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DIRECT ESTIMATES IN SIMULTANEOUS APPROXIMATION FOR DURRMEYER TYPE OPERATORS

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Abstract. In the present paper, we study a Durrmeyer type integral modification of the well-known Baskakov operators with the weight function of Beta basis function. Some approximation properties of these operators were recently studied by Finta [2]. Here we study simultaneous approximation properties for these operators. We estimate local direct result in terms of modulus of continuity. The operators considered in this paper reproduce not only the constant functions but also the linear ones, due to this property we can improve the order of approximation for these operators by applying the iterative combinations, which were first studied by Micchelli [7]. We establish an asymptotic formula and error estimation in terms of higher order modulus of continuity in simultaneous approximation for the Micchelli combinations of these operators.

1. Introduction

The new type of Baskakov-Durrmeyer operator is defined as

$$B_n(f(t),x)=\sum_{v=1}^{\infty}p_{n,v}(x)\int_0^{\infty}b_{n,v}(t)f(t)dt+(1+x)^{-n}f(0)=\int_0^{\infty}K_n(x,t)f(t)dt, \quad (1)$$

where $p_{n,v}(x)=\binom{n+v-1}{v}\frac{x^v}{(1+x)^{n+v}}$; $b_{n,v}(t)=\frac{1}{B(n+1,v)}\ \frac{t^{v-1}}{(1+t)^{n+v+1}}$ and $K_n(x,t)=\sum\limits_{v=1}^{\infty}p_{n,v}(x)$ $b_{n,v}(t)+(1+x)^{-n}\delta(t)$, $\delta(t)$ being the Dirac delta function.

The operators defined by (1) are the Durrmeyer integral modification of the well known Baskakov operators having weight functions of Beta basis functions. Some approximation properties of the operators $B_n(f,x)$ were recently discussed by Finta [2]. The operators B_n have different approximation properties than the other usual Baskakov Durrmeyer operators studied in [4], [8] and [9] etc. These operators reproduce not only the constant functions but also the linear functions, which is the interesting property of these operators, while the usual Baskakov Durrmeyer operators reproduce only the constant functions. However it turns out that the order of approximation for the operators (1) is at best $O(n^{-1})$ even for smooth functions. To improve the order

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of approximation, we consider the iterative combinations due to Micchelli [7] of these operators.

We define the class $C_\gamma[0,\infty) \equiv \{f \in C[0,\infty) : |f(t)| \leqslant Mt^\gamma$ for some $M > 0, \gamma > 0\}$. The norm $\|.\|_\gamma$ on $C_\gamma[0,\infty)$ is defined as $\|f\|_\gamma = \sup_{0<t<\infty} |f(t)|t^{-\gamma}$. For $f \in C_\gamma[0,\infty)$, we introduce the operators $B_{n,k}(f,x)$, which are defined by

$$B_{n,k}(f,x) = (I-(I-B_n)^k)(f(t),x) = \sum_{p=1}^{k}(-1)^{p+1}\binom{k}{p}B_n^p(f,x),$$

where $B_n^p(f,x)$, $p \in N$ denotes the p-th iterate, and $B_n^0(f,x) = I$. For some other operators such type of iterative combinations were recently considered in [1] and [6].

For sufficiently small $\delta > 0$, linear approximating function viz. Steklov mean $f_{\eta,2k}(t)$ of 2k-th order corresponding to $f \in C_\gamma[0,\infty)$ is defined by

$$f_{\eta,2k}(t) = \delta^{-2k}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \ldots\ldots \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \big(f(t) - \Delta_\eta^{2k} f(t)\big)dt_1 dt_2 \ldots . dt_{2k},$$

where $\eta = \frac{1}{2}\sum_{i=1}^{2k} t_i, t \in [a,b]$ and $\Delta_\eta^{2k} f(t)$ is the $2k$-th order forward difference of f with step length η. It is easily checked for $0 < a_1 < a_2 < b_2 < b_1 < \infty$ that:

