omega = h_A$.

Proof. 1) $D \geqslant H_A$: This inequality is trivial.

2) $H_A \geqslant \omega$: Let $v \in \mathbb{S}^1$. The maximal chord defined by a pair of points in ∂K with direction v does not in general divide K into two subsets of equal area. But because of continuity (Lemma 1) if we rotate v in $\mathbb{S}^1$ there is at least a particular direction v_0 in $\mathbb{S}^1$ such that the diametral chord in this direction divides K into two subsets of equal area. Then

$$\omega \leqslant l(K, v_0) = h_A(K, v_0) \leqslant H_A.$$

3) $\omega \geqslant h_A$: For every v, $l(K,v) \geqslant h_A(K,v)$. It follows immediately that

$$\omega = \min_{v\in\mathbb{S}^1} l(K,v) \geqslant \min_{v\in\mathbb{S}^1} h_A(K,v) = h_A. \qquad \square$$

COROLLARY 1.

$$+\infty \geqslant \frac{H_A}{h_A} \geqslant 1$$

The upper bound equality is attained only by segments and in the lower bound equality is attained by discs, and by other convex sets whose boundary is a Zindler curve.

Another consequence from Proposition 2 is the behavior of H_A and h_A under central symmetrization.

The central symmetrization is a transformation that assigns to any convex body K another convex body $C(K) := \frac{1}{2}(K+(-K))$. It is well known that central symmetrization preserves the diameter, the minimal width and the perimeter and does not decrease the area ([3]). It is also easy to prove that it does not decrease the inradius and it does not increase the circumradius.

PROPOSITION 2. *$H_A(C(K)) \geqslant H_A(K)$ and $h_A(C(K)) \geqslant h_A(K)$*

Proof. $H_A(C(K)) = D(C(K)) = D(K) \geqslant H_A(K)$, and similarly $h_A(C(K)) = \omega(C(K)) = \omega(K) \geqslant h_A(K)$ (cf. [11, Lemma 15]). □

3. Bounds derived from the classical inequalities

Combining Proposition 2 with some classical inequalities we can deduce many inequalities comparing H_A and h_A with the six classical geometric measures.

PROPOSITION 3.

$$a)\ \frac{D}{H_A} \geqslant 1, \qquad b)\ \frac{R}{H_A} \geqslant \frac{1}{2}, \qquad c)\ \frac{4\pi}{3} > \frac{p}{H_A} \geqslant 2,$$

$$d)\ 1 \geqslant \frac{\omega}{H_A} \geqslant 0,\ \ e)\ \infty \geqslant \frac{H_A^2}{A} \geqslant 4\pi,\ \ f)\ \frac{1}{2} \geqslant \frac{r}{H_A} \geqslant 0.$$

Equality in $a)$ and $b)$ is attained by many sets (for instance by centrally symmetric convex sets).

Equality in the lower bound of $c)$ is attained only by segments. The upper bound is not tight.

Equality in $d)$ is attained only by segments for the lower bound. In the upper bound case, equality is attained by many sets: sets of constant width, equilateral triangle, sets bounded by Zindler curves,...

Equality in $e)$ is attained only by discs.

Equality in $f)$ is attained only by segments for the lower bound and only by discs for the upper bound.

Proof. $a)$ The bound is proved in Proposition 2.

$b)$, $c)$ The lower bounds are obtained by plugging (a) into the known tight inequalities $D \leqslant 2R$ and $D \leqslant \frac{p}{2}$. The upper bound in b) stems from combining the known inequality $p \leqslant \pi D$ with Radziszewski's result $H_A \geqslant \frac{3}{4}D$. It is not tight because the equality $p = \pi D$ holds only for curves of constant width (see e.g. Cauchy's surface area formula [11, Lemma 7]), and by Proposition 1 these curves satisfy $H_A = D$.

$d)$ The bound is proved in Proposition 2.

$e)$ Goodey [13] proved that $\int_0^{2\pi} h_A^2(K, v(\theta))d\theta \geqslant 8A$ (equality holding for centrally symmetric sets). As obviously $H_A \geqslant h_A(K, v(\theta))$ (equality holding everywhere only for sets of constant halving distance), we conclude e). Equality holds only for centrally symmetric sets of constant halving distance. Clearly, only circles satisfy both conditions.

