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A REMARK ON BETTER λ–INEQUALITY

JAN VYBÍRAL

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Abstract. We generalize the inequality of R. J. Bagby and D. S. Kurtz [1] to a wider class of potentials defined in terms of Young's functions. We make use of a certain submultiplicativity condition. We show that this condition cannot be omited.

1. Introduction

The classical Riesz potentials are defined for every real number $0 < \gamma < n$ as a convolution operators $(I_\gamma f)(x) = (\tilde{I}_\gamma * f)(x)$, where $\tilde{I}_\gamma(x) = |x|^{\gamma-n}$. This definition coincides with the usual one up to some multiplicative constant c_γ which is not interesting for our purpose. Burkholder and Gundy invented in [2] the technique involving distribution function later known as *good* λ *-inequality*. This inequality dealt with level sets of singular integral operators and of maximal operator. Later, Bagby and Kurtz discovered in [1] that the reformulation of good λ -inequality in terms of non-increasing rearrangement contains more information.

We generalize their approach in the following way. For every Young's function Φ satisfying the Δ_2 -condition we define the Riesz potential

$$(I_\Phi f)(x) = \int_{\mathbb{R}^n} \tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right) f(y)\mathrm{d}y,$$

where $\tilde{\Phi}$ is Young's function conjugated to Φ and $\tilde{\Phi}^{-1}$ is its inverse. Instead of the classical Hardy-Littlewood maximal operator we work with a generalized maximal operator

$$(M_\varphi f)(x) = \sup_{Q\ni x} \frac{1}{\varphi(|Q|)} \int_Q |f(y)|\mathrm{d}y,$$

where φ is a given nonnegative function on $(0,\infty)$ and the supremum is taken over all cubes Q containing x with sides parallel to the coordinate axes such that $\varphi(|Q|) > 0$. For every measurable set $\Omega \subset \mathbb{R}^n$ we denote by $|\Omega|$ its Lebesgue measure.

We prove that under some restrictive condition on function Φ one can obtain an inequality combining the nonincreasing rearrangement of $I_\Phi f$ and $M_{\tilde{\Phi}^{-1}} f$. We also show that this restrictive condition cannot be left out.

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2. Better λ –inequality

Before we state our main result, we give some definitions and recall some very well known facts about Young's functions and non-increasing rearrangements.

Lebesgue measure will be denoted by μ or simply be an absolute value. Let Ω be a subset of $\mathbb{R}^n$, $n \geqslant 1$. We denote by $\mathscr{M}$ the collection of all extended scalar-valued Lebesgue measurable functions on Ω and by $\mathscr{M}_0$ the class of functions in $\mathscr{M}$ that are finite μ-a.e. Further let $\mathscr{M}^+$ be the cone of nonnegative functions from $\mathscr{M}$ and $\mathscr{M}_0^+$ the class of nonnegative functions from $\mathscr{M}_0$. We shall also write $\mathscr{M}(\Omega), \mathscr{M}^+(\Omega)$ and so on when we want to emphasize the underlying space Ω.

The letter c denotes a general constant which does not depend on the parameters involved. It may change from one occurrence to another.

DEFINITION 2.1. 1. Let $\phi : [0,\infty) \to [0,\infty)$ be a non-decreasing and right-continuous function with $\phi(0) = 0$ and $\phi(\infty) = \lim_{t\to\infty} \phi(t) = \infty$. Then the function Φ defined by

$$\Phi(t) = \int_0^t \phi(s)\mathrm{d}s, \quad t \geqslant 0$$

is said to be a *Young's function*.

2. A Young's function is said to satisfy the Δ_2*–condition* if there is $c > 0$ such that

$$\Phi(2t) \leqslant c\,\Phi(t), \quad t \geqslant 0.$$

3. A Young's function is said to satisfy the ∇_2*–condition* if there is $l > 1$ such that

$$\Phi(t) \leqslant \frac{1}{2l}\Phi(lt), \quad t \geqslant 0.$$

4. Let Φ be a Young's function, represented as the indefinite integral of ϕ. Let

$$\psi(s) = \sup\{u : \phi(u) \leqslant s\}, \quad s \geqslant 0.$$

Then the function

$$\tilde{\Phi}(t) = \int_0^t \psi(s)\mathrm{d}s, \quad t \geqslant 0,$$

is called the *complementary Young's function* of Φ.

