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SOME EXTENSIONS OF UNIVALENT CONDITIONS FOR CERTAIN INTEGRAL OPERATOR

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Abstract. Three subclasses $\mathcal{S}(p)$, $\mathcal{T}_2$ and $\mathcal{T}_{2,\mu}$ of analytic functions $f(z)$ in the open unit disk $\mathbb{U}$ are introduced. The object of the present paper is to discuss some extensions of univalent conditions for an integral operator $F_{\alpha,\beta}(z)$ of $f(z)$ belonging to the classes $\mathcal{S}(p)$, $\mathcal{T}_2$ and $\mathcal{T}_{2,\mu}$.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} \mid |z| < 1\}$. Also, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are univalent in $\mathbb{U}$. For some real number p with $0 < p \leqq 2$, we define the subclass $\mathcal{S}(p)$ of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leqq p \qquad (z \in \mathbb{U}).$$

Singh [3] has shown that if $f(z) \in \mathcal{S}(p)$, then $f(z)$ satisfies

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| \leqq p|z|^2 \qquad (z \in \mathbb{U}).$$

Furthermore, we define the subclass $\mathcal{T}_{2,\mu}$ of $\mathcal{S}$ consisting of functions $f(z)$ given by

$$f(z) = z + \sum_{k=3}^{\infty} a_k z^k \qquad (a_2 = 0)$$

which satisfy

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < \mu \qquad (z \in \mathbb{U})$$

for some μ $(0 < \mu \leqq 1)$. Let us denote by $\mathcal{T}_{2,1} \equiv \mathcal{T}_2$ when $\mu = 1$.

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To discuss our problems, we have to recall here the following results for the classes $\mathcal{T}_2$, $\mathcal{S}(p)$ and $\mathcal{T}_{2,\mu}$.

SCHWARZ LEMMA *Let the function* $g(z)$ *be regular in the open unit disc* $\mathbb{U}$ *with* $g(0)=0$. *If* $|g(z)| \leqslant 1$ $(z \in \mathbb{U})$, *then*

$$|g(z)| \leqslant |z| \qquad (z \in \mathbb{U}).$$

Equality holds only for $g(z) = \varepsilon z$ $(|\varepsilon| = 1)$.

THEOREM 1. [2] *Let* $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ *and* $f(z) \in \mathcal{A}$. *If* $f(z)$ *satisfies*

$$\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\left|\frac{zf''(z)}{f'(z)}\right| \leqq 1 \qquad (z \in \mathbb{U}),$$

then the integral operator

$$F_\beta(z) = \left\{\beta \int_0^z t^{\beta-1} f'(t)\, dt\right\}^{1/\beta} \tag{1}$$

is univalent in $\mathbb{U}$.

For $f(z) \in \mathcal{T}_2$, we know

THEOREM 2. [1] *Let* $f_i(z) \in \mathcal{T}_2$ *and*

$$f_i(z) \;=\; z + \sum_{k=3}^{\infty} a_k^i z^k \tag{2}$$

for $\forall i = \overline{1,n}, n \in N^*$. *If* $|f_i(z)| \leqq 1$ $(z \in \mathbb{U})$, *then, for* $\beta \in \mathbb{C}$,

$$F_{\alpha,\beta}(z) \;=\; \left\{\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{1/\alpha} dt\right\}^{1/\beta} \;\in \mathcal{S}, \tag{3}$$

where α, $\beta \in \mathbb{C}$, $\mathrm{Re}(\alpha) \geqq \dfrac{3n}{|\alpha|}$, *and* $\mathrm{Re}(\beta) \geqq \mathrm{Re}(\alpha)$.

Further, for $f(z) \in \mathcal{T}_{2,\mu}$, we see:

THEOREM 3. [1] *Let* $f_i(z)$ *defined by* (2) *be in the class* $\mathcal{T}_{2,\mu}$ *for* $\forall i = \overline{1,n}, n \in N^*$. *If* $|f_i(z)| \leqq 1$ $(z \in \mathbb{U})$, *then, for* $\beta \in \mathbb{C}$, *the integral operator* $F_{\alpha,\beta}$ *defined by* (3) *is in the class* $\mathcal{S}$, *where* α, $\beta \in \mathbb{C}$, $\mathrm{Re}(\alpha) \geqq \dfrac{n(\mu+2)}{|\alpha|}$, *and* $\mathrm{Re}(\beta) \geqq \mathrm{Re}(\alpha)$.