$f)$ The lower bound is trivial. The upper bound is consequence from $e)$ and from the classical inequality $\sqrt{A} \geqslant \sqrt{\pi}r$. As in both inequalities, equality is attained only by discs, also in the upper bound of f) equality is attained only by discs. □

In the proof of Proposition 4 we will use the following lemma provided by Hammer and Smith:

LEMMA 2. ([15]) *If K is a planar convex set of constant width such that each of its area halving chords is diametral, then K is a circular disc.*

Actually, Hammer and Smith prove a more general statement: If every area halving chord of a planar convex set K is diametral, then K is centrally symmetric. The lemma above is an easy consequence, because circles are the only centrally symmetric sets of constant width.

PROPOSITION 4.

$$a)\ \frac{r}{h_A} \geqslant \frac{1}{3}, \qquad b)\ +\infty \geqslant \frac{D}{h_A} \geqslant 1, \quad c)\ +\infty \geqslant \frac{R}{h_A} \geqslant \frac{1}{2},$$
$$d)\ +\infty \geqslant \frac{p}{h_A} \geqslant \pi, \quad e)\ \frac{\omega}{h_A} \geqslant 1.$$

The bound in $a)$ is not optimal.

Equality in $b)$, $c)$ and $d)$ is attained only by segments for the upper bound. The equality in the lower bound is attained only by discs.

Equality in $e)$ is attained by many sets for instance by all centrally symmetric convex sets.

Proof. $a)$ This bound is a consequence from the inequality $h_A \leqslant \omega$ of Proposition 2 and from the classical inequality $\omega \leqslant 3r$ (see for instance [20]). The inequality $a)$ cannot be tight, because $\omega = 3r$ is only attained by the equilateral triangle and its minimum halving distance h_A is strictly smaller than its width.

$b)$ The lower bound is proved in Proposition 2. In the equality case ($D = h_A$) two other equalities must hold: $D = \omega$ and $H_A = h_A$; this means first that K is a set of constant width and second that each of the diametral chords bisects the area; so as a consequence of Lemma 6 equality $D = h_A$ is attained only by discs.

$c)$ The lower bound follows from Proposition 2 ($h_A \leqslant \omega$) and from the classical inequality $R \geqslant \frac{\omega}{2}$ (see for instance [20]). As $R = \omega/2$ is attained only by discs, the extremal sets attaining $R = h_A/2$ are only discs.

$d)$ The result is a consequence from Proposition 2 ($h_A \leqslant \omega$) and from the classical inequality $\pi\omega \leqslant p$ (see for instance [20]). The equality $\pi\omega = p$ is attained only by sets of constant width, see for instance Cauchy's surface area formula [11, Lemma 7]. In this case $h_A = \omega$ holds only if every halving chord is diametral. Hence, from Lemma 6 we conclude that $p = \pi h_A$ is attained only by discs.

$e)$ The bound is proved in Proposition 2. □

4. Minimum halving distance h_A and width ω

In [8, Lemma 6] Dumitrescu et al. prove the inequality $h_p \geqslant \omega/2$ for the perimeter halving distance h_p. The proof can be transferred to the area halving distance. We take advantage of the following known inequality which was proved by Kubota [16] in 1923 and is listed in [20].

LEMMA 3. (Kubota [16]) *If K is a convex body in $\mathbb{R}^2$, then $A \geqslant Dw/2$.*

Proof. Without loss of generality assume that C admits a horizontal diameter D, which divides C into two parts C_1 and C_2, above and below D respectively. Then $y_1 + y_2 = w(\pi/2) \geqslant \omega$, where y_i is the extent of C_i in the vertical direction. By the convexity of C,

$$A \geqslant \frac{Dy_1 + Dy_2}{2} \geqslant \frac{Dw}{2}. \qquad \square$$

We will combine this known inequality with the following new result.