The following theorem puts these three notions together. For the proof see [3].

THEOREM 2.2. *Let Φ be a Young's function and $\tilde{\Phi}$ be its complementary Young's function. Then Φ satisfies the Δ_2–condition if and only if $\tilde{\Phi}$ satisfies the ∇_2–condition.*

We shall need following Lemma.

LEMMA 2.3. *Let Φ be a Young's function satisfying the Δ_2—condition. Then there is a constant $c > 0$ such that*

$$\int_0^t \tilde{\Phi}^{-1}\left(\frac{1}{u}\right)\mathrm{d}u \leqslant c\,t\tilde{\Phi}^{-1}\left(\frac{1}{t}\right), \quad 0 < t < \infty$$

Proof. If Φ satisfies the Δ_2—condition, then $\tilde{\Phi}$ satisfies the ∇_2—condition. It means that there is a real number $k > 1$ such that $\tilde{\Phi}(t) \leqslant \frac{1}{2k}\tilde{\Phi}(kt)$ for every $t > 0$. When we pass to inverses we get $\tilde{\Phi}^{-1}\left(\frac{1}{u}\right) \leqslant \frac{l}{2}\tilde{\Phi}^{-1}\left(\frac{1}{lu}\right)$, where $l = 2k > 2$ and $u > 0$. Now setting $h(s) = \tilde{\Phi}^{-1}\left(\frac{1}{s}\right)$ and $H(u) = \int_0^u h(s)\mathrm{d}s$ we get $2h(s) \leqslant lh(ls)$ and integrating this inequality from 0 to t we obtain $2H(t) \leqslant H(lt)$. To show that $H(t)$ is finite for all $t > 0$, write

$$\begin{aligned} H(t) &= \int_0^t h(s)\mathrm{d}s = \sum_{k=0}^{\infty} \int_{t/l^{k+1}}^{t/l^k} h(s)\mathrm{d}s \\ &\leqslant \sum_{k=0}^{\infty} \int_{t/l^{k+1}}^{t/l^k} \frac{l^k}{2^k} h(l^k s)\mathrm{d}s \\ &= \sum_{k=0}^{\infty} \frac{1}{2^k} \int_{t/l}^{t} h(u)\mathrm{d}u < \infty. \end{aligned}$$

Because h is a decreasing function, we can calculate

$$lth(t) \geqslant \int_t^{lt} h(s)\mathrm{d}s = H(lt) - H(t) \geqslant 2H(t) - H(t) = H(t),$$

which can be rewritten as

$$lt\tilde{\Phi}^{-1}\left(\frac{1}{t}\right) \geqslant \int_0^t \tilde{\Phi}^{-1}\left(\frac{1}{u}\right)\mathrm{d}u.$$

DEFINITION 2.4. The *distribution function* μ_f of a function f in $\mathscr{M}_0(\Omega)$ is given by

$$\mu_f(\lambda) = \mu(\{x \in \Omega : |f(x)| > \lambda\}), \quad \lambda \geqslant 0.$$

For every $f \in \mathscr{M}_0(\Omega)$ we define its *nonincreasing rearrangement* f^* by

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leqslant t\}, \quad 0 \leqslant t < \infty$$

and its *maximal function* f^{**} by

$$f^{**}(t) = t^{-1}\int_0^t f^*(u)\mathrm{d}u, \quad 0 < t < \infty.$$

Assume now that Young's function Φ satisfies the Δ_2–condition. Using the classical O'Neil inequality (see [4]) and Lemma 2.3 we obtain

$$(I_\Phi f)^*(t) \leqslant c\left\{\tilde{\Phi}^{-1}\left(\frac{1}{t}\right)\int_0^t f^*(u)\mathrm{d}u + \int_t^{\infty} f^*(u)\tilde{\Phi}^{-1}\left(\frac{1}{u}\right)\mathrm{d}u\right\}, \tag{1}$$

We shall derive a better λ -inequality connecting the operators I_Φ and $M_{\tilde{\Phi}^{-1}}$.

THEOREM 2.5. *Let us suppose that a Young's function Φ satisfies the Δ_2–condition. Let us further suppose that there is a constant $c_1 > 0$ such that*

$$\tilde{\Phi}^{-1}(s)\tilde{\Phi}^{-1}(1/s) < c_1, \quad s > 0. \tag{2}$$

Then there is a constant $c_2 > 0$*, such that for every function* f *and every positive number* t

$$(I_\Phi f)^*(t) \leqslant (I_\Phi |f|)^*(t) \leqslant c_2\,(M_{\tilde{\Phi}^{-1}} f)^*(t/2) + (I_\Phi |f|)^*(2t) \tag{3}$$

Proof. We may assume that given function f is nonnegative.

