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# ON WIRTINGER'S INEQUALITY AND ITS ELEMENTARY PROOF

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*Abstract.* By an elementary method, we exactly determine the best possible constant and its attaining function which satisfy $\|f\|_q \leqslant C_q\|f'\|_q$ $(1 < q < \infty)$ for certain class of continuously differentiable functions on the unit interval $[0,1]$.

## 1. Introduction and results

Let $1 \leqslant q \leqslant \infty$ and $\|f\|_q$ the $L^q$-norm of a continuous function $f$ on the unit interval $[0,1]$. Let us denote by $C^1[0,1]$ the class of continuously differentiable real-valued functions on $[0,1]$. The original Wirtinger's inequalities (cf. [2, p. 184–185]) assert that

$(a)$ $\|f\|_2 \leqslant \frac{2}{\pi}\|f'\|_2$ holds for all $f \in C^1[0,1]$ with $f(0) = 0$ and the equality is attained if and only if $f$ is a multiple of $\sin \frac{\pi}{2}t$.

$(b)$ $\|f\|_2 \leqslant \frac{1}{\pi}\|f'\|_2$ for all $f \in C^1[0,1]$ with $f(0) = f(1) = 0$ and the equality is attained if and only if $f$ is a multiple of $\sin \pi t$.

After that, many mathematicians have investigated Wirtinger-type inequalities. The purpose of this paper is to investigate Wirtinger's inequality for $L^q$-norm by an elementary method used in [3, 4]. Let $1 < p, q < \infty$ with $1/p + 1/q = 1$. Let us consider the function

$$F_p(s) = \int_0^s \frac{du}{|u|^p + 1} \qquad (-\infty < s < \infty).$$

Set

$$T_{p,\omega}(t) = F_p^{-1}((1-\omega t)\theta_p) \qquad (0 < t < 2/\omega),$$

where $\omega = 1, 2$ and $\theta_p = \int_0^\infty (u^p + 1)^{-1}\,du$. Of course $\theta_p = \pi/(p \sin \pi/p)$ as is well-known. Also we can observe that $T_{p,\omega}$ is a strictly decreasing $C^1$-function on the open unit interval $(0, 2/\omega)$ by Lemma 1. Note that $T_{p,\omega}(1/\omega) = 0$. Our main result is stated in the following theorem.

THEOREM 1. *Let $1 < p, q < \infty$ with $1/p + 1/q = 1$ and $\omega = 1, 2$. Then $\|f\|_q \leqslant C_{q,\omega}\|f'\|_q$ holds for all $f \in C^1[0,1]$ with $f(0) = f(\omega - 1) = 0$, where $C_{q,\omega} = (q-1)^{-1/q}\frac{q}{\omega\pi}\sin\frac{\pi}{p}$. The equality is attained if and only if $f$ is a multiple of $(|T_{p,\omega}|^p + 1)^{-1/q}$.*

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REMARK 1. Since $C_{2,\omega} = \frac{2}{\omega\pi}$ and $(T_{2,\omega}(t)^2+1)^{-1/2} = \sin(\omega\pi t/2)$, it follows that the original Wirtinger's inequalities $(a)$ and $(b)$ are special cases of Theorem 1. Also the contents of Theorem 1 in case when $\omega = 1$ and $q$ is an even positive integer can be observed in [2, p. 182, Theorem 256] which is implied by higher method.

REMARK 2. By a simple computation, we can see that $\lim_{q\searrow 1} C_{q,\omega} = \lim_{q\to\infty} C_{q,\omega} = 1/\omega$. Then $\|f\|_q \leqslant \frac{1}{\omega}\|f'\|_q$ $(q = 1,\infty)$ holds for all $f \in C^1[0,1]$ with $f(0) = f(\omega-1) = 0$ by Theorem 1. In more detail,

$(i)$ $\|f\|_\infty \leqslant \|f'\|_\infty$ holds for all $f \in C^1[0,1]$ with $f(0) = 0$. The equality is attained if and only if $f$ is a multiple of $t$.

$(ii)$ $\|f\|_\infty < \frac{1}{2}\|f'\|_\infty$ holds for all $f \in C^1[0,1]$ with $f(0) = f(1) = 0$ unless $f$ is the zero function.

