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AN EXTENSION OF BUZANO'S INEQUALITY IN INNER PRODUCT SPACES

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Abstract. An extension of Buzano's inequality in inner product spaces and applications for discrete and integral inequalities are given.

1. Introduction

In [1] M. L. Buzano obtained the following extension of Schwarz's inequality in a real or complex inner product space $(H, \langle \cdot, \cdot \rangle)$:

$$|\langle a,x\rangle \langle x,b\rangle| \leqslant \frac{1}{2}[\|a\|\,\|b\| + |\langle a,b\rangle|]\|x\|^2, \tag{1.1}$$

for any $a,b,x \in H$. For $a = b$, the above inequality becomes the standard Schwarz inequality

$$|\langle a,x\rangle|^2 \leqslant \|a\|^2\|x\|^2, \quad a,x \in H, \tag{1.2}$$

with equality if and only if there exists a scalar $\lambda \in K$ ($K = \mathbb{R}$ or $\mathbb{C}$) such that $x = \lambda a$.

M. Fujii and F. Kubo [2] proved that the case of equality holds in (1.1) if

$$x = \begin{cases} \alpha\left(\dfrac{a}{\|a\|} + \dfrac{\langle a,b\rangle}{|\langle a,b\rangle|}\cdot\dfrac{b}{\|b\|}\right), & \text{when} \quad \langle a,b\rangle \neq 0 \\ \alpha\left(\dfrac{a}{\|a\|} + \beta\dfrac{b}{\|b\|}\right), & \text{when} \quad \langle a,b\rangle = 0,\ |\beta| = 1. \end{cases}$$

Refinements of Buzano's inequality are given by S. S. Dragomir in [3].

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2. Main result

The following theorem is the main result of the present paper.

THEOREM 2.1. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be an inner product space over the real or complex number field* K. *For all* $\alpha,\beta\in(0,\infty)$ *and* $x,a,b\in H$ *one has the inequality*

$$|\langle a,x\rangle|^{\alpha}|\langle b,x\rangle|^{\beta}\leqslant t_1^{\frac{\alpha}{2}}t_2^{\frac{\beta}{2}}\|x\|^{\alpha+\beta} \tag{2.1}$$

where

$$\begin{aligned} t_1 &= \frac{2\alpha\|a\|^2\|b\|^2+(\beta-\alpha)|\langle a,b\rangle|^2}{2\|b\|^2(\alpha+\beta)} \\ &\quad+\frac{|\langle a,b\rangle|\sqrt{|\langle a,b\rangle|^2(\alpha-\beta)^2+4\alpha\beta\|a\|^2\|b\|^2}}{2\|b\|^2(\alpha+\beta)} \\ t_2 &= \frac{2\beta\|a\|^2\|b\|^2+(\alpha-\beta)|\langle a,b\rangle|^2}{2\|a\|^2(\alpha+\beta)} \\ &\quad+\frac{|\langle a,b\rangle|\sqrt{|\langle a,b\rangle|^2(\alpha-\beta)^2+4\alpha\beta\|a\|^2\|b\|^2}}{2\|a\|^2(\alpha+\beta)} \end{aligned} \tag{2.2}$$

The case of equality holds in (2.1) *if*

$$x=\begin{cases} \dfrac{\lambda}{(\alpha+\beta)\sqrt{t_1}}\left(\alpha a+\dfrac{(\alpha+\beta)t_1-\alpha\|a\|^2}{\langle b,a\rangle}b\right), & \text{when} \quad \langle a,b\rangle\neq 0 \\ \lambda\left(\lambda_1\sqrt{\dfrac{\alpha}{\alpha+\beta}}\dfrac{a}{\|a\|}+\lambda_2\sqrt{\dfrac{\beta}{\alpha+\beta}}\dfrac{b}{\|b\|}\right), & \text{when} \quad \langle a,b\rangle=0 \end{cases}$$

where $\lambda_1,\lambda_2\in K$ *and* $|\lambda_1|=|\lambda_2|=1$.

Proof. It is sufficient to prove (2.1) for $x\in H$ with $\|x\|=1$. In this case there exists $z\in H$, $\|z\|=1$ such that

$$\max_{\|x\|=1}|\langle a,x\rangle|^{\alpha}|\langle b,x\rangle|^{\beta}=|\langle a,z\rangle|^{\alpha}|\langle b,z\rangle|^{\beta}. \tag{2.3}$$

For a fixed arbitrary $y\in H$, the function $f=f_y$ on $\mathbb{R}$ is defined by

$$f(\lambda)=\left|\left\langle a,\frac{z+\lambda y}{\|z+\lambda y\|}\right\rangle\right|^{\alpha}\left|\left\langle b,\frac{z+\lambda y}{\|z+\lambda y\|}\right\rangle\right|^{\beta}.$$

Since

$$f(\lambda)\leqslant|\langle a,z\rangle|^{\alpha}|\langle b,z\rangle|^{\beta}=f(0)$$

by (2.3), it has a maximum at $\lambda=0$.

