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THE STABILITY PROBLEM OF THE HERMITE–HADAMARD INEQUALITY

KAZIMIERZ NIKODEM, THOMAS RIEDEL AND PRASANNA K. SAHOO

(*communicated by Zs. Páles*)

Abstract. The problem of the Hyers-Ulam stability of the Hermite-Hadamard inequality posed by Zs. Páles is solved. It is shown that for continuous functions $f : I \to \mathbb{R}$ neither the inequality $f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{y-x}\int_x^y f(t)\,dt + \epsilon$ nor $\frac{1}{y-x}\int_x^y f(t)\,dt \leqslant \frac{f(x)+f(y)}{2} + \epsilon$ implies the $c\epsilon-$ convexity of f (with any $c > 0$). However, if f is continuous and satisfies both of the above inequalities simultaneously, then it is 4ϵ-convex.

1. Introduction

It is well known that if a function $f : I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}$ is convex, that is if

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y), \quad x, y \in I,\ t \in [0,1], \tag{1.1}$$

then it satisfies the following Hermite-Hadamard inequality

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{y-x}\int_x^y f(t)\,dt \leqslant \frac{f(x)+f(y)}{2}, \quad x, y \in I,\ x < y. \tag{1.2}$$

Moreover, for continuous functions f the validity of the left or the right-hand side inequality in (1.2) is equivalent to the convexity of f (cf. e.g. [1], [3]).

By the classical Hyers-Ulam stability theorem [2] we also know that if $f : I \to \mathbb{R}$ is ϵ-convex, that is if it satisfies

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) + \epsilon, \quad x, y \in I,\ t \in [0,1], \tag{1.3}$$

with an $\epsilon > 0$, then there exists a convex function g such that $|f - g| \leqslant \frac{\epsilon}{2}$ on I. Using the above results one can show easily that ϵ-convex functions satisfy also the following inequalities

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{y-x}\int_x^y f(t)\,dt + \epsilon \tag{1.4}$$

$$\frac{1}{y-x}\int_x^y f(t)\,dt \leqslant \frac{f(x)+f(y)}{2} + \epsilon, \tag{1.5}$$

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for all $x < y$ in I. The problem, if for continuous functions the validity of (1.4) or (1.5) implies the $c\epsilon$-convexity of f with any constant $c > 0$ independent of f, was posed recently by Zs. Páles [5].

In this note we show that the answer to this question is negative. However, if a continuous function f satisfies simultaneously both inequalities (1.4) and (1.5), then it is 4ϵ-convex. Then by the Hyers-Ulam theorem, there exists a convex function g such that $|f - g| \leqslant 2\epsilon$.

2. Counterexamples

COUNTEREXAMPLE 1. The function $f(x) = \ln x$, $x > 0$, satisfies inequality (1.4) with $\epsilon = 1$, but it is not c-convex with any $c > 0$.

Proof. We will prove first that f satisfies (1.4) with $\epsilon = 1$, that is

$$(y-x)\ln\frac{x+y}{2} \leqslant \int_x^y \ln t\,dt + (y-x). \tag{2.1}$$

Fix x, y with $x < y$. The right-hand side of (2.1) is equal to

$$\begin{aligned}\int_x^y \ln t\,dt + (y-x) &= y(\ln y - 1) - x(\ln x - 1) + y - x \\ &= y\ln y - x\ln x \\ &= \ln\frac{y^y}{x^x}.\end{aligned} \tag{2.2}$$

Since $0 < x < \frac{x+y}{2} < y$, we have

$$\left(\frac{x+y}{2}\right)^y x^x \leqslant y^y x^x \leqslant y^y \left(\frac{x+y}{2}\right)^x,$$

or, equivalently,

$$\left(\frac{x+y}{2}\right)^{y-x} \leqslant \frac{y^y}{x^x}.$$

Consequently

$$(y-x)\ln\frac{x+y}{2} \leqslant \ln\frac{y^y}{x^x},$$

which together with (2.2) proves (2.1).