$(iii)$ $\|f\|_1 < \frac{1}{\omega}\|f'\|_1$ holds for all $f \in C^1[0,1]$ with $f(0) = f(\omega-1) = 0$ unless $f$ is the zero function.

In fact, $(i)$ and $(ii)$ can be obtained by an easy observation. To see $(iii)$, let $1 < q < \infty$, $\omega = 1,2$ and $f \in C^1[0,1]$ with $f(0) = f(\omega-1) = 0$. Then

$$\begin{aligned}\int_0^1 |f(t)|^q\,dt &= -\frac{1}{\omega}\left(\Big[|f(t)|^q(1-\omega t)\Big]_0^1 - \int_0^1 (|f(t)|^q)'(1-\omega t)\,dt\right)\\ &= \frac{1}{\omega}\int_0^1 (|f(t)|^q)'(1-\omega t)\,dt\\ &= \frac{q}{\omega}\int_0^1 (\operatorname{sgn} f(t))|f(t)|^{q-1}f'(t)(1-\omega t)\,dt \quad \text{(by Lemma 3)}\\ &\leqslant \frac{q}{\omega}\int_0^1 |f(t)|^{q-1}|f'(t)||1-\omega t|\,dt.\end{aligned}$$

By letting $q \searrow 0$, we have

$$\int_0^1 |f(t)|\,dt \leqslant \frac{1}{\omega}\int_0^1 |f'(t)||1-\omega t|\,dt.$$

This inequality implies easily $(iii)$.

REMARK 3. Theorem 1 implies easily the following fact: For $1 \leqslant p \leqslant q \leqslant \infty$ and $n \geqslant 1$, there exists a positive constant $K(p,q;n)$ such that $\|f\|_p \leqslant K(p,q;n)\|f^{(n)}\|_q$ holds for all $C^n$-functions $f$ on $[0,1]$ with $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$. In fact putting $K(p,q;n) = C_{p,1}^n$, we obtain the desired result by Theorem 1. The similar holds for the case of $f(0) = f(1) = f'(0) = f'(1) = \cdots = f^{(n-1)}(0) = f^{(n-1)}(1) = 0$. Refer to J. Brink [2] for another detail.

## 2. Lemmas

Unless explicitly stated otherwise, $p$, $q$ and $\omega$ will be such that $1 < p,q < \infty$ with $1/p + 1/q = 1$ and $0 < \omega \leqslant 2$.

LEMMA 1. $T'_{p,\omega}(t) = -\omega\theta_p(|T_{p,\omega}(t)|^p + 1)$ $(0 < t < 2/\omega)$.

*Proof.* Since $F_p(T_{p,\omega}(t)) = (1-\omega t)\theta_p$, it follows that $F_p'(T_{p,\omega}(t))T_{p,\omega}'(t) = -\omega\theta_p$ and hence

$$\frac{T_{p,\omega}'(t)}{|T_{p,\omega}(t)|^p+1} = -\omega\theta_p,$$

which implies the desired inequality. □

LEMMA 2.

$$\lim_{t\searrow 0} t^{q-1}T_{p,\omega}(t) = \left(\frac{q-1}{\omega\theta_p}\right)^{1/(p-1)} \tag{1}$$

$$\lim_{t\nearrow 1}(1-t)^{q-1}T_{p,\omega}(t) = 0 \qquad (\omega < 2) \tag{2}$$

$$\lim_{t\nearrow 2/\omega}(2/\omega - t)^{q-1}T_{p,\omega}(t) = -\left(\frac{q-1}{\omega\theta_p}\right)^{1/(p-1)}. \tag{3}$$

*Proof.* Set $s = T_{p,\omega}(t)$. Then $t = \frac{\theta_p - F_p(s)}{\omega\theta_p}$. Hence $t \searrow 0$ if and only if $s \to \infty$. Note that one has

$$t^{q-1}T_{p,\omega}(t) = \left(\frac{1}{\omega\theta_p}\right)^{q-1} s(\theta_p - F_p(s))^{q-1} \qquad (0 \leqslant s < \infty)$$

and so $\lim_{t\searrow 0} t^{q-1}T_{p,\omega}(t)$ exists if and only if $\lim_{s\to\infty} s(\theta_p - F_p(s))^{q-1}$ exists. Set $h(s) = s(\theta_p - F_p(s))^{q-1}$ $(s \geqslant 0)$. Then

$$h'(s) = (\theta_p - F_p(s))^{q-2}(\theta_p - F_p(s) - (q-1)\frac{s}{s^p+1}) \quad (s \geqslant 0).$$