First we shall estimate the size of the level set $G = \{x \in \mathbb{R}^n : (I_\Phi g)(x) > \lambda\}$ for function $g \in L^1(\mathbb{R}^n)$. According to (1), $|G| < \infty$. Hence we can find a real number $R \geqslant 0$ such that $|G| = |B(0,R)|$. We can write

$$\begin{aligned}
\lambda|G| &= \int_G \lambda \leqslant \int_G (I_\Phi g)(x)\mathrm{d}x \\
&= \int_G \int_{\mathbb{R}^n} g(y)\tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right)\mathrm{d}y\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \int_G \tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right)\mathrm{d}x g(y)\mathrm{d}y \\
&\leqslant ||g||_1 \int_{B(0,R)} \tilde{\Phi}^{-1}\left(\frac{1}{|x|^n}\right)\mathrm{d}x \\
&= ||g||_1 \alpha_n \int_0^{|G|/\alpha_n} \tilde{\Phi}^{-1}(1/s)\mathrm{d}s.
\end{aligned}$$

Dividing this inequality by $|G|$ and using the Lemma 2.3 we obtain

$$\lambda \leqslant ||g||_1 \frac{\alpha_n}{|G|}\int_0^{|G|/\alpha_n} \tilde{\Phi}^{-1}(1/s)\mathrm{d}s \leqslant \tilde{c}\,||g||_1 \tilde{\Phi}^{-1}\left(\frac{1}{|G|}\right).$$

This can be rewritten as

$$|G| \leqslant \frac{1}{\tilde{\Phi}\left(\frac{\lambda}{\tilde{c}||g||_1}\right)}, \tag{4}$$

where $\tilde{c}$ is independent of g and λ.

We can now pass to the proof of our theorem which is mainly based on [1]. For a given function $f \geqslant 0$ and a real number $t > 0$ we shall denote by E the set $\{x \in \mathbb{R}^n : (I_\Phi f)(x) > (I_\Phi f)^*(2t)\}$. Then $|E| \leqslant 2t$ and we can find an open set Ω, $|\Omega| < 3t, E \subset \Omega$. Now using Whitney covering theorem (see [5]) we can find cubes Q_k with disjoint interiors, such that $\Omega = \cup_{k=1}^{\infty} Q_k$ and diam $Q_k \leqslant$ dist $(Q_k, \mathbb{R}^n \setminus \Omega) \leqslant$ 4 diam Q_k.

We want to show that there is a constant $C > 0$ such that for every f, t and for every corresponding cube Q_k

$$|\{x \in Q_k : I_\Phi f(x) > C(M_{\tilde{\Phi}^{-1}} f)(x) + (I_\Phi f)^*(2t)\}| \leqslant \frac{1}{6}|Q_k|. \tag{5}$$

Then we would have

$$|\{x \in \mathbb{R}^n : I_\Phi f(x) > C(M_{\tilde{\Phi}^{-1}} f)(x) + (I_\Phi f)^*(2t)\}| \leqslant 1/6 \sum |Q_k| \leqslant t/2$$

and thus

$$\begin{aligned}|\{x \in \mathbb{R}^n : I_\Phi f(x) > C(M_{\tilde{\Phi}^{-1}} f)^*(t/2) + (I_\Phi f)^*(2t)\}| \\ \leqslant |\{x \in \mathbb{R}^n : I_\Phi f(x) > C(M_{\tilde{\Phi}^{-1}} f)(x) + (I_\Phi f)^*(2t)\}| \\ + |\{x \in \mathbb{R}^n : (M_{\tilde{\Phi}^{-1}} f)(x) > (M_{\tilde{\Phi}^{-1}} f)^*(t/2)\}| \\ \leqslant t/2 + t/2 = t,\end{aligned}$$

which finishes the proof.

