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CONTROLLABILITY OF NONAUTONOMOUS SEMILINEAR INTEGRODIFFERENTIAL INCLUSIONS IN BANACH SPACES

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Abstract. In this paper, we establish a sufficient condition for the controllability of nonautonomous integrodifferential inclusions with nonlocal conditions in Banach spaces. The approach used is the Sadovskii's fixed point theorem with the theory of resolvent operators.

1. Introduction

Controllability of nonlinear systems represented by ordinary differential equations in Banach spaces has been extensively studied by many authors, see survey paper [1] by Balachandran and Dauer. Recently, several authors have established the controllability results for differential inclusions or integrodifferential inclusions in Banach spaces, such as [2, 3, 7, 10] and the references therein. In [3], the authors studied the following semilinear integrodifferential inclusions with nonlocal conditions of the form

$$y'(t) \in A\left[y(t) + \int_0^t G(t-s)\,y(s)\,ds\right] + F(t,y) + (Bu)(t), \quad t \in J,$$

$$y(0) + \sum_{k=1}^{p} c_k y(t_k) = y_0,$$

where $F : J \times X \to \mathcal{P}(X)$ is a multi-valued map, $\mathcal{P}(X)$ is the family of all nonempty subsets of X, $G(t), t \in J$, is a bounded operator, $J = [0,b]$, $b > 0$, $y_0 \in X$, $0 \leqslant t_1 < t_2 < \cdots < t_p \leqslant b$, $p \in \mathbb{N}$, $c_k \neq 0$, $k = 1, 2, \cdots, p$, A is the infinitesimal generator of a linear semigroup, and X is a real separable Banach space with the norm $|\cdot|$. Also, the control function $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space. B is a bounded linear operator from U into X. They proved controllability results in the cases when the multivalued map F has convex values or nonconvex values, respectively.

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Motivated by excellent work in [3], in this paper we consider the controllability of the following nonautonomous semilinear functional integrodifferential inclusions

$$y'(t) \in A(t)\left[y(t) + \int_0^t G(t,s)\,y(s)\,ds\right] + F(t,y) + (Bu)(t), \quad t \in J, \tag{1.1}$$

$$y(0) + g(y) = y_0, \tag{1.2}$$

where $A(\cdot)$ generates a strongly continuous semigroups, and $g : C(J,X) \to X$, $G(t,s), t,s \in J,$ is a bounded operator, F, B, u, y_0 are defined as before. By using Sadovskii's fixed point theorem combined with the theory of resolvent operators, we establish a nonlocal controllability result for mild solutions of the system $(1.1)-(1.2)$. Instead of the usual assumption that F is convex valued, we assume that F has decomposable values. The result obtained here can be seen as a continuation and an extension of autonomous control system in [3].

2. Preliminaries

In this section, we shall introduce some basic definitions, lemmas which are used throughout this paper.

Let $C(J,X)$ be the Banach spaces of all continuous functions from J into X with the norm

$$\|y\| := \sup\{|y(t)| : t \in J\}$$

and $B(X)$ denotes the Banach space of bounded linear operators from X into itself.

A measurable function $y : J \to X$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. For properties of the Bochner integral see Yosida [14].

Let $L^1(J,X)$ denote the Banach space of continuous functions $y : J \to X$ which are Bochner integrable, normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|\,dt, \text{ for all } y \in L^1(J,X).$$

Let $(X, |\cdot|)$ be a Banach space. Then a multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed, compact) valued if $G(x)$ is convex (closed, compact) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for any bounded set B of X (i.e. $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$).

Let Z be a nonempty closed subset of X, and $G : Z \to \mathcal{P}(X)$ be a multi-valued map with nonempty closed values. G is lower semi-continuous (l.s.c.) on Z if the set $\{x \in Z : G(x) \cap C \neq \emptyset\}$ is open for any open set C in X.