To show that f is not c-convex, fix arbitrary $c > 0$ and $x_0 > 0$. For every $n > \frac{1}{x_0}$ we can express x_0 as the convex combination of the points $\frac{1}{n}$ and $2x_0 - \frac{1}{n}$

$$x_0 = \frac{1}{2}\cdot\frac{1}{n} + \frac{1}{2}\cdot\left(2x_0 - \frac{1}{n}\right).$$

Since $\ln\frac{1}{n}$ tends to $-\infty$ if $n \to +\infty$, for large enough $n \in \mathbb{N}$ we have

$$\ln\left(\frac{1}{2}\cdot\frac{1}{n} + \frac{1}{2}\cdot\left(2x_0 - \frac{1}{n}\right)\right) = \ln x_0 > \frac{1}{2}\ln\frac{1}{n} + \frac{1}{2}\ln\left(2x_0 - \frac{1}{n}\right) + c,$$

which proves that f is not c-convex. □

In the proof of the next statement we will use the following simple facts which we formulate here as lemmas.

LEMMA 2.1. *For every $n \in \mathbb{N}$ and $h \geqslant 0$ such that $e^{-n} + h \leqslant 1$*

$$\int_{e^{-n}}^{e^{-n}+h} -\ln t\,dt \leqslant h - h\ln(e^{-n}+h).$$

LEMMA 2.2. *If $a \geqslant c$ and $d \geqslant b$, then $ab + cd \leqslant \frac{1}{2}(a+c)(b+d)$.*

COUNTEREXAMPLE 2. For every $n \in \mathbb{N}$ the function

$$f_n(x) \;=\; \begin{cases} -\ln(\,|x| + e^{-n}\,) & \text{if } |x| \leqslant 1 - e^{-n} \\ 0 & \text{if } 1 - e^{-n} < |x| \leqslant 1 \end{cases}$$

satisfies inequality (1.5) with $\epsilon = 1$, but it is not c-convex with any $c < n$.

Proof. First we will prove that f_n satisfy (1.5) with $\epsilon = 1$, that is

$$\int_x^y f_n(t)\,dt \leqslant \frac{f_n(x) + f_n(y)}{2}\,(y-x) + (y-x). \tag{2.3}$$

We will consider three cases.

Case 1. Suppose $x, y \in [0,1]$ or $x, y \in [-1, 0]$. Then (2.3) holds because f_n restricted to each of these intervals is convex and therefore satisfies the Hermite-Hadamard inequality.

Case 2. Suppose $-1 + e^{-n} \leqslant x \leqslant 0 \leqslant y \leqslant 1 - e^{-n}$. Without loss of generality we may assume additionally that $|x| \geqslant y$. By Lemma 2.1 we obtain

$$\begin{aligned} \int_x^y f_n(t)\,dt &= \int_x^0 f_n(t)\,dt + \int_0^y f_n(t)\,dt \\ &= \int_{e^{-n}}^{e^{-n}-x} -\ln t\,dt + \int_{e^{-n}}^{e^{-n}+y} -\ln t\,dt \\ &\leqslant -x + x\ln(e^{-n}-x) + y - y\ln(e^{-n}+y). \end{aligned}$$

Now, using Lemma 2.2 for $a = -x$, $b = -\ln(e^{-n} - x)$, $c = y$ and $d = -\ln(e^{-n} + y)$, we get

$$\begin{aligned} \int_x^y f_n(t)\,dt &\leqslant \frac{1}{2}\left(-\ln(e^{-n}-x) - \ln(e^{-n}+y)\right)(y-x) + (y-x) \\ &= \frac{f_n(x) + f_n(y)}{2}(y-x) + (y-x). \end{aligned}$$

Case 3. Suppose $-1 \leqslant x < -1 + e^{-n}$, $0 \leqslant y \leqslant 1 - e^{-n}$ (the symmetric case: $-1 + e^{-n} \leqslant x \leqslant 0$, $1 - e^{-n} < y \leqslant 1$, and the case: $-1 \leqslant x < -1 + e^{-n}$, $1 - e^{-n} < y \leqslant 1$ can be proved analogously). Using the fact proved in Case 2, we obtain

$$\begin{aligned}
\int_x^y f_n(t)\,dt &= \int_{-1+e^{-n}}^y f_n(t)\,dt \\
&\leqslant \frac{f_n(-1+e^{-n})+f_n(y)}{2}\big(y-(-1+e^{-n})\big)+\big(y-(-1+e^{-n})\big) \\
&\leqslant \frac{f_n(x)+f_n(y)}{2}(y-x)+(y-x).
\end{aligned}$$

This finishes the proof of (2.3).