We here remark that $f(\lambda)$ is differentiable and so

$$f'(0)=0. \tag{2.4}$$

As a matter of fact, using the identity

$$\left|\left\langle a, \frac{z+\lambda y}{\|z+\lambda y\|}\right\rangle\right| = \left(\frac{\lambda^2|\langle a,y\rangle|^2 + \lambda(\langle a,y\rangle\langle z,a\rangle + \langle a,z\rangle\langle y,a\rangle) + |\langle a,z\rangle|^2}{1+\lambda(\langle z,y\rangle + \langle y,z\rangle) + \lambda^2\|y\|^2}\right)^{\frac{1}{2}}$$

we obtain

$$\begin{aligned}\frac{f'(\lambda)}{f(\lambda)} = & \frac{\alpha(2\lambda|\langle a,y\rangle|^2 + \langle a,y\rangle\langle z,a\rangle + \langle a,z\rangle\langle y,a\rangle)}{2|\langle a,z+\lambda y\rangle|^2} \\ & + \frac{\beta(2\lambda|\langle b,y\rangle|^2 + \langle b,y\rangle\langle z,b\rangle + \langle b,z\rangle\langle y,b\rangle)}{2|\langle b,z+\lambda y\rangle|^2} \\ & - \frac{(\alpha+\beta)(\langle z,y\rangle + \langle y,z\rangle + 2\lambda\|y\|^2)}{2\|z+\lambda y\|^2}\end{aligned} \tag{2.5}$$

From (2.4) and (2.5) we have:

$$\alpha\left(\frac{\langle a,y\rangle}{\langle a,z\rangle} + \frac{\langle y,a\rangle}{\langle z,a\rangle}\right) + \beta\left(\frac{\langle b,y\rangle}{\langle b,z\rangle} + \frac{\langle y,b\rangle}{\langle z,b\rangle}\right) = (\alpha+\beta)(\langle z,y\rangle + \langle y,z\rangle). \tag{2.6}$$

The above relation is true for every $y \in H$.

If we put in (2.6) instead of y the element iy we get:

$$\begin{aligned}\alpha\left(-\frac{\langle a,y\rangle}{\langle a,z\rangle} + \frac{\langle y,a\rangle}{\langle z,a\rangle}\right) + \beta\left(-\frac{\langle b,y\rangle}{\langle b,z\rangle} + \frac{\langle y,b\rangle}{\langle z,b\rangle}\right) \\ = (\alpha+\beta)(-\langle z,y\rangle + \langle y,z\rangle).\end{aligned} \tag{2.7}$$

From (2.6) and (2.7) we obtain:

$$\alpha\frac{\langle a,y\rangle}{\langle a,z\rangle} + \beta\frac{\langle b,y\rangle}{\langle b,z\rangle} = (\alpha+\beta)\langle z,y\rangle, \tag{2.8}$$

for every $y \in H$.

Taking $y = a, b$ in (2.8), we have:

$$\frac{\alpha\|a\|^2}{\langle a,z\rangle} + \frac{\beta\langle b,a\rangle}{\langle b,z\rangle} = (\alpha+\beta)\langle z,a\rangle \tag{2.9}$$

and

$$\frac{\alpha\langle a,b\rangle}{\langle a,z\rangle} + \frac{\beta\|b\|^2}{\langle b,z\rangle} = (\alpha+\beta)\langle z,b\rangle. \tag{2.10}$$

If we put in (2.8)

$$y_0 = \frac{\alpha}{\langle a,z\rangle}a + \frac{\beta}{\langle b,z\rangle}b - (\alpha+\beta)z$$

then we obtain $y_0 = 0$ and so

$$\frac{\alpha}{\langle a,z\rangle}a + \frac{\beta}{\langle b,z\rangle}b = (\alpha+\beta)z. \tag{2.11}$$

If $\langle a,b\rangle \neq 0$ from (2.9) we get

$$\langle b,z\rangle = \frac{\beta\langle b,a\rangle\langle a,z\rangle}{(\alpha+\beta)|\langle z,a\rangle|^2-\alpha\|a\|^2}.$$