Set $g(s) = \theta_p - F_p(s) - (q-1)s/(s^p+1)$ $(0 \leqslant s < \infty)$. Then one has $g(0) = \theta_p$ and $\lim_{s\to\infty} g(s) = 0$. By differentiation

$$g'(s) = -\frac{1}{s^p+1} - (q-1)\frac{s^p+1-ps^p}{(s^p+1)^2} = \frac{-q}{(s^p+1)^2} < 0.$$

It implies $g(s) > 0$ and thus $h'(s) > 0$ $(0 \leqslant s < \infty)$ so that $h(s)$ is an increasing function. Since

$$\begin{aligned} h(s) = s(\theta_p - F_p(s))^{q-1} &= s\left(\int_s^\infty \frac{dt}{t^p+1}\right)^{q-1} \\ &\leqslant s\left(\frac{s^{1-p}}{p-1}\right)^{q-1} = \frac{1}{(p-1)^{q-1}}, \end{aligned}$$

it follows that $h(s)$ is bounded on $[0,\infty)$. Therefore $\lim_{s\to\infty} s(\theta_p - F_p(s))^{q-1}$, say, $\alpha$ exists and $\alpha > 0$. Therefore $\lim_{t\to 0} t^{q-1}T_{p,\omega}(t)$, say, $\beta$ exists and one has $\beta = \alpha\cdot(\omega\theta_p)^{-q+1}$. By L'Hospital's theorem, we have

$$\begin{aligned} \beta = \lim_{t\searrow 0} t^{q-1}T_{p,\omega}(t) &= \lim_{t\searrow 0}\frac{-\omega\theta_p(T_{p,\omega}(t)^p+1)}{(1-q)t^{-q}} \\ &= \frac{\omega\theta_p}{q-1}\lim_{t\searrow 0} t^q T_{p,\omega}(t)^p = \frac{\omega\theta_p\beta^p}{q-1}. \end{aligned}$$

Hence $\beta = (\frac{q-1}{\omega\theta_p})^{1/(p-1)}$.

If $\omega < 2$, then $T_{p,\omega}(1) < \infty$, so we have $\lim_{t\nearrow 1}(1-t)^{q-1}T_{p,\omega}(t) = 0$. To see (3) we remark that, since $F_p^{-1}(t)$ $(-\theta_p < t < \theta_p)$ is an odd function,

$$T_{p,\omega}(2/\omega - t) = -T_{p,\omega}(t) \quad (0 < t < 2/\omega).$$

Hence

$$\begin{aligned}\lim_{t\nearrow 2/\omega}(2/\omega - t)^{q-1}T_{p,\omega}(t) &= -\lim_{t\searrow 0} t^{q-1}T_{p,\omega}(t) \\ &= -\left(\frac{q-1}{\omega\theta_p}\right)^{1/(p-1)}. \qquad \square\end{aligned}$$

LEMMA 3. *Let $f \in C^1[0,1]$. Then $|f|^q \in C^1[0,1]$ and*

$$(|f|^q)'(t) = q(\operatorname{sgn} f(t))|f(t)|^{q-1}f'(t) \qquad (0 \leqslant t \leqslant 1).$$

*Proof.* Since $x \mapsto |x|^q$ for $q > 1$ is $C^1(-\infty,\infty)$ and $f \in C^1[0,1]$ is real-valued, $|f|^q$ is well-defined and belongs to $C^1[0,1]$. The formula follows directly from $(|x|^q)' = q(\operatorname{sgn} x)|x|^{q-1}$. $\square$

LEMMA 4. *Let $f \in C^1[0,1]$ with $f(0) = 0$. Then $\lim_{t\searrow 0}(\operatorname{sgn} f(t))f'(t)$, say $\alpha_f$ exists and is evaluated as $\alpha_f = |f'(0)|$. Additionally if $f(1) = 0$, then $\lim_{t\nearrow 1}(\operatorname{sgn} f(t))f'(t)$ exists and is evaluated as $|f'(1)|$.*