Also, for $f(z) \in \mathcal{S}(p)$, we introduce:

THEOREM 4. [1] *Let* $f_i(z)$ *defined by* (2) *be in the class* $\mathcal{S}(p)$ *for* $\forall i = \overline{1,n}, n \in N^*$. *If* $|f_i(z)| \leqq 1$ $(z \in \mathbb{U})$, *then, for* $\beta \in \mathbb{C}$, *the integral operator* $F_{\alpha,\beta}$ *defined by* (3) *is in the class* $\mathcal{S}$, *where* α, $\beta \in \mathbb{C}$, $\mathrm{Re}(\alpha) \geqq \dfrac{n(p+2)}{|\alpha|}$, *and* $\mathrm{Re}(\beta) \geqq \mathrm{Re}(\alpha)$.

2. Some extensions

We discuss about some extensions for the results introduced in the previous section. The first extension is contained in

THEOREM 5. *Let* $f_i(z)$ *defined by* (2) *be in the class* $\mathcal{T}_2$ *for* $\forall i = \overline{1,n}, n \in N^*$. *If* $|f_i(z)| \leqq M$ $(M \geqq 1\,;\, z \in \mathbb{U})$, *then, for* $\beta \in \mathbb{C}$, *the integral operator* $F_{\alpha,\beta}$ *defined by* (3) *is in the class* $\mathcal{S}$, *where* $\alpha,\ \beta \in \mathbb{C}$, $\operatorname{Re}(\alpha) \geqq \dfrac{(2M+1)n}{|\alpha|}$, *and* $\operatorname{Re}(\beta) \geqq \operatorname{Re}(\alpha)$.

Proof. Let us define the function $h(z)$ by

$$h(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{1/\alpha} dt$$

for $f_i(z) \in \mathcal{T}_2$. Since

$$h'(z) = \prod_{i=1}^n \left(\frac{f_i(z)}{z}\right)^{1/\alpha},$$

we see that $h(0) = 0$ and $h'(0) = 1$. Further, noting that

$$h''(z) = \sum_{i=1}^n \left(\frac{B_i}{\alpha}\, h'(z)\right)$$

with

$$B_i = \left(\frac{z}{f_i(z)}\right) \frac{z f_i'(z) - f_i(z)}{z^2} \qquad (i = 1, 2, 3, \ldots, n),$$

we obtain that

$$\frac{zh''(z)}{h'(z)} = \frac{z\dfrac{1}{\alpha}h'(z)\displaystyle\sum_{i=1}^n B_i}{h'(z)} = \frac{z}{\alpha}\sum_{i=1}^n B_i \qquad (z \in \mathbb{U}). \tag{4}$$

Replacing B_i in the formula (4), we obtain

$$\frac{zh''(z)}{h'(z)} = \frac{1}{\alpha}\sum_{i=1}^n \left(\frac{zf_i'(z)}{f_i(z)} - 1\right). \tag{5}$$

It follows from (5) that

$$\frac{1-|z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \left|\frac{zh''(z)}{h'(z)}\right| \leqq \frac{1-|z|^{2\operatorname{Re}(\alpha)}}{|\alpha|\operatorname{Re}(\alpha)} \sum_{i=1}^n \left(\left|\frac{z^2 f_i'(z)}{(f_i(z))^2}\right| \left|\frac{f_i(z)}{z}\right| + 1\right).$$

Since $|f_i(z)| \leqq M$ $(z \in \mathbb{U})$, applying the Schwarz lemma, we know that

$$\left|\frac{f_i(z)}{z}\right| \leqq M \qquad (z \in \mathbb{U}).$$

Therefore, we obtain that

$$\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\left|\frac{zh''(z)}{h'(z)}\right| \leqq \frac{1-|z|^{2\mathrm{Re}(\alpha)}}{|\alpha|\,\mathrm{Re}(\alpha)}\sum_{i=1}^{n}\left(\left|\frac{z^2 f_i'(z)}{(f_i(z))^2}\right| M+1\right). \tag{6}$$

Note that $f_i(z) \in \mathcal{T}_2$ implies

$$\begin{aligned}\left|\frac{z^2 f_i'(z)}{(f_i(z))^2}\right| &= \left|\frac{z^2 f_i'(z)}{(f_i(z))^2}-1+1\right| \\ &\leqq \left|\frac{z^2 f_i'(z)}{(f_i(z))^2}-1\right|+1 < 2 \qquad (z\in\mathbb{U}).\end{aligned} \tag{7}$$