Let $\overline{A}$ be a subset of $J \times X$. We say that $\overline{A}$ is $\mathcal{L} \otimes \mathcal{B}$ measurable if $\overline{A}$ belongs to the σ-algebra generated by all sets of the form $\mathcal{N} \times \mathcal{D}$, where $\mathcal{N}$ is Lebesgue measurable in J and $\mathcal{D}$ is Borel measurable in X. A subset S of $L^1(J,X)$ is decomposable if, for all $u, v \in S$ and all measurable subsets $\mathcal{N}$ of J, the function $u\chi_{\mathcal{N}} + v\chi_{J-\mathcal{N}} \in S$, where χ denotes the characteristic function.

Let $F : J \times X \to \mathcal{P}(X)$ be a multi-valued map with nonempty compact values. Assign to F the multi-valued operator

$$\mathcal{F} : C(J,X) \to \mathcal{P}(L^1(J,X))$$

by letting

$$\mathcal{F}(y) = \{w \in L^1(J,X) : w(t) \in F(t,y) \text{ for a.e. } t \in J\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F.

G has a fixed point if there is $x \in X$ such that $x \in G(x)$. For more details on multi-valued maps, see the books of Deimling [5] and Hu and Papageorgious [9].

To set the framework for our main controllability result, we make use of the following definitions and lemmas.

DEFINITION 2.1. Let Y be a separable metric space and let $G : Y \to \mathcal{P}(L^1(J,X))$ be a multi-valued operator. We say G has property (BC) if

(i) G is lower semi-continuous (l.s.c);

(ii) G has nonempty closed and decomposable values.

DEFINITION 2.2. Let $F : J \times X \to \mathcal{P}(X)$ be a multi-valued map with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is lower semi continuous and has nonempty closed and decomposable values.

DEFINITION 2.3. [11] A resolvent operator for the system $(1.1)-(1.2)$ is a operator-valued function $R(t,s) \in B(X)$, $0 \leqslant s \leqslant t \leqslant b$, having the following properties:

(i) $R(s,s) = I$, the identity operator on X, $0 \leqslant s \leqslant b$. $R(t,s)$ is strongly continuous in s and t, $\|R(t,s)\| \leqslant Me^{\beta(t-s)}$ for some constants M and β.

(ii) $R(t,s)\,Y \subset Y, R(t,s)$ is strongly continuous in s and t on Y, and Y is the Banach space formed from $D(A)$, the domain of $A(t)$, endowed with the graph norm.

(iii) For each $y \in Y$, $R(t,s)\,y$ is continuously differentiable in s and t, and

$$\frac{\partial}{\partial t} R(t,s)\,y = A(t)\left[R(t,s)\,y + \int_s^t G(t,r)\,R(r,s)\,y dr\right], \quad t \in J.$$

DEFINITION 2.4. A function $y(\cdot) \in C(J,X)$ is called a mild solution of the problem $(1.1)-(1.2)$, if there exists a function $f \in L^1(J,X)$ such that $f(t) \in F(t,y(t))$ a.e. in J and

$$y(t) = R(t,0)\,[y_0 - g(y)] + \int_0^t R(t,s)\,[f(s) + (Bu)(s)]\,ds.$$

DEFINITION 2.5. The system $(1.1)-(1.2)$ is said to be nonlocally controllable on the interval J if, for every $y_0, y_1 \in X$, there exists a control $u \in L^2(J,U)$ such that the mild solution $y(\cdot)$ of the problem $(1.1)-(1.2)$ satisfies $y(b) + g(y) = y_1$.

LEMMA 2.1. ([4]) *Let Y be a separable metric space and $G : Y \to \mathcal{P}(L^1(J,X))$ be a multi-valued operator which has property (BC). Then G has a continuous selection, i.e. there exists a continuous function (single-valued) $g : Y \to L^1(J,X)$ such that $g(y) \in G(y)$ for every $y \in Y$.*

LEMMA 2.2. ((Sadovskii's fixed point theorem [13])) *Let P be a condensing operator on a Banach space X, i.e., P is continuous and takes bounded sets into bounded sets, and $\alpha(P(B)) < \alpha(B)$ for every bounded set B of X with $\alpha(B) > 0$. If $P(D) \subset D$ for a convex, closed and bounded set D of X, then P has a fixed point in D (where $\alpha(\cdot)$ denotes Kuratowski's measure of non-compactness).*

We remark that a completely continuous operator is the easiest example of a condensing map.