*Proof.* Since $f'$ is continuous on $[0,1]$, it follows that $\lim_{t\searrow 0}(\operatorname{sgn} f(t))f'(t) = 0$ when $f'(0) = 0$. Suppose $f'(0) > 0$. Then $\lim_{t\searrow 0} f(t)/t = f'(t) > 0$. Hence $f(t) > 0$ holds for sufficiently small $t > 0$. This implies the assertion. The case $f'(0) < 0$ is similar. $\square$

LEMMA 5. *For all $f \in C^1[0,1]$ with $f(0) = 0$, we assume $f(1) = 0$ when $\omega \neq 1$. Then it holds*

$$\int_0^1 T_{p,\omega}{}'(t)|f(t)|^q\,dt = -\int_0^1 T_{p,\omega}(t)(|f(t)|^q)'\,dt.$$

*Proof.* Let $f \in C^1[0,1]$ be as such. Since

$$\begin{aligned}\lim_{t\searrow 0} T_{p,\omega}(t)(|f(t)|^q)' &= q\lim_{t\searrow 0} T_{p,\omega}(t)(\operatorname{sgn} f(t))|f(t)|^{q-1}f'(t) \qquad \text{(by Lemma 3)} \\ &= q\lim_{t\searrow 0} t^{q-1}T_{p,\omega}(t)(\operatorname{sgn} f(t))f'(t)\left|\frac{f(t)}{t}\right|^{q-1} \\ &= q\left(\frac{q-1}{\omega\theta_p}\right)^{1/(p-1)}|f'(0)|^q \qquad \text{(by Lemmas 2 and (4))},\end{aligned}$$

it follows that $\lim_{t\searrow 0} T_{p,\omega}(t)(|f(t)|^q)'$ exists. If $\omega < 2$, it is clear that $\lim_{t\nearrow 1} T_{p,\omega}(t)(|f(t)|^q)'$ exists. If $\omega = 2$ then

$$\begin{aligned}\lim_{t\nearrow 1} T_{p,2}(t)(|f(t)|^q)' &= q\lim_{t\nearrow 1} T_{p,2}(t)(\operatorname{sgn} f(t))|f(t)|^{q-1}f'(t)\\ &= q\lim_{t\nearrow 1}(1-t)^{q-1}T_{p,2}(t)(\operatorname{sgn} f(t))f'(t)\left|\frac{f(t)}{t-1}\right|^{q-1}\\ &= -q\left(\frac{q-1}{\omega\theta_p}\right)^{1/(p-1)}|f'(1)|^{q-1} \qquad \text{(by Lemma 4)}\end{aligned}$$

and so $\lim_{t\nearrow 1} T_{p,2}(t)(|f(t)|^q)'$ exists. Hence the function $t\to T_{p,\omega}(t)(|f(t)|^q)'$ has the unique continuous extension to $[0,1]$, so that the integral of the right hand side should converge. To show the equality, we use the integration by parts.

$$\int_{\epsilon_0}^{\epsilon_1} T'_{p,\omega}(t)|f(t)|^q\,dt = \Big[T_{p,\omega}(t)|f(t)|^q\Big]_{\epsilon_0}^{\epsilon_1} - \int_{\epsilon_0}^{\epsilon_1} T_{p,\omega}(t)(|f(t)|^q)'\,dt. \tag{4}$$

Similar calculation shows

$$\begin{aligned}\lim_{t\searrow 0} T_{p,\omega}(t)|f(t)|^q &= \lim_{t\searrow 0} t\cdot t^{q-1}T_{p,\omega}(t)\left|\frac{f(t)}{t}\right|^q\\ &= \lim_{t\searrow 0} t\cdot\left(\frac{q-1}{\omega\theta_p}\right)^{1/(p-1)}\cdot|f'(0)|^q = 0.\end{aligned}$$