The last equality and (2.10) lead to the equation:

$$\begin{aligned}&\|b\|^2[(\alpha+\beta)|\langle z,a\rangle|^2-\alpha\|a\|^2]^2\\&\quad+(\alpha-\beta)|\langle a,b\rangle|^2[(\alpha+\beta)|\langle z,a\rangle|^2-\alpha\|a\|^2]-\alpha\beta\|a\|^2|\langle a,b\rangle|^2=0.\end{aligned} \tag{2.12}$$

In the same way we get:

$$\begin{aligned}&\|a\|^2[(\alpha+\beta)|\langle z,b\rangle|^2-\beta\|b\|^2]^2\\&\quad+(\beta-\alpha)|\langle a,\rangle|^2[(\alpha+\beta)|\langle z,b\rangle|^2-\beta\|b\|^2]-\alpha\beta\|b\|^2|\langle a,b\rangle|^2=0.\end{aligned} \tag{2.13}$$

From relations (2.12) and (2.13) we obtain:

$$\begin{aligned}|\langle a,z\rangle|^2&=\frac{2\alpha\|a\|^2\|b\|^2+(\beta-\alpha)|\langle a,b\rangle|^2}{2\|b\|^2(\alpha+\beta)}\\&\qquad\pm\frac{|\langle a,b\rangle|\sqrt{|\langle a,b\rangle|^2(\alpha-\beta)^2+4\alpha\beta\|a\|^2\|b\|^2}}{2\|b\|^2(\alpha+\beta)}\\|\langle b,z\rangle|^2&=\frac{2\beta\|a\|^2\|b\|^2+(\alpha-\beta)|\langle a,b\rangle|^2}{2\|a\|^2(\alpha+\beta)}\\&\qquad\pm\frac{|\langle a,b\rangle|\sqrt{|\langle a,b\rangle|^2(\alpha-\beta)^2+4\alpha\beta\|a\|^2\|b\|^2}}{2\|a\|^2(\alpha+\beta)}.\end{aligned} \tag{2.14}$$

The maximum in (2.3) is obtained for

$$\begin{aligned}|\langle a,z\rangle|^2&=t_1\\|\langle b,z\rangle|^2&=t_2.\end{aligned} \tag{2.15}$$

The inequality (2.1) follows from (2.15).
Since $\langle a,b\rangle \neq 0$, it follows from (2.9) that

$$\frac{\beta\langle z,b\rangle}{t_2}=\frac{\langle z,a\rangle}{\langle b,a\rangle}\left((\alpha+\beta)-\frac{\alpha\|a\|^2}{t_1}\right). \tag{2.16}$$

Using (2.16) in (2.11) we get

$$z=\frac{\langle z,a\rangle}{\alpha+\beta}\left(\frac{\alpha}{t_1}a+\frac{(\alpha+\beta)t_1-\alpha\|a\|^2}{t_1\langle b,a\rangle}b\right), \tag{2.17}$$

so that

$$z=\frac{\lambda}{(\alpha+\beta)\sqrt{t_1}}\left(\alpha a+\frac{(\alpha+\beta)t_1-\alpha\|a\|^2}{\langle b,a\rangle}b\right)\text{ where }|\lambda|=1. \tag{2.18}$$

Finally, if $\langle a, b \rangle = 0$, then (2.9) and (2.10) imply that:

$$\begin{aligned} |\langle a, z \rangle|^2 &= \tfrac{\alpha \|a\|^2}{\alpha+\beta} \\ |\langle b, z \rangle|^2 &= \tfrac{\beta \|b\|^2}{\alpha+\beta} \end{aligned} \tag{2.19}$$

From (2.19) and (2.11) we obtain

$$z = \lambda_1 \sqrt{\frac{\alpha}{\alpha+\beta}} \frac{a}{\|a\|} + \lambda_2 \sqrt{\frac{\beta}{\alpha+\beta}} \frac{b}{\|b\|}, \quad \text{where} \quad |\lambda_1| = |\lambda_2| = 1. \tag{2.20}$$

The relations (2.18) and (2.20) prove the theorem.

COROLLARY 2.2. *Let a, b be two orthogonal vectors and $\alpha, \beta > 0$. Then*

$$|\langle a, x \rangle|^\alpha |\langle b, x \rangle|^\beta \leqslant \frac{\alpha^{\frac{\alpha}{2}} \beta^{\frac{\beta}{2}}}{(\alpha+\beta)^{\frac{\alpha+\beta}{2}}} \|a\|^\alpha \|b\|^\beta \|x\|^{\alpha+\beta}. \tag{2.21}$$

REMARK. If $\alpha = \beta = 1$, (2.1) becomes Buzano's inequality (1.1).

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