3. Controllability result

In this section, we shall present and prove our main result. Let us list the following assumptions:

$(H1)$ The operator $R(t,s)$ is compact when $t-s>0$ and there exists a positive constant M_1 such that $\|R(t,s)\| \leqslant M_1$.

$(H2)$ The linear operator $W : L^2(J,U) \to X$ define by

$$Wu = \int_0^b R(b,s)(Bu)(s)\,ds$$

has an induced inverse operator W^{-1} which takes values in $L^2(J,U)/\ker W$, and there exist positive constants M_2, M_3 such that

$$\|B\| \leqslant M_2, \ \left\|W^{-1}\right\| \leqslant M_3.$$

$(H3)$ Let $F : J \times X \to \mathcal{P}(X)$ be a nonempty, compact-valued multi-valued map such that (i) $(t,u) \mapsto F(t,u)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable; and (ii) $u \mapsto F(t,u)$ is lower semi-continuous for a.e. $t \in J$.

$(H4)$ For each $r>0$, there exists $h_r \in L^1(J,\mathbb{R}_+)$ such that for a.e. $t \in J$ and $u \in X$ with $|u| \leqslant r$,

$$\|F(t,u)\| = \sup\{|v| : v \in F(t,u)\} \leqslant h_r(t)$$

and

$$\liminf_{r\to+\infty} \frac{1}{r}\int_0^b h_r(t)\,dt = \gamma < \infty.$$

$(H5)$ $g : C(J,X) \to X$ is a completely continuous operator and there exists a nondecreasing function $m : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(y)| \leqslant m(\|y\|) \text{ for all } y \in C(J,X)$$

and

$$\liminf_{r\to+\infty} \frac{m(r)}{r} = 0.$$

REMARK 3.1. The construction of the operator W and its inverse is studied by Quinn and Carmichael in reference [12].

REMARK 3.2. The existence of the resolvent operator $R(t,s)$ is discussed in references [8] and [11].

REMARK 3.3. The function $m : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $(H5)$ does exist, such as, we may take $m(\|y\|) = c + d\|y\|^{\alpha}$, where constants $c, d \in \mathbb{R}, \alpha \in [0,1)$.

Before stating our main result, we give a lemma which will be needed in the sequel.

LEMMA 3.1. ([6]) *Let $F : J \times X \rightarrow \mathcal{P}(X)$ be a multi-valued map with nonempty, compact values. Assume $(H3)$ and $(H4)$ hold. Then F is of l.s.c. type.*

THEOREM 3.1. *Assume that $(H1)-(H5)$ are satisfied, then the system $(1.1)-(1.2)$ is nonlocally controllable on J, provided*

$$\delta\gamma < 1, \tag{3.1}$$

where $\delta = M_1(1 + bM_1M_2M_3)$.

Proof. Assumptions $(H3)$ and $(H4)$ imply by Lemma 3.1 that F is of the lower semi-continuous type. Then from Lemma 2.1 there exists a continuous function $f : C(J,X) \rightarrow L^1(J,X)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in C(J,X)$. Using the assumption (H2), for an arbitrary function $y(\cdot)$ define the control

$$u_y(t) = W^{-1}\left\{y_1 - g(y) - R(b,0)(y_0 - g(y)) - \int_0^b R(b,s)f(y)(s)\,ds\right\}(t).$$