Thus, it follows from (6) and (7) that

$$\begin{aligned}\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\left|\frac{zh''(z)}{h'(z)}\right| &\leqq \frac{\left(1-|z|^{2\mathrm{Re}(\alpha)}\right)(2M+1)n}{|\alpha|\mathrm{Re}(\alpha)} \\ &\leqq \frac{(2M+1)n}{|\alpha|\mathrm{Re}(\alpha)} \leqq 1 \qquad (z\in\mathbb{U})\end{aligned}$$

because $\mathrm{Re}(\alpha) \geqq \frac{(2M+1)n}{|\alpha|}$. Finally, applying Theorem 1 for the function $h(z)$, we prove that $F_{\alpha,\beta} \in \mathcal{S}$.

REMARK 1. If we take $M = 1$ in Theorem 5, then we have Theorem 2 in [1]. Therefore, Theorem 5 is an extension of Theorem 2.

Next, we derive

THEOREM 6. *Let $f_i(z)$ defined by (2) be in the class $\mathcal{T}_{2,\mu}$ for $\forall i = \overline{1,n}, n \in N^*$. If $|f_i(z)| \leqq M$ $(M \geqq 1\,;\, z\in\mathbb{U})$, then, for $\beta\in\mathbb{C}$, the integral operator $F_{\alpha,\beta}$ defined by (3) is in the class $\mathcal{S}$, where $\alpha,\ \beta \in \mathbb{C}$, $\mathrm{Re}(\alpha) \geqq \frac{((\mu+1)M+1)n}{|\alpha|}$, and $\mathrm{Re}(\beta) \geqq \mathrm{Re}(\alpha)$.*

Proof. Defining the function $h(z)$ by

$$h(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{1/\alpha} dt,$$

we take the same steps as in the proof of Theorem 5. Then, we obtain that

$$\begin{aligned}\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\left|\frac{zh''(z)}{h'(z)}\right| &\leqq \frac{1-|z|^{2\mathrm{Re}(\alpha)}}{|\alpha|\,\mathrm{Re}(\alpha)}\sum_{i=1}^{n}\left(\left|\frac{z^2 f_i'(z)}{(f_i(z))^2}-1\right| M+M+1\right) \\ &\leqq \frac{1-|z|^{2\mathrm{Re}(\alpha)}}{|\alpha|\,\mathrm{Re}(\alpha)}\left((\mu+1)M+1\right)n \leqq 1\end{aligned}$$

for $f_i(z) \in \mathcal{T}_{2,\mu}$. In view of Theorem 1, we know that $F_{\alpha,\beta}(z) \in \mathcal{S}$.

REMARK 2. We see that Theorem 6 is a generalization of Theorem 3 in [1].

Finally, we discuss

THEOREM 7. *Let* $f_i(z)$ *defined by* (2) *be in the class* $\mathcal{S}(p)$ *for* $\forall i = \overline{1,n}, n \in N^*$. *If* $|f_i(z)| \leqq M$ $(M \geqq 1\,;\, z \in \mathbb{U})$, *then, for* $\beta \in \mathbb{C}$, *the integral operator* $F_{\alpha,\beta}$ *defined by* (3) *is in the class* $\mathcal{S}$, *where* $\alpha,\ \beta \in \mathbb{C}$, $\mathrm{Re}(\alpha) \geqq \dfrac{((p+1)M+1)n}{|\alpha|}$, *and* $\mathrm{Re}(\beta) \geqq \mathrm{Re}(\alpha)$.

Proof. Considering

$$h(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{1/\alpha} dt,$$

and spending the same way as in the proof of Theorem 5, we see that

$$\begin{aligned}\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\left|\frac{zh''(z)}{h'(z)}\right| &\leqq \frac{1-|z|^{2\mathrm{Re}(\alpha)}}{|\alpha|\,\mathrm{Re}(\alpha)}\sum_{i=1}^n\left(\left|\frac{z^2 f_i'(z)}{(f_i(z))^2}-1\right| M+M+1\right)\\ &\leqq \frac{1-|z|^{2\mathrm{Re}(\alpha)}}{|\alpha|\,\mathrm{Re}(\alpha)}\big((p|z|^2+1)M+1\big)\,n \leqq 1\end{aligned}$$

for $f_i(z) \in \mathcal{S}(p)$. Therefore, $F_{\alpha,\beta}(z) \in \mathcal{S}$ follows from Theorem 1.

REMARK 3. Letting $M = 1$ in Theorem 7, we see that Theorem 7 is a generalization of Theorem 4 in [1].

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