To prove (5) fix $k \in \mathbb{N}$ and choose $x_k \in (\mathbb{R}^n \setminus \Omega)$ so that $\operatorname{dist}(x_k, Q_k) \leqslant 4 \operatorname{diam}(Q_k)$. Let Q be a cube with center at x_k having diameter $20 \operatorname{diam}(Q_k)$. Split $f = g + h = f\chi_Q + f\chi_{\mathbb{R}^n \setminus Q}$. We may assume that $g \in L^1(\mathbb{R}^n)$, otherwise the right-hand side of (3) would be infinite.

We shall prove that for C_1 and C_2 large enough

$$|\{x \in Q_k : (I_\Phi g)(x) > C_1(M_{\tilde{\Phi}^{-1}} f)(x)\}| \leqslant 1/6|Q_k|, \tag{6}$$

and, for every $x \in Q_k$,

$$I_\Phi h(x) \leqslant C_2(M_{\tilde{\Phi}^{-1}} f)(x) + I_\Phi f(x_k) \leqslant C_2(M_{\tilde{\Phi}^{-1}} f)(x) + (I_\Phi f)^*(2t), \tag{7}$$

which together gives (5).

For the first inequality, notice that for $x \in Q_k$

$$(M_{\tilde{\Phi}^{-1}} f)(x) \geqslant \frac{1}{\tilde{\Phi}^{-1}(|Q|)} \int_Q g = \frac{||g||_1}{\tilde{\Phi}^{-1}(|Q|)}.$$

Using (4) now gives

$$\begin{aligned}|\{x \in Q_k : (I_\Phi g)(x) > C_1(M_{\tilde{\Phi}^{-1}} f)(x)\}| &\leqslant \left|\left\{x \in Q_k : (I_\Phi g)(x) > \frac{C_1||g||_1}{\tilde{\Phi}^{-1}(|Q|)}\right\}\right| \\ &\leqslant \frac{1}{\tilde{\Phi}\left(\frac{C_1}{\tilde{c}\tilde{\Phi}^{-1}(|Q|)}\right)},\end{aligned}$$

where $\tilde{c}$ is the constant from (4). The last expression is less then $|Q_k|/6$ for C_1 big enough (here we use (2) again).

In the proof of the second inequality we shall use two observations. The first is that

$$\left|\tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right) - \tilde{\Phi}^{-1}\left(\frac{1}{|x_k-y|^n}\right)\right| \leqslant c\,\frac{|x_k - x|}{|x-y|}\tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right) \tag{8}$$

with c independent of k, $y \in (\mathbb{R}^n \setminus Q)$ and $x \in Q_k$.

The second is that for any $\delta > 0$ and any $x \in \mathbb{R}^n$

$$\int_{y:|x-y|>\delta} \delta \frac{f(y)}{|x-y|}\tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right) \mathrm{d}y \leqslant c\,M_{\tilde{\Phi}^{-1}} f(x). \tag{9}$$

The proof of (7) now follows easily. For every $x \in Q_k$ we get

$$\begin{aligned} I_\Phi h(x) - I_\Phi f(x_k) &\leqslant I_\Phi h(x) - I_\Phi h(x_k) \\ &\leqslant \int_{\mathbb{R}^n \setminus Q} \left| \tilde\Phi^{-1}\left(\frac{1}{|x-y|^n}\right) - \tilde\Phi^{-1}\left(\frac{1}{|x_k-y|^n}\right)\right| f(y)\mathrm{d}y \\ &\leqslant c|x_k - x| \int_{\mathbb{R}^n \setminus Q} \frac{1}{|x-y|} \tilde\Phi^{-1}\left(\frac{1}{|x-y|^n}\right) f(y)\mathrm{d}y \\ &\leqslant c M_{\tilde\Phi^{-1}} f(x). \end{aligned}$$

It remains to prove (8) and (9). Proof of (9) is a combination of definition of $M_{\tilde\Phi^{-1}}$ and (2).

To prove (8) let us write $\tilde\Phi(t) = \int_0^t \tilde\varphi(u)\mathrm{d}u$ and $A(t) = \tilde\Phi^{-1}(t^{-n})$ for $t > 0$. Then

$$\frac{1}{s}\int_0^s \tilde\varphi(u)\mathrm{d}u \leqslant \tilde\varphi(s), \quad s > 0$$

or, equivalently, $\tilde\Phi(s) \leqslant s\tilde\Phi'(s)$ for $s > 0$. Now we set $s = A(t)$ and obtain

$$-tA'(t) = \frac{nt^{-n}}{\tilde\Phi'(A(t))} \leqslant cA(t).$$

Finally the left hand side of (8) can be estimated by

$$|A(|x-y|) - A(|x_k-y|)| \leqslant c\left|\int_{|x-y|}^{|x_k-y|} \frac{A(t)}{t}\mathrm{d}t\right| \leqslant c\frac{|x_k-x|}{|x-y|}A(|x-y|).$$

In the following example we will show that the assumption (2) cannot be omitted.