LEMMA 4. *If K is a convex body in $\mathbb{R}^2$, then $A \leqslant h_A D$. This bound cannot be improved.*

Proof. Without loss of generality we assume that a bisecting chord pq of minimum length h_A lies on the y-axis, p on top and q at the bottom (see Figure 2). Let A_- be the area of the part of K left of the y-axis, and let $A_+ := A - A_-$ be the remaining area of K. We have $A_+ = A_- = A/2$ because pq is a halving chord.

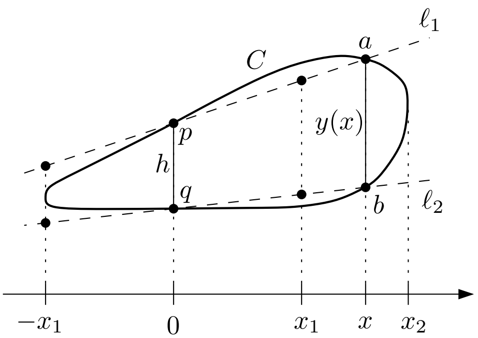


Figure 2. Proving by contradiction that $y(x) \leqslant h_A$ for every x in $[x_1, x_2]$.

Let $-x_1$ and x_2 denote the minimum and maximum x-coordinate of C. Note that x_1 has a positive value. We assume that $x_2 > x_1$. Otherwise we could reflect K at the y-axis. Let $y(x)$ be the length of the vertical line segment of x-coordinate x inside K, for every $x \in [-x_1, x_2]$. These definitions result in $x_1 + x_2 \leqslant D$ and $A = \int_{-x_1}^{x_2} y(x)\mathrm{d}x$. Furthermore, the convexity of K implies

$$\forall x \in [0, x_1]: \quad y(-x) + y(x) \leqslant 2h_A, \tag{1}$$

because the intersection of the convex hull of the vertical line segments at x and -x with the y-axis has length $(y(-x) + y(x))/2$ and must be contained in K.

As a next step, we want to show that

$$\forall x \in [x_1, x_2] : \quad y(x) \leqslant h_A. \tag{2}$$

We assume that $y(x) > h_A$. Let ab be the vertical segment of x-coordinate x inside C, a on top and b at the bottom. Then, the left part of K is enclosed in between the line ℓ_1 through p and a and the line ℓ_2 through q and b, while K contains the parts of ℓ_1 and ℓ_2 with x-coordinates in $[0, x_1]$. This implies $A_- \leqslant x_1 \cdot h_A < A_+$. This contradicts to pq being a halving chord, and the proof of (2) is completed.

Now we can plug everything together and get

$$A = \int_{-x_1}^{x_2} y(x)\mathrm{d}x = \int_{0}^{x_1} y(-x) + y(x)\mathrm{d}x + \int_{x_1}^{x_2} y(x)\mathrm{d}x$$
$$\overset{(1),(2)}{\leqslant} x_1 \cdot 2h_A + (x_2 - x_1)h_A = (x_1 + x_2)h_A \leqslant Dh_A.$$

Now consider a rectangle of side lengths a and 1, $a \ll 1$. Then, $D = \sqrt{a^2+1}$, $h_A = a$ and $A = a$. Obviously $A/(Dh_A) = 1/\sqrt{a^2+1} \nearrow 1$ for $a \to 0$. Hence, the bound cannot be improved. $\square$

Lemma 3 and Lemma 4 immediately imply

$$h_A \geqslant \frac{w}{2}.$$

However, we can show a better lower bound. We prove the following lemma which implies $h_A \geqslant \omega/\sqrt{2}$.

LEMMA 5. *If $K \subset \mathbb{R}^2$ is a convex body and $v \in \mathbb{S}^1$ is an arbitrary direction, then $h_A(v) \geqslant l(v)/\sqrt{2}$. This bound cannot be improved.*

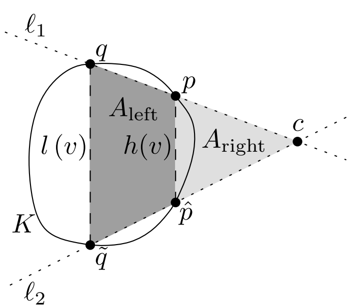


Figure 3. By convexity of K we can prove $h_A(v) \geqslant l(v)/\sqrt{2}$.
(Of course this figure is unrealistic because $p\hat{p}$ is no halving chord.)