If $\omega<2$ then

$$\lim_{t\nearrow 1} T_{p,\omega}(t)|f(t)|^q = T_{p,\omega}(1)|f(1)|^q = 0.$$

Else if $\omega = 2$ then

$$\begin{aligned}\lim_{t\nearrow 1} T_{p,2}(t)|f(t)|^q &= \lim_{t\nearrow 1}(1-t)\cdot(1-t)^{q-1}T_{p,2}(t)\left|\frac{f(t)-f(1)}{t-1}\right|^q\\ &= -\lim_{t\nearrow 1}(1-t)\cdot\left(\frac{q-1}{\omega\theta_p}\right)^{1/(p-1)}\cdot|f'(1)|^q = 0.\end{aligned}$$

Taking the limit $\epsilon_0\searrow 0$, $\epsilon_1\nearrow 0$, we see the left hand side of (4) thus converges, which also show the equality of the assertion. □

Next we consider the function $S_{p,\omega}$ on $(0,2/\omega)$ defined as follows:

$$S_{p,\omega}(t) = (|T_{p,\omega}(t)|^p+1)^{-1/q} \qquad (0<t<2/\omega).$$

As observed in the proof of Lemma 3, $S_{p,\omega}$ is a $C^1$-function on $(0,2/\omega)$. Since $\lim_{t\searrow 0} T_{p,\omega}(t)=\infty$, it follows that $\lim_{t\searrow 0} S_{p,\omega}(t)=0$. We also have $\lim_{t\nearrow 2/\omega} T_{p,\omega}(t) = -\infty$ and so $\lim_{t\nearrow 2/\omega} S_{p,\omega}(t)=0$. Therefore we can regard $S_{p,\omega}$ as a function on $[0,2/\omega]$ such that $S_{p,\omega}(2/\omega)=S_{p,\omega}(0)=0$. Actually we have more.

LEMMA 6. *$S_{p,\omega}$ is a $C^1$-function on $[0,2/\omega]$ such that $S_{p,\omega}{}'(0) = -S_{p,\omega}{}'(2/\omega) = \omega(p-1)\theta_p$.*

*Proof.* Set $s = T_{p,\omega}(t)$. Since

$$S_{p,\omega}{}'(t) = \omega\theta_p \frac{p}{q} \cdot (s^p+1)^{-1/q} s^{p-1} = \omega(p-1)\theta_p \left(\frac{s^p}{(s^p+1)}\right)^{1/q}$$

for sufficiently small $t > 0$, it follows that

$$\lim_{t \searrow 0} S_{p,\omega}{}'(t) = \omega(p-1)\theta_p.$$

Hence $\lim_{t\searrow 0} S_{p,\omega}(t)/t$, say $S_{p,\omega}{}'(0)$ exists and is evaluated as $\omega(p-1)\theta_p$. From the oddness property of $T_{p,\omega}(t)$, we have $S_{p,\omega}(t) = S_{p,\omega}(2/\omega - t)$ and so $S_{p,\omega}{}'(t) = -S_{p,\omega}{}'(2/\omega - t)$, which implies the desired equality. □

We remark that $S_{p,\omega}(1/\omega) = 1$ and $S_{p,\omega}{}'(1/\omega) = 0$.

LEMMA 7. *We have*

$$S_{p,\omega}(t) = \begin{cases} \exp\left(\omega(1-p)\theta_p \int_t^{1/\omega} T_{p,\omega}(u)^{p-1}\,du\right) & (0 < t \leqslant 1/\omega) \\ \exp\left(\omega(p-1)\theta_p \int_t^{1/\omega} (-T_{p,\omega}(u))^{p-1}\,du\right) & (1/\omega < t < 2/\omega). \end{cases}$$

*Proof.* Set $s = T_{p,\omega}(t)$. Then $dt = -ds/\omega\theta_p(|s|^p+1)$. If $0 < t \leqslant 1/\omega$, then $s \geqslant 0$ and

$$\begin{aligned} \omega(1-p)\theta_p \int_t^{1/\omega} T_{p,\omega}(u)^{p-1}\,du &= \omega(1-p)\theta_p \int_{T_{p,\omega}(t)}^{T_{p,\omega}(1)} s^{p-1} \frac{-ds}{\omega\theta_p(s^p+1)} \\ &= \frac{1-p}{p} \int_0^{T_{p,\omega}(t)} \frac{ps^{p-1}}{s^p+1}\,ds \\ &= -\frac{1}{q} \int_0^{T_{p,\omega}(t)^p} \frac{dx}{x+1} \qquad (x = s^p) \\ &= -\frac{1}{q} \log(T_{p,\omega}(t)^p + 1), \end{aligned}$$