(i) $f_{\eta,2k}$ has continuous derivatives up to order $2k$ on $[a_1,b_1]$,

(ii) $\|f_{\eta,2k}^{(r)}\|_{C[a_2,b_2]} \leqslant \widehat{M}_1\delta^{-r}\omega_{2k}(f,\delta,a_1,b_1)$,

(iii) $\|f - f_{\eta,2k}\|_{C[a_2,b_2]} \leqslant \widehat{M}_2\omega_{2k}(f,\delta,a_1,b_1)$,

(iv) $\|f_{\eta,2k}\|_{C[a_2,b_2]} \leqslant \widehat{M}_3\|f\|_\gamma$,

where $\widehat{M}_i, i = 1,2,3$ are certain constants that depend on $[a,b]$ but are independent of f and n. These properties are also mentioned in [6].

By $C_B[0,\infty)$ we denote the space of all real valued continuous bounded functions f on $[0,\infty)$ endowed with the norm $\|f\| = \sup_{x\geqslant 0}|f(x)|$. Let $\omega(f,\delta) = \sup_{0<h\leqslant\delta}\sup_{x\geqslant 0}|f(x+h)-f(x)|$ be the usual modulus of continuity of $f \in C_B[0,\infty)$.

In the present paper we investigate and study simultaneous approximation, in first main result, we establish a local direct result in terms of ordinary modulus of continuity. Second and third main result in the present paper are respectively Voronovskaja type asymptotic formula and an estimation of error for the iterative combinations of the operators (1).

Our main results are as follows:

THEOREM 1.1. *Let $n > r+1 \geqslant 2$ and $f^{(i)} \in C_B[0,\infty)$ for $i \in \{0,1,2,...,r\}$. Then*

$$|B_n^{(r)}(f,x) - f^{(r)}(x)| \leqslant \left(\frac{(n+r-1)!(n-r)!}{n!(n-1)!} - 1\right)\|f^{(r)}\| + 2\frac{(n+r-1)!(n-r)!}{n!(n-1)!}\omega\big(f^{(r)},\delta(n,r,x)\big),$$

where $\delta(n,r,x) = \left\{\frac{2n+4r(1+r)}{(n-r)(n-r-1)}x^2 + \frac{2n+4r(1+r)}{(n-r)(n-r-1)}x + \frac{r(1+r)}{(n-r)(n-r-1}\right\}^{\frac{1}{2}}$, *and* $x \in [0,\infty)$.

THEOREM 1.2. *Let* $f \in C_\gamma[0,\infty)$ *and* $f^{(2k+r)}$ *exists at a point* $x \in (0,\infty)$. *Then*

$$\lim_{n\to\infty} n^k[B_{n,k}^{(r)}(f(t),x) - f^{(r)}(x)] = \sum_{j=r}^{2k+r} P(j,k,r,x) f^{(j)}(x),$$

where $P(j,k,r,x)$ *are certain polynomials in* x.

THEOREM 1.3. *Let* $f \in C_\gamma[0,\infty)$ *and suppose* $0 < a_1 < a_2 < b_2 < b_1 < \infty$. *Then for all* n *sufficiently large, we have*

$$\|B_{n,k}^{(r)}(f,\bullet) - f^{(r)}\|_{C[a_2,b_2]} \leqslant M\{\omega_{2k}(f^{(r)}, n^{-\frac{1}{2}}, a_1, b_1) + n^{-k}\|f\|_\gamma\},$$

where M *is a constant independent of* f *and* n.

2. Basic results

In this section we mention certain lemmas which will be used in the sequel.

LEMMA 2.1. [4] *Let* $m \in N \cup \{0\}$*., If the* m^{th} *order moment is defined as*

$$U_{n,m}(x) = \sum_{v=0}^{\infty} p_{n,v}(x)\left(\frac{v}{n} - x\right)^m,$$

then $U_{n,0}(x) = 1, U_{n,1}(x) = 0$ *and also there holds the recurrence relation:*

$$nU_{n,m+1}(x) = x(1+x)\lfloor U_{n,m}^{(1)}(x) + mU_{n,m-1}(x)\rfloor.$$

Consequently we have $U_{n,m}(x) = O\big(n^{-[(m+1)/2]}\big)$.