Proof. Let $p\hat{p}$ the halving chord in direction v. And let $q\tilde{q}$ be the longest chord in direction v. By definition $h_A(v) = |p\hat{p}|$ and $l(v) = |q\tilde{q}|$. If $h_A(v) = l(v)$, we are done. Otherwise the lines ℓ_1 through p and q, and ℓ_2 through $\hat{p}$ and $\tilde{q}$ intersect in a point c.

We may assume that v is vertical and that $q\tilde{q}$ lies to the left of $p\hat{p}$. Let A_{right} denote the area of the triangle $\triangle(c,p,\hat{p})$, and let A_{left} be the area of the quadrilateral $\square(p,\hat{p},\tilde{q},q)$. By convexity, the part of K left of $p\hat{p}$ contains A_{left}, and the part of K right of $p\hat{p}$ is contained in A_{right}. Because $p\hat{p}$ is a bisecting chord, this implies

$$A_{\text{left}} \leqslant A_{\text{right}}. \tag{3}$$

The triangles $\triangle(c,p,\hat{p})$ and $\triangle(c,q,\tilde{q})$ are similar by construction. We get

$$\frac{A_{\text{left}} + A_{\text{right}}}{l^2(v)} = \frac{A_{\text{right}}}{h_A^2(v)}. \tag{4}$$

Plugging everything together yields

$$\frac{h_A^2(v)}{l^2(v)} \overset{(4)}{=} \frac{A_{\text{right}}}{A_{\text{left}} + A_{\text{right}}} \overset{(3)}{\geqslant} \frac{A_{\text{right}}}{A_{\text{right}} + A_{\text{right}}} = \frac{1}{2}.$$

Now assume $K = \triangle(c,q,\tilde{q})$. Then, inequality (3) becomes an equality and we get $h_A(v) = l(v)/\sqrt{2}$. Hence, the inequality is tight. $\square$

COROLLARY 2. *If $K \subset \mathbb{R}^2$ is a convex body, then $h_A \geqslant \omega/\sqrt{2}$. This bound cannot be improved.*

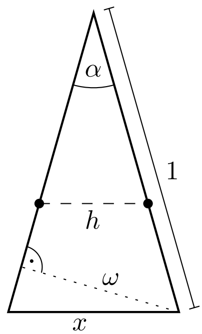


Figure 4. In a thin isosceles triangle $h_A/\omega \searrow 1/\sqrt{2}$ if $\alpha \to 0$.

Proof. Let $v \in \mathbb{S}^1$ be a direction such that $h_A = h_A(v)$. Then, we get

$$h_A = h_A(v) \overset{\text{Lem. 5}}{\geqslant} \frac{l(v)}{\sqrt{2}} \geqslant \frac{w}{\sqrt{2}}.$$

To prove the tightness of the inequality, for example consider the isosceles triangle in Figure 4. We get

$$h_A = \frac{1}{\sqrt{2}}x = \frac{1}{\sqrt{2}} \cdot 2\sin\frac{\alpha}{2} = \sqrt{2}\sin\frac{\alpha}{2}$$

because the area of the triangles is halved if we scale it by $1/\sqrt{2}$. On the other hand basic trigonometry yields

$$\omega = \sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}.$$

This results in

$$\frac{h_A}{w} = \frac{1}{\sqrt{2}\cos\frac{\alpha}{2}} \searrow \frac{1}{\sqrt{2}}. \qquad \square$$

5. Minimum halving distance h_A and inradius r

Corollary 2 further implies the following corollary.

COROLLARY 3. *If $K \subset \mathbb{R}^2$ is a convex body, then $h_A \geqslant \sqrt{2}r$. This bound is tight.*

Proof. We only have to combine Corollary 2 with the known inequality $r \leqslant \omega/2$. This inequality is trivial because if K contains a circular disk of radius r then the width of K is bigger than the width of the disk which obviously equals $2r$.