which implies the desired equality. Consider next the case $1/\omega < t < 2/\omega$. By the oddness property, we have

$$\begin{aligned} S_{p,\omega}(t) = S_{p,\omega}(2/\omega - t) &= \exp\left(\omega(1-p)\theta_p \int_{2/\omega - t}^{1/\omega} T_{p,\omega}(u)^{p-1}\,du\right) \\ &= \exp\left(\omega(1-p)\theta_p \int_t^{1/\omega} T_{p,\omega}(2/\omega - v)^{p-1}\,(-dv)\right) \\ &= \exp\left(\omega(1-p)\theta_p \int_t^{1/\omega} (-T_{p,\omega}(v))^{p-1}\,(-dv)\right). \end{aligned}$$

So we have the lemma. □

## 3. Proof of Theorem 1

Let $f \in C^1[0,1]$ with $f(0)=0$. Assume $f(1)=0$ if $\omega \neq 1$. Set

$$A_{p,\omega} = -\left(\frac{\omega p \theta_p}{q}\right)^{1/(p-1)} = -(\omega(p-1)\theta_p)^{1/(p-1)}.$$

Then one has $\omega A_{p,\omega}\theta_p + (q/p)|A_{p,\omega}|^p = 0$. By the Hölder-Rogers and Young inequalities, we have

$$\begin{aligned}
\int_0^1 A_{p,\omega}T_{p,\omega}{}'(t)|f(t)|^q\,dt &= -\int_0^1 A_{p,\omega}T_{p,\omega}(t)(|f(t)|^q)'\,dt \qquad \text{(by Lemma 5)}\\
&= -q\int_0^1 A_{p,\omega}T_{p,\omega}(t)(\operatorname{sgn} f(t))|f(t)|^{q-1}f'(t)\,dt \qquad \text{(by Lemma 3)}\\
&\leqslant q\int_0^1 |A_{p,\omega}T_{p,\omega}(t)|\cdot|f(t)|^{q-1}|f'(t)|\,dt\\
&\leqslant q\left(\int_0^1 |A_{p,\omega}T_{p,\omega}(t)|^p|f(t)|^q\,dt\right)^{1/p}\left(\int_0^1 |f'(t)|^q\,dt\right)^{1/q}\\
&\leqslant q\left(\frac{1}{p}\int_0^1 |A_{p,\omega}T_{p,\omega}(t)|^p|f(t)|^q\,dt + \frac{1}{q}\int_0^1 |f'(t)|^q\,dt\right).
\end{aligned}$$

On the other hand, by Lemma 1,

$$\begin{aligned}
A_{p,\omega}T_{p,\omega}{}'(t) - \frac{q}{p}|A_{p,\omega}T_{p,\omega}(t)|^p &= -\omega A_{p,\omega}\theta_p(|T_{p,\omega}(t)|^p+1) - \frac{q}{p}|A_{p,\omega}|^p|T_{p,\omega}(t)|^p\\
&= -|T_{p,\omega}(t)|^p\left(\omega A_{p,\omega}\theta_p + \frac{q}{p}|A_{p,\omega}|^p\right) - \omega A_{p,\omega}\theta_p\\
&= -\omega A_{p,\omega}\theta_p
\end{aligned}$$

for all $0<t<1$. Then we obtain

$$\int_0^1 (-\omega A_{p,\omega}\theta_p)|f(t)|^q\,dt \leqslant \int_0^1 |f'(t)|^q\,dt.$$

By definition of $A_{p,\omega}$,

$$\begin{aligned}
\frac{1}{-\omega A_{p,\omega}\theta_p} &= \frac{1}{\omega\theta_p}\left(\frac{q}{\omega p\theta_p}\right)^{\frac{1}{p-1}} = \left(\frac{q}{p}\right)^{q/p}\left(\frac{1}{\omega\theta_p}\right)^q\\
&= \left(\frac{q}{p}\right)^{q/p}\left(\frac{p\sin(\pi/p)}{\omega\pi}\right)^q = \left(\frac{q}{p}\right)^{q/p}\cdot p^q\cdot\left(\frac{\sin(\pi/q)}{\omega\pi}\right)^q\\
&= \frac{1}{(q-1)}\left(\frac{q\sin(\pi/q)}{\omega\pi}\right)^q.
\end{aligned}$$