LEMMA 2.2. *Let the function* $T_{n,m}(x), m \in N \cup \{0\}$*, be defined as*

$$T_{n,m}(x) = B_n\big((t-x)^m, x\big) = \sum_{v=1}^{\infty} p_{n,v}(x) \int_0^\infty b_{n,v}(t)(t-x)^m dt + (1+x)^{-n}(-x)^m.$$

Then $T_{n,0}(x) = 1, T_{n,1} = 0, T_{n,2}(x) = \frac{2x(1+x)}{n-1}$. *Also, there holds the recurrence relation*

$$(n-m)T_{n,m+1}(x) = x(1+x)\lfloor T_{n,m}^{(1)}(x) + 2mT_{n,m-1}(x)\rfloor + m(1+2x)T_{n,m}(x), n > m.$$

From the above recurrence relation, it is easily verified that for all $x \in [0,\infty)$*, we have*

$$T_{n,m}(x) = O\big(n^{-[(m+1)/2]}\big).$$

The proof of Lemma 2.2 can easily be done along the lines of the proof of [5, Lemma 2.2].

REMARK 1. It is easily verified from Lemma 2.2 that for each $x \in (0,\infty)$

$$B_n(t^i, x) = \frac{(n+i+1)!(n-i)!}{n!(n-1)!}x^i + i(i-1)\frac{(n+i-2)!(n-i)!}{n!(n-1)!}x^{i-1} + O(n^{-2}).$$

COROLLARY 2.1. *Let δ be a positive number. Then for every $\gamma > 0,\ x \in (0,\infty)$, there exists a constant $M(s,x)$ independent of n and depending on s and x such that*

$$\left\| \int_{|t-x|>\delta} K_n(x,t)t^{\gamma}\,dt \right\|_{C[a,b]} \leqslant M(s,x)n^{-s}, \quad s = 1,2,3....$$

The m-th order moment for the operators $B_n^p(f,x)$ is denoted by $T_{n,m}^{(p)}(x)$ and defined as $T_{n,m}^{(p)}(x) = B_n^p((t-x)^m,x)$. In particular $T_{n,m}^{(1)}(x)$ reduces to $T_{n,m}(x)$, defined in Lemma 2.2.

LEMMA 2.3. *For $p \in N$, there holds the following relation*

$$T_{n,m}^{\{p+1\}}(x) = \sum_{j=0}^{m} \binom{m}{j} \sum_{i=0}^{m-j} \frac{1}{i!} T_{n,i+j}(x) \frac{\partial^i}{\partial x^i} T_{n,m-j}^{\{p\}}(x).$$

LEMMA 2.4. *We have*

$$T_{n,m}^{(p)}(x) = O\left(n^{-[(m+1)/2]}\right).$$

Proof. For $p = 1$, the result follows easily from Lemma 2.2. By using Lemma 2.3, and the fact that $T_{n,m-j}^{\{p\}}(x)$ is a polynomial in x of degree at most $m-j$, the result, in the general case, follows immediately by the principle of mathematical induction.

LEMMA 2.5. *For $l \in N$, we have*

$$B_{n,k}((t-x)^l,x) = O(n^{-k}).$$

Proof. For $k = 1$ the result easily follows from Lemma 2.2, and in the general case it follows immediately on applying Lemma 2.3, Lemma 2.4, and the induction hypothesis.

LEMMA 2.6. [4] *There exist the polynomials $Q_{i,j,r}(x)$, independent of n and v such that*

$$\{x(1+x)\}^r D^r\left[p_{n,v}(x)\right] = \sum_{\substack{2i+j\leqslant r \\ i,j\geqslant 0}} n^i(v-nx)^j Q_{i,j,r}(x)p_{n,v}(x),$$

where $D \equiv \frac{d}{dx}$.