THEOREM 2.6. *There is a Young's function Φ satisfying the Δ_2—condition for which*

$$\sup_{f,t>0} \frac{(I_\Phi f)^*(t) - (I_\Phi f)^*(2t)}{(M_{\tilde\Phi^{-1}} f)^*(t/2)} = \infty$$

Proof. Set

$$\tilde\Phi(u) = \begin{cases} u^3 & \text{if } 0 < u < 1 \\ \frac{3}{2}u^2 - \frac{1}{2} & \text{if } 1 < u < \infty \end{cases}, \qquad \tilde\varphi(u) = \begin{cases} 3u^2 & \text{if } 0 < u < 1 \\ 3u & \text{if } 1 < u < \infty \end{cases}.$$

Then

$$\Phi(u) = \begin{cases} \frac{2}{3\sqrt{3}}u^{3/2} & \text{if } 0 < u < 3 \\ \frac{u^2}{6} + \frac{1}{2} & \text{if } 3 < u < \infty \end{cases}, \qquad \varphi(u) = \begin{cases} \sqrt{\frac{u}{3}} & \text{if } 0 < u < 3 \\ \frac{u}{3} & \text{if } 3 < u < \infty \end{cases}.$$

Finally $\tilde\Phi^{-1}(u) = \sqrt[3]{u}$ for $0 < u < 1$ and $\tilde\Phi^{-1}(u) = \sqrt{2/3(u+1/2)}$ for $u > 1$.

Let $n=1$. For any integer $m>0$ set $t_m=1/m$, $f_m(x)=\chi_{(0,t_m)}(x)$. Then

$$(M_{\tilde{\Phi}^{-1}}f_m)^*(t_m/2)=(M_{\tilde{\Phi}^{-1}}f_m)(0)=\sup_{0<s<1/m}\frac{1}{\tilde{\Phi}^{-1}(s)}\int_0^s 1=m^{-2/3},$$

$$(I_\Phi f_m)^*(t_m)=(I_\Phi f_m)(0)=\int_0^{1/m}\tilde{\Phi}^{-1}(1/s)\mathrm{d}s=\sqrt{\frac{2}{3}}\int_0^{1/m}\sqrt{\frac{1}{u}+\frac{1}{2}}\mathrm{d}u,$$

$$(I_\Phi f_m)^*(2t_m)=(I_\Phi f_m)\Big(\frac{3}{2}t_m\Big)=\int_{1/(2m)}^{3/(2m)}\tilde{\Phi}^{-1}(1/s)\mathrm{d}s=\sqrt{\frac{2}{3}}\int_{1/(2m)}^{3/(2m)}\sqrt{\frac{1}{u}+\frac{1}{2}}\mathrm{d}u.$$

We can now estimate

$$\begin{aligned}&\frac{(I_\Phi f_m)^*(t_m)-(I_\Phi f_m)^*(2t_m)}{(M_{\tilde{\Phi}^{-1}}f_m)^*(t_m/2)}\\ &\geqslant\sqrt{\frac{2}{3}}m^{2/3}\left\{\int_0^{1/(2m)}\sqrt{\frac{1}{u}}\mathrm{d}u-\int_{1/m}^{3/(2m)}\sqrt{m+\frac{1}{2}}\mathrm{d}u\right\}\\ &=\sqrt{\frac{2}{3}}m^{2/3}\left\{\frac{\sqrt{2}}{\sqrt{m}}-\frac{\sqrt{m+\frac{1}{2}}}{2m}\right\}=\sqrt{\frac{2}{3}}m^{1/6}\left\{\sqrt{2}-\frac{1}{2}\sqrt{1+\frac{1}{2m}}\right\}.\end{aligned}$$

The last expression tends to infinity as m tends to infinity.

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Friedrich-Schiller Universität Jena
Mathematisches Institut
Ernst Abbe Platz 1-4
D-07737 Jena
Germany
e-mail: vybiral@minet.uni-jena.de