Thus the former part of the theorem follows from

$$\int_0^1 |f(t)|^q\,dt \leqslant \frac{1}{(q-1)}\left(\frac{q\sin(\pi/q)}{\omega\pi}\right)^q\int_0^1 |f'(t)|^q\,dt,$$

that is

$$\|f\|_q \leqslant C_{q,\omega}\|f'\|_q. \tag{5}$$

To see the latter part, we see that the equality of (5) is attained if and only if all of the following three conditions hold:

$$\int_0^1 |A_{p,\omega}T_{p,\omega}(t)|^p|f(t)|^q\,dt = \int_0^1 |f'(t)|^q\,dt \tag{6}$$

$$|A_{p,\omega}T_{p,\omega}(t)|^p|f(t)|^q = B|f'(t)|^q \ \ (0<t<1)\text{, for some non-negative number } B \tag{7}$$

$$-A_{p,\omega}T_{p,\omega}(t)(\operatorname{sgn} f(t))|f(t)|^{q-1}f'(t) \geqslant 0 \quad (0<t<1). \tag{8}$$

Without loss of generality, we can assume that $f \neq 0$. Then (6) and (7) imply $B=1$. Then (7) holds if and only if

$$|A_{p,\omega}T_{p,\omega}|^{p/q}|f(t)| = |f'(t)| \qquad (0<t<1).$$

Note that (8) holds if and only if $(\operatorname{sgn} f(t))f'(t) \geqslant 0 \quad (0<t\leqslant \min(1,1/\omega))$ and $(\operatorname{sgn} f(t))f'(t) \leqslant 0 \quad (\min(1,1/\omega) \leqslant t<1)$. By this observation, the equality of (5) holds if and only if

$$|A_{p,\omega}T_{p,\omega}(t)|^{p/q}f(t) = f'(t) \quad (0<t\leqslant \min(1,1/\omega))$$

and

$$|A_{p,\omega}T_{p,\omega}(t)|^{p/q}f(t) = -f'(t) \quad (\min(1,1/\omega)\leqslant t<1).$$

Since $|A_{p,\omega}|^{p/q} = \omega p\theta_p/q$ and $p/q = p-1$, it follows that the equality of (5) holds if and only if

$$\omega(p-1)\theta_p T_{p,\omega}(t)^{p-1}f(t) = f'(t) \quad (0<t\leqslant \min(1,1/\omega))$$

and

$$\omega(p-1)\theta_p(-T_{p,\omega}(t))^{p-1}f(t) = -f'(t) \quad (\min(1,1/\omega)<t<1).$$

Then the equality of (5) holds if and only if

$$f(t) = \begin{cases} \lambda_0 \exp\left(\omega(1-p)\theta_p \displaystyle\int_t^{\min(1,1/\omega)} T_{p,\omega}(u)^{p-1}\,du\right) & (0<t\leqslant \min(1,1/\omega)) \\ \lambda_1 \exp\left(\omega(p-1)\theta_p \displaystyle\int_t^{\min(1,1/\omega)} (-T_{p,\omega}(u))^{p-1}\,du\right) & (\min(1,1/\omega)\leqslant t<1) \end{cases}$$

for some real numbers $\lambda_0,\lambda_1$. In this case one finds $\lambda_0=\lambda_1=f(\min(1,1/\omega))$. We see that this function is a multiple of $S_{p,\omega}$ by Lemma 7. By Lemma 6, $S_{p,\omega}$ is a $C^1$-function on $[0,1]$ such that $S_{p,\omega}(0)=0$. When $\omega=2$ or $\omega<2$, one has $S_{p,\omega}(1)=0$ or $S_{p,\omega}(1)\neq 0$ respectively. By the condition of $f$ the equality of (5) holds if and only if $\omega=1,2$ and $f$ is a multiple of $S_{p,\omega}$. Now we finish the proof of the theorem.

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