Now, take $x=-1$, $y=1$ and $c<n$. Then

$$f_n\left(\frac{1}{2}\cdot(-1)+\frac{1}{2}\cdot 1\right)=f_n(0)=n>\frac{1}{2}f_n(-1)+\frac{1}{2}f_n(1)+c,$$

which shows that f_n is not c-convex with any $c<n$. □

The referee provided another example of such a function: $f(x)=\ln(1+|x|)$, $x\in\mathbb{R}$ also satisfies inequality (1.5) with $\epsilon=1$, and it is not $c\epsilon$-convex. Unlike the function in Counterexample 2, this function is “universal” in the sense that it does not depend on the constant c. The integral of f on compact subintervals, is given by

$$\int_x^y f(t)\,dt=\begin{cases}(1+x)\ln(1+x)-(1+y)\ln(1+y)+(y-x) & \text{if}\quad 0\leqslant x<y;\\ -(1-x)\ln(1-x)-(1+y)\ln(1+y)+(y-x) & \text{if}\quad x<0<y;\\ -(1-x)\ln(1-x)+(1-y)\ln(1-y)+(y-x) & \text{if}\quad x<y\leqslant 0.\end{cases}$$

Comparing this with the corresponding form of the right side of the inequality (1.5) with $\epsilon=1$ and applying equivalent rearrangements, it suffice to check the validity of the inequalities

$$\begin{aligned}
(x+y+2)\ln(1+x) &\leqslant (x+y+2)\ln(1+y) \quad (0\leqslant x<y),\\
(x+y-2)\ln(1-x) &\leqslant (x+y+2)\ln(1+y) \quad (x<0<y),\\
(x+y-2)\ln(1-x) &\leqslant (x+y-2)\ln(1-y) \quad (x<y\leqslant 0).
\end{aligned}$$

The first and the last ones are quite trivial. The second can be proved after some further rearrangements:

$$0\leqslant \ln(1-x)^2(1+y)^2\left(\frac{1+y}{1-x}\right)^{x+y}.$$

Clearly, $1-x>1$ and $1+y>1$. On the other hand, $1+y\geqslant 1-x$ if and only if $x+y\geqslant 0$, therefore the last term in the argument is always greater than or equal to 1. To complete the proof, observe that $f(0)=0$, while $\frac{1}{2}f(-n)+\frac{1}{2}f(n)=-\ln(n+1)$ tends to $-\infty$ as $n\to\infty$ which together show that f cannot be $c\epsilon$-convex.

3. Conclusion

Having presented counterexamples we now conclude with a positive result and some remarks.

THEOREM 3.1. *If a function $f: I\to\mathbb{R}$ is continuous and satisfies inequalities* (1.4) *and* (1.5)*, then it is* 4ϵ*-convex.*

Proof. By (1.4) and (1.5) we obtain that f is 2ϵ-midconvex, that is

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2} + 2\epsilon, \quad x, y \in I.$$

Since f is continuous, this implies that it is 4ϵ-convex (cf. [4], Thm. 2). □

REMARK 3.2. Since the Ng-Nikodem theorem used in the above proof holds under much weaker assumptions than the continuity of f (for instance, for f locally bounded from above at a point or Lebesgue measurable (cf. [4]), Theorem 3.1) also holds under such assumptions (provided the integrals in (1.4) and (1.5) are defined).

REMARK 3.3. As was mentioned at the beginning, the left as well as the right-hand side inequality in the Hermite-Hadamard inequality (1.2) is (for continuous f) equivalent to (1.1), and hence each of them defines the convexity of f. However, the above counterexamples show that these inequalities are not stable, whereas the inequality (1.1) is stable by the Hyers-Ulam theorem. Also the whole Hermite-Hadamard inequality is stable by Theorem 3.1. This shows that the stability of convex functions is connected with an individual inequality defining the convexity but not with the convexity property itself.

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Kazimierz Nikodem
Katedra Matematyki i Informatyki
Akademia Techniczno-Humanistyczna
43-309 Bielsko-Biala
Poland
e-mail: knikodem@ath.bielsko.pl

Thomas Riedel and Prasanna K. Sahoo
Department of Mathematics
University of Louisville
Louisville, Kentucky 40292
USA
e-mail: Thomas.Riedel@louisville.edu
e-mail: sahoo@louisville.edu