LEMMA 2.7. *If f is r times differentiable on $[0,\infty)$, and $f^{(r-1)} = O(t^{\alpha}), \alpha > 0$ as $t \to \infty$ then for $r = 1,2,3,....$ and $n > \alpha + r$ we have*

$$B_n^{(r)}(f,x) = \frac{(n+r-1)!(n-r)!}{n!(n-1)!} \sum_{v=0}^{\infty} p_{n+r,v}(x) \int_0^{\infty} b_{n-r,v+r}(t) f^{(r)}(t)\,dt.$$

The proof of the above lemma easily follows along the lines of the proof of [5, Lemma 2.3].

3. Proofs

In this section we present the proofs of main results.

Proof of Theorem 1.1. Applying Lemma 2.7, we get

$$B_n^{(r)}(f,x)-f^{(r)}(x)=\left[\frac{(n+r-1)!(n-r)!}{n!(n-1)!}-1\right]f^{(r)}(x)$$
$$+\frac{(n+r-1)!(n-r)!}{n!(n-1)!}\sum_{v=0}^{\infty}p_{n+r,v}(x)\int_0^{\infty}b_{n-r,v+r}(t)\left[f^{(r)}(t)-f^{(r)}(x)\right]dt,$$

because $\int_0^{\infty}b_{n-r,v+r}(t)dt=1$ and $\sum_{v=0}^{\infty}p_{n+r,v}(x)=1$.

Using the inequality $\omega\big(f^{(r)},\lambda\delta\big)\leqslant(1+\lambda)\omega\big(f^{(r)},\delta\big)$, $\lambda\geqslant 0$ we get

$$|B_n^{(r)}(f,x)-f^{(r)}(x)|\leqslant\left[\frac{(n+r-1)!(n-r)!}{n!(n-1)!}-1\right]\|f^{(r)}\|$$
$$+\frac{(n+r-1)!(n-r)!}{n!(n-1)!}\sum_{v=0}^{\infty}p_{n+r,v}(x)\int_0^{\infty}b_{n-r,v+r}(t)|f^{(r)}(t)-f^{(r)}(x)|dt$$
$$\leqslant\frac{(n+r-1)!(n-r)!}{n!(n-1)!}\sum_{v=0}^{\infty}p_{n+r,v}(x)\int_0^{\infty}b_{n-r,v+r}(t)\big(1+\delta^{-1}|t-x|\big)\omega\big(f^{(r)},\delta\big)dt$$
$$+\left[\frac{(n+r-1)!(n-r)!}{n!(n-1)!}-1\right]\|f^{(r)}\|. \tag{2}$$

Using Cauchy-Schwarz inequality, we obtain

$$\sum_{v=0}^{\infty}p_{n+r,v}(x)\int_0^{\infty}b_{n-r,v+r}(t)|t-x|dt\leqslant\left(\sum_{v=0}^{\infty}p_{n+r,v}(x)\int_0^{\infty}b_{n-r,v+r}(t)(t-x)^2dt\right)^{\frac{1}{2}}. \tag{3}$$

Also, by easy computation, we are led to

$$\sum_{v=0}^{\infty}p_{n+r,v}(x)\int_0^{\infty}b_{n-r,v+r}(t)(t-x)^2dt$$
$$=\frac{2n+4r(1+r)}{(n-r)(n-r-1)}x^2+\frac{2n+4r(1+r)}{(n-r)(n-r-1)}x+\frac{r(1+r)}{(n-r)(n-r-1)}. \tag{4}$$

On combining (2)-(4), we get

$$\left|B_n^{(r)}(f,x)-f^{(r)}(x)\right|$$
$$\leqslant\frac{(n+r-1)!(n-r)!}{n!(n-1)!}\left(1+\delta^{-1}\left[\frac{2n+4r(1+r)}{(n-r)(n-r-1)}x^2\right.\right.$$
$$\left.\left.+\frac{2n+4r(1+r)}{(n-r)(n-r-1)}.x+\frac{r(r+1)}{(n-r)(n-r-1)}\right]^{\frac{1}{2}}\right)\omega(f^{(r)},\delta)$$
$$+\left[\frac{(n+r-1)!(n-r)!}{n!(n-1)!}-1\right]\|f^{(r)}\|.$$

Finally, if we choose $\delta = \delta(n,r,x)$, we obtain the assertion of the theorem.