Now, we shall show that, when using this control, the operator $N : C(J,X) \rightarrow C(J,X)$ defined by

$$N(y)(t) = R(t,0)(y_0 - g(y)) + \int_0^t R(t,s)[f(y)(s) + (Bu_y)(s)]\,ds$$

has a fixed point. This fixed point is then a mild solution of the system $(1.1)-(1.2)$. Clearly, $y_1 - g(y) \in N(y)(b)$ and

$$|u_y(t)| \leqslant M_3\left[|y_1| + m(\|y\|) + M_1(|y_0| + m(\|y\|)) + M_1\int_0^b |f(y)(s)|\,ds\right]. \tag{3.2}$$

Next, we shall show that N satisfies the conditions of Lemma 2.2. The proof will be given in several steps.

Step 1. For each constant $r > 0$, let $B_r = \{y \in C(J,X) : \|y\| \leqslant r\}$, clearly B_r is a bounded closed convex set in $C(J,X)$. We claim that for some positive number $r^*, N(B_{r^*}) \subseteq B_{r^*}$.

If it is not true, then for each positive number r, there exists a function $y_r(\cdot) \in B_r$ such that $\|N(y_r(t^r))\| > r$ for some $t^r \in J$. However, on the other hand, we have

$$\begin{aligned} r &< \|N(y_r(t^r))\| \\ &= \left\|R(t^r,0)(y_0 - g(y_r)) + \int_0^{t^r} R(t^r,s)[f(y_r)(s) + (Bu_{y_r})(s)]\,ds\right\| \\ &\leqslant M_1(|y_0| + m(r)) + M_1\int_0^{t^r} h_r(s)\,ds + M_1M_2\int_0^{t^r} |u_{y_r}|\,ds \end{aligned}$$

$$
\begin{aligned}
&\leqslant M_1\left(|y_0|+m(r)\right)+M_1\int_0^{t^r} h_r(s)\,ds+bM_1M_2M_3\left(|y_1|+m(r)\right)\\
&\quad+bM_1^2M_2M_3\left(|y_0|+m(r)\right)+bM_1^2M_2M_3\int_0^b h_r(s)\,ds\\
&=\overline{M}+\delta\int_0^b h_r(s)\,ds+\zeta m(r)\\
&\Rightarrow 1<\frac{1}{r}\left[\overline{M}+\delta\int_0^b h_r(s)\,ds+\zeta m(r)\right],
\end{aligned}
\tag{3.3}
$$

where $\overline{M}$ is independent of r, $\delta = M_1(1+bM_1M_2M_3)$, $\zeta = M_1(1+bM_2M_3+bM_1M_2M_3)$. Observing $(H4)$, $(H5)$ and by passing to the lower limit as $r\to\infty$ in inequality (3.3), we get

$$\delta\gamma \geqslant 1,$$

which contradicts (3.1). Hence, there exists a positive number r^* such that $N(B_{r^*})\subseteq B_{r^*}$.

Step 2. N is continuous on B_{r^*}.

Let $\{y_n\}$ be a sequence such that $y_n\to y$ in B_{r^*}. Then

$$
\begin{aligned}
|N(y_n)(t)-N(y)(t)| &\leqslant |R(t,0)[g(y)-g(y_n)]|\\
&\quad+\left|\int_0^t R(t,s)[f(y_n)(s)-f(y)(s)]\,ds\right|\\
&\quad+\left|\int_0^t R(t,s)B[u_{y_n}(s)-u_y(s)]\,ds\right|\\
&\leqslant M_1|g(y_n)-g(y)|+M_1\int_0^t |f(y_n)(s)-f(y)(s)|\,ds\\
&\quad+M_1M_2\int_0^t |u_{y_n}(s)-u_y(s)|\,ds.
\end{aligned}
$$

Since f, g are continuous and $u_{y_n}(s)\to u_y(s)$, $n\to\infty$, by the dominated convergence theorem we have

$$\|N(y_n)-N(y)\| = \sup_{t\in J}|N(y_n)(t)-N(y)(t)|\to 0,\ \text{as } n\to\infty,$$

i.e. N is continuous on B_{r^*}.