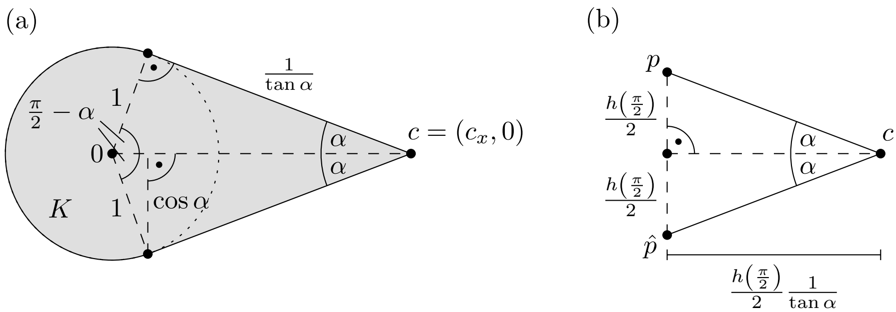


Figure 5. The grey convex body K attains $h_A/r \searrow \sqrt{2}$ for $\alpha \searrow 0$, i.e. $c_x \nearrow \infty$.

Now consider the convex body K of Figure 5 (a). It is the convex hull of a unit circle centered at the origin and a point $c = (c_x, 0)$, $c_x \geqslant 1$. Let α denote half the interior angle of K in c. Basic trigonometry yields

$$A(K) = \pi - \left(\frac{\pi}{2} - \alpha\right) + \frac{1}{\tan\alpha} = \frac{\pi}{2} + \alpha + \frac{1}{\tan\alpha}.$$

For small α, c_x gets big and the vertical halving chord $p\hat{p}$ of K is located in the triangular part of K. Remember that $h(\pi/2)$ denotes $|p\hat{p}|$ in this situation. Consider Figure 5 (b). The area of K right of $p\hat{p}$ equals

$$A_{\text{right}} = \frac{h\left(\frac{\pi}{2}\right)}{2} \cdot \frac{h\left(\frac{\pi}{2}\right)}{2}\frac{1}{\tan\alpha}.$$

Because $p\hat{p}$ is a halving chord, we get

$$\begin{aligned}
&\frac{h^2\left(\frac{\pi}{2}\right)}{4}\frac{1}{\tan\alpha} = A_{\text{right}} = \frac{1}{2}A = \frac{1}{2}\left(\frac{\pi}{2} + \alpha + \frac{1}{\tan\alpha}\right)\\
\Rightarrow \quad & h^2\left(\frac{\pi}{2}\right) = (\pi + 2\alpha)\tan\alpha + 2\\
\Rightarrow \quad & h\left(\frac{\pi}{2}\right) = \sqrt{(\pi + 2\alpha)\tan\alpha + 2}.
\end{aligned}$$

If we let α tend to 0, we get the tightness of the lower bound.

$$h_A \leqslant h\left(\frac{\pi}{2}\right) = \sqrt{(\pi+2\alpha)\tan\alpha + 2} \overset{\alpha\to 0}{\searrow} \sqrt{2} = \sqrt{2}r \qquad \square$$

6. Open problems

Obviously, $h_A/\sqrt{A}$ can become arbitrarily close to 0, as one can see by considering a rectangle of side lengths 1 and x where x tends to infinity. Pal's Theorem $\omega \leqslant \sqrt[4]{3}\sqrt{A}$ and $h_A \leqslant \omega$ from Proposition 2 imply the upper bound

$$\frac{h_A}{\sqrt{A}} \leqslant \sqrt[4]{3} \cong 1.31607...$$

This inequality is not tight.

Santaló asked (see [7, A26]) whether $\frac{h_A}{\sqrt{A}} \leqslant \sqrt{\frac{4}{\pi}} \cong 1.12837...$ where the discs would be extremal sets. As we have mentioned in the introduction this conjecture is false, and the problem of finding the greatest possible value of $\frac{h_A}{\sqrt{A}}$ is still open.

The following questions are also open problems:

Which is the smallest possible value of r/h_A?

Which is the smallest possible value of H_A/p?

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