Proof of Theorem 1.2. By Taylor expansion of f, we have

$$f(t) = \sum_{i=0}^{2k+r} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{2k+r},$$

where $\varepsilon(t,x) \to 0$ as $t \to x$.

Note that,

$$\begin{aligned} B_{n,k}^{(r)}(f,x) &= \sum_{p=1}^{k}(-1)^{p+1}\binom{k}{p}\frac{\partial^r}{\partial x^r}B_n^p(f,x) \\ &= \sum_{p=1}^{k}(-1)^{p+1}\binom{k}{p}\int_0^\infty K_n^{(r)}(x,y)\times \\ &\quad \times\left\{\sum_{j=r}^{2k+r}\frac{f^{(j)}(x)}{j!}B_n^{p-1}((t-x)^i,y) + B_n^{p-1}(\varepsilon(t,x)(t-x)^{2k+r},y)\right\}dy \\ &=: E_1 + E_2. \end{aligned}$$

Using Lemma 2.2, we get

$$\begin{aligned} E_1 &= \sum_{j=r}^{2k+r}\frac{f^{(j)}(x)}{j!}\sum_{i=0}^{j}\binom{j}{i}(-x)^{j-i}B_{n,k}^{(r)}(t^i,x) \\ &= \sum_{j=r}^{2k+r}\frac{f^{(j)}(x)}{j!}\sum_{i=0}^{j}\binom{j}{i}(-x)^{j-i}\left\{\frac{\partial^r}{\partial x^r}x^i + n^{-k}\frac{\partial^r}{\partial x^r}\left(\frac{P(j,k,x)}{j!}\frac{\partial^j}{\partial x^j}x^j\right) + o(n^{-k})\right\} \\ &= \sum_{j=r}^{2k+r}\frac{f^{(j)}(x)}{j!}r!\sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}\binom{j}{r}x^{j-i} + n^{-k}\sum_{j=r}^{2k+r}P(j,k,r,x)f^{(j)}(x) + o(n^{-k}) \\ &= f^{(r)}(x) + n^{-k}\sum_{j=r}^{2k+r}P(j,k,r,x)f^{(j)}(x) + o(n^{-k}), \end{aligned}$$

in view of the identities

$$\sum_{i=0}^{j}(-1)^i\binom{j}{i}\binom{i}{r} = \begin{cases} 0 & j > r \\ (-1)^r & j = r. \end{cases}$$

To estimate E_2, note that if

$$I = \int_0^\infty K_n^{(r)}(x,y)B_n^{r-1}(\varepsilon(t,x)(t-x)^{2k+r},y)\,dy,$$

then on applying Lemma 2.6, we get

$$|I| \leqslant \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i \frac{|Q_{i,j,r}(x)|}{\{x(1+x)\}^r} \sum_{v=1}^{\infty} |v-nx|^j p_{n,v}(x) \int_0^{\infty} b_{n,v}(y) B_n^{p-1}(|\varepsilon(t,x)||t-x|^{2k+r}, y)dy$$
$$+ \frac{(n+r+1)!}{(n-1)!}(1+x)^{-n-r}|\varepsilon(0,x)|x^{2k+r}.$$