Step 3. $\{N(y): y\in B_{r^*}\}$ is an equicontinuous family of functions.

Let $\varepsilon>0$ small, $0<\tau_1<\tau_2$, then

$$
\begin{aligned}
|N(y_n)(\tau_2)-N(y)(\tau_1)| &\leqslant \|R(\tau_2,0)-R(\tau_1,0)\|\,|y_0-g(y)|\\
&\quad+\int_0^{\tau_1-\varepsilon}\|R(\tau_2,s)-R(\tau_1,s)\|\,h_{r^*}(s)\,ds\\
&\quad+\int_{\tau_1-\varepsilon}^{\tau_1}\|R(\tau_2,s)-R(\tau_1,s)\|\,h_{r^*}(s)\,ds+M_1\int_{\tau_1}^{\tau_2}h_{r^*}(s)\,ds
\end{aligned}
$$

$$+M_2 \int_0^{\tau_1-\varepsilon} \|R(\tau_2,s)-R(\tau_1,s)\| \, |u_y(s)| \, ds$$
$$+M_2 \int_{\tau_1-\varepsilon}^{\tau_1} \|R(\tau_2,s)-R(\tau_1,s)\| \, |u_y(s)| \, ds + M_1 M_2 \int_{\tau_1}^{\tau_2} |u_y(s)| \, ds.$$

In view of (3.2), $|g(y)| \leqslant m(r^*)$ and (H1), as $\tau_2 \to \tau_1$ and ε sufficiently small, the right-hand side of the above inequality tends to zero, since the compactness of $R(t,s)$ $(t-s>0)$ implies the continuity of $R(t,s)$ in t in the uniform operator topology. Hence, N maps B_{r^*} into a family of equicontinuous functions.

Step 4. The set $V(t) = \{N(y)(t) : y \in B_{r^*}\}$ is relatively compact in X.

Let $0 < t \leqslant b$ be fixed, $0 < \varepsilon < t$, for $y \in B_{r^*}$, we define

$$N_\varepsilon(y)(t) = R(t,0)(y_0 - g(y)) + \int_0^{t-\varepsilon} R(t,s)\left[f(y)(s) + (Bu_y)(s)\right] ds.$$

Using the estimation on $|u_y(s)|$ as (3.2) and by the compactness of $R(t,s)$ $(t-s>0)$, we obtain the set $V_\varepsilon(t) = \{N_\varepsilon(y)(t) : y \in B_{r^*}\}$ is relatively compact in X for every $\varepsilon, 0 < \varepsilon < t$. Moreover for each $y \in B_{r^*}$ we have

$$\begin{aligned} |N(y)(t) - N_\varepsilon(y)(t)| &= \left| \int_{t-\varepsilon}^{t} R(t,s)\left[f(y)(s) + (Bu_y)(s)\right] ds \right| \\ &\leqslant M_1 \int_{t-\varepsilon}^{t} h_{r^*}(s)\, ds + M_1 M_2 \int_{t-\varepsilon}^{t} |u_y(s)| \, ds. \end{aligned}$$

Therefore, there are relatively compact sets arbitrarily close to the set $V(t) = \{N(y)(t) : y \in B_{r^*}\}$. Hence, the set $V(t) = \{N(y)(t) : y \in B_{r^*}\}$ is relatively compact in X.

As a consequence of Step 1 to Step 4, $(H5)$, together with the Arzela-Ascoli theorem, we conclude that N is completely continuous on B_{r^*}, therefore, a condensing map. In view of Lemma 2.2, N has a fixed point on B_{r^*}, which is in turn a mild solution of $(1.1)-(1.2)$. Therefore, the system $(1.1)-(1.2)$ is nonlocally controllable on J.

REMARK 3.4. If $(H4)$ holds with $\gamma = 0$ (which is used in [10]), then the system $(1.1)-(1.2)$ is nonlocally controllable on J.

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