The second term in the right hand side of above expression multiplied by n^k tends to zero as $n \to \infty$. Since $\varepsilon(t,x) \to 0$ as $t \to x$ for a given $\varepsilon > 0$ there exists a $\delta > 0$, such that $|\varepsilon(t,x)| < \varepsilon$ whenever $0 < |t-x| < \delta$. Therefore, for $|t-x| \leqslant \delta$, we have $|\varepsilon(t,x)(t-x)^{2k+r}| \leqslant Mt^{\gamma}$ for some $M > 0$. Hence

$$|I| \leqslant \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i \frac{|Q_{i,j,r}(x)|}{\{x(1+x)\}^r} \sum_{v=1}^{\infty} |v-nx|^j p_{n,v}(x) \times$$
$$\times \left\{ \varepsilon \int_{|t-x|<\delta} b_{n,v}(y) B_n^{p-1}(|t-x|^{2k+r}, y)dy + \int_{|t-x|\geqslant\delta} b_{n,v}(y) B_n^{p-1}(Mt^{\gamma}, y)dy \right\}$$
$$=: I_1 + I_2.$$

If we apply Cauchy-Schwarz inequality, Lemma 2.1 and Lemma 2.4, we get that $I_1 = \varepsilon O(n^{-k})$. Now, proceeding in a similar way by applying Cauchy-Schwarz inequality and Corollary 2.1, we obtain $I_2 = o(n^{-k})$. Since $\varepsilon > 0$ is arbitrary, we get $I = o(n^{-k})$, and this completes the proof of the theorem.

Proof of Theorem 1.3. Using the linearity property, we get

$$||B_{n,k}^{(r)}(f,\bullet) - f^{(r)}||_{C[a_2,b_2]}$$
$$\leqslant ||B_{n,k}^{(r)}(f - f_{\eta,2k},\bullet)||_{C[a_2,b_2]} + ||B_{n,k}^{(r)}(f_{\eta,2k},\bullet) - f_{\eta,2k}^{(r)}||_{C[a_2,b_2]} + ||f^{(r)} - f_{\eta,2k}^{(r)}||_{C[a_2,b_2]}$$
$$=: J_1 + J_2 + J_3.$$

Since $f_{\eta,2k}^{(r)} = \left(f^{(r)}\right)_{\eta,2k}(t)$, by the property (iii) of the Steklov mean, we have

$$J_3 \leqslant M_1 \omega_{2k}(f^{(r)}, \delta, a_1, b_1).$$

Next on applying Theorem 1.2, we get

$$J_2 \leqslant M_2 n^{-k} \sum_{j=r}^{2k+r} \|f_{\eta,2k}^{(j)}\|_{C[a_1,b_1]}.$$

If we now apply the interpolation property due to Goldberg and Meir [3], for each $j = r$, $r+1, \ldots, 2k+r$, it follows that

$$\|f_{\eta,2k}^{(j)}\|_{C[a_2,b_2]} \leqslant M_3 \left\{ \|f_{\eta,2k}\|_{C[a_1,b_1]} + \|f_{\eta,2k}^{(2k+r)}\|_{C[a_1,b_1]} \right\},$$

and by applying properties (iii) and (iv) of the Steklov mean, we obtain

$$J_2 \leqslant M_4.n^{-k}\big\{\|f\|_\gamma + \delta^{-2k}\omega_{2k}(f^{(r)},\delta)\big\}.$$

Finally we shall estimate J_1, choosing a^*, b^* satisfying the conditions $0 < a_1 < a^* < a_2 < b_2 < b^* < b_1 < \infty$, and for this if $\psi(t)$ denotes the characteristic function of the interval $[a^*, b^*]$, then

$$J_1 \leqslant \|B_{n,k}^{(r)}(\psi(t)(f(t)-f_{2,\delta}(t)),\bullet)\|_{C[a_1,b_1]}+\|B_{n,k}^{(r)}((1-\psi(t))(f(t)-f_{2,\delta}(t)),\bullet)\|_{C[a_1,b_1]}$$
$$=: J_4 + J_5.$$

We may note here that to estimate J_4 and J_5, it is enough to consider their expressions without the iterative combinations. By using Lemma 2.7, it is clear that

$$B_n^{(r)}\big(\psi(t)(f(t)-f_{\eta,2k}(t)),x\big)$$
$$= \frac{(n+r-1)!(n-r)!}{n!(n-1)!}\sum_{v=0}^{\infty} p_{n+r,v}(x)\int_0^\infty b_{n-r,v+r}(t)\psi(t)(f^{(r)}(t)-f_{\eta,2k}^{(r)}(t))dt.$$

Hence

$$\|B_{n,k}^{(r)}(\psi(t)(f(t)-f_{\eta,2k}(t)),\bullet)\|_{C[a_1,b_1]} \leqslant M_5\|f^{(r)}-f_{\eta,2k}^{(r)}\|_{C[a^*,b^*]}.$$

Next for $x \in [a_1, b_1]$ and $t \in [0,\infty)\setminus[a^*,b^*]$, we choose a $\delta_1 > 0$ satisfying $|t-x| \geqslant \delta_1$.

Therefore, by applying Lemma 2.6 and Cauchy-Schwarz inequality, we get that if

$$I \equiv B_n^{(r)}((1-\psi(t))(f(t)-f_{\eta,2k}(t)),x),$$

then

$$|I| \leqslant \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i\frac{|Q_{i,j,r}(x)|}{\{x(1+x)\}^r}\sum_{v=1}^{\infty}p_{n,v}(x)|v-nx|^j\int_0^\infty b_{n,v}(t)(1-\psi(t))|f(t)-f_{\eta,2k}(t)|dt$$
$$+\frac{(n+r-1)!}{(n-1)!}(1+x)^{-n-r}(1-\psi(0))|f(0)-f_{\eta,2k}(0)|.$$

Note that for sufficiently large n, the second term tends to zero. Hence

$$|I| \leqslant M_6\|f\|_\gamma \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i\sum_{v=1}^{\infty}p_{n,v}(x)|v-nx|^j\int_{|t-x|\geqslant\delta_1} b_{n,v}(t)dt$$
$$\leqslant M_6\|f\|_\gamma\delta_1^{-2s}\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i\sum_{v=1}^{\infty}p_{n,v}(x)|v-nx|^j\left(\int_0^\infty b_{n,v}(t)dt\right)^{\frac{1}{2}}\left(\int_0^\infty b_{n,v}(t)(t-x)^{4s}dt\right)^{\frac{1}{2}}$$
$$\leqslant M_6\|f\|_\gamma\delta_1^{-2m}\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i\left\{\sum_{v=1}^{\infty}p_{n,v}(x)(v-nx)^{2j}\right\}^{\frac{1}{2}}\left\{\sum_{v=1}^{\infty}p_{n,v}(x)\int_0^\infty b_{n,v}(t)(t-x)^{4s}dt\right\}^{\frac{1}{2}}.$$

If we now use Lemmas 2.1 and 2.2, we get

$$|I| \leqslant M_7 \|f\|_\gamma \delta_1^{-2s} O(n^{(i+\frac{j}{2}-s)}) \leqslant M_7 n^{-q} \|f\|_\gamma,$$

where $q = s - \frac{r}{2}$. Next choosing $s > 0$ satisfying $q > k$ we obtain

$$|I| \leqslant M_7 n^{-k} \|f\|_\gamma.$$

Therefore by property (iii) of the function $f_{\eta,2k}(t)$, we get

$$\begin{aligned} J_1 &\leqslant M_8 \|f^{(r)} - f_{\eta,2k}^{(r)}\|_{C[a^*,b^*]} + M_7 n^{-k} \|f\|_\gamma \\ &\leqslant M_9 \omega_{2k}(f^{(r)}, \delta, a_1, b_1) + M_7 n^{-k} \|f\|_\gamma, \end{aligned}$$

and now, on choosing $\delta = n^{-\frac{1}{2}}$, the Theorem 1.3 follows.

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