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FURUTA INEQUALITY OF INDEFINITE TYPE

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Abstract. In this article, we study matrix inequalities on an (indefinite) inner product space, including a generalization of Furuta inequality: let A,B be J-selfadjoint matrices with non-negative eigenvalues and $I \geqq^J A \geqq^J B$. Then for each $r \geqq 0$,

$$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geqq^J (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

holds for $p \geqq 0, q \geqq 1$ with $(1+r)q \geqq p+r$.

1. Introduction

In [2], T. Ando studies matrix inequalities on an (indefinite) inner product space; he shows the following:

(1.1) ([2, Theorem 4]) Let J be a selfadjoint involution, and A,B J-selfadjoint matrices with $\sigma(A),\sigma(B) \subseteqq (\alpha,\beta)$. Then

$$A \geqq^J B \Rightarrow f(A) \geqq^J f(B)$$

for any operator monotone function $f(t)$ on (α,β).

(1.2) ([2, Theorem 6]) Let A,B be J-selfadjoint matrices with non-negative eigenvalues. If

$$I \geqq^J A \geqq^J B,$$

then J-selfadjoint square roots $A^{\frac{1}{2}}$, $B^{\frac{1}{2}}$ are well defined and

$$I \geqq^J A^{\frac{1}{2}} \geqq^J B^{\frac{1}{2}}.$$

In this article, we would like to show a generalization of (1.2) for α-powers (Theorem 2.4); moreover Furuta inequality of indefinite type (Theorem 3.3). Although we can show them as well as the corresponding ones on a Hilbert space, we think that careful argument is required; therefore, we have this article.

In the remainder of this section, we recall basic facts about matrices on an (indefinite) inner product space. We refer the reader to [3, 1].

Let $M_n(\mathbb{C})$ be the set of all complex n-square matrices, and $\langle \cdot,\cdot \rangle$ the standard inner product on $\mathbb{C}^n$. For a pair of selfadjoint matrices A,B, $A \geqq B$ means that $A-B$

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is positive semidefinite. For a selfadjoint involution J: $J = J^*, J^2 = I$, we consider an (indefinite) inner product $[\cdot,\cdot]$ on $\mathbb{C}^n$ given by

$$[x,y] := \langle Jx, y\rangle \qquad (x,y \in \mathbb{C}^n).$$

The J-adjoint matrix $A^\sharp$ of A is defined as

$$[Ax,y] = [x, A^\sharp y] \qquad (x,y \in \mathbb{C}^n).$$

In other words, $A^\sharp = JA^*J$. A matrix A is said to be J-selfadjoint if $A^\sharp = A$ or JA is selfadjoint: $JA = A^*J$. For a pair of J-selfadjoint matrices A, B, the J-order, denoted as $A \geqq^J B$, is defined by

$$[Ax,x] \geqq [Bx,x] \qquad (x \in \mathbb{C}^n),$$

i.e., $JA \geqq JB$. A matrix A is called J-positive if $[Ax,x] \geqq 0$ $(x \in \mathbb{C}^n)$, or equivalently $JA \geqq O$. A matrix A is said to be a J-contraction if $I \geqq^J A^\sharp A$ or $[x,x] \geqq [Ax,Ax]$ for $x \in \mathbb{C}^n$. Remark that $I \geqq^J A$ implies that all eigenvalues of A are real. Hence, for a J-contraction A all eigenvalues of $A^\sharp A$ are real. In fact, by a result of Potapov-Ginzburg (see [3, Chapter 2, Section 4]), all eigenvalues of $A^\sharp A$ are non-negative.

If all eigenvalues of a J-selfadjoint matrix A are real and $\sigma(A) \subseteqq (\alpha,\beta)$, where $\sigma(A)$ means the set of all eigenvalues of A, then for any real-valued function $f(t)$ on (α,β) with analytic continuation, we can define $f(A)$ by the Dunford integral

$$f(A) := \int_\Gamma f(\zeta)(\zeta I - A)^{-1} d\zeta,$$

where Γ is a closed rectifiable contour in the domain of analytic continuation of $f(t)$ with positive direction surrounding $\sigma(A)$ in its interior. Note that $f(A)$ is J-selfadjoint.

2. Inequality for powers

In Section 1, we recall the inequalities (1.1) and (1.2) by T. Ando. In this section, we have a generalization of (1.2): Theorem 2.4.

LEMMA 2.1. (cf., [2, Lemma 5]) *Suppose that $A \in M_n(\mathbb{C})$ is J-selfadjoint with non-negative eigenvalues. If $1 \geqq^J A$, that is, $J \geqq JA$, then the integral*

$$\frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-1} A(\lambda I + A)^{-1} d\lambda$$

converges for $0 < \alpha < 1$.

The integral is denoted by A^α. Remark that for a J-selfadjoint matrix A with positive eigenvalues the integral is just the Dunford integral $f(A)$: $f(t) = t^\alpha$ on $(0,\infty)$.

The following proof is given for the reader's convenience; it is the same as that for [2, Lemma 5].

Proof. When $\mathcal{M} := \ker A = \{0\}$, then the assertion follows immediately from the above remark. Hence, we assume that $\mathcal{M} \neq \{0\}$. Let $C := J(I - A)$. The assumption $C \geqq O$ yields that

$$\|C\|\langle Cx, x\rangle \geqq \|Cx\|^2 \quad (x \in \mathbb{C}^n).$$

This means that

$$\|C\|[y,y] \geqq \|y\|^2 \qquad (y \in \mathcal{M}).$$

Hence, $\mathcal{M}$ is a J- positive subspace. Then it is known (see [3, Chapter 1, Section 7] that positive definiteness of $\mathcal{M}$ implies projective completeness of $\mathcal{M}$: $\mathbb{C}^n$ is the algebraic direct sum of $\mathcal{M}$ and its J-orthocomplement $\mathcal{N}$, defined by

$$\mathcal{N} := \{z; [y,z] = 0 \quad (\forall y \in \mathcal{M})\},$$

both of which are invariant for A. By definition, $\sigma(A_{|\mathcal{N}}) \subseteqq (0,\infty)$. Any vector $x \in \mathbb{C}^n$ is uniquely written as

$$x = y + z \qquad (y \in \mathcal{M}, z \in \mathcal{N}),$$

and for any $\lambda > 0$,

$$A(\lambda I + A)^{-1}x = (A_{|\mathcal{N}})(\lambda I + (A_{|\mathcal{N}}))^{-1}z,$$

which guarantees the convergence of the integral

$$\frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-1} A(\lambda I + A)^{-1} x \, d\lambda,$$

and the proof is complete. □

LEMMA 2.2. *Let $A \in M_n(\mathbb{C})$ be a J-positive matrix. Then $X^\sharp AX$ is J-positive for any $X \in M_n(\mathbb{C})$.*

Proof. By definition, $JA \geqq O$. Hence, $JX^\sharp AX = X^*(JA)X \geqq O$ and the proof is complete. □

LEMMA 2.3. *Let A, B be J-selfadjoint matrices with positive eigenvalues. If*

$$A \geqq^J B,$$

then

$$B^{-1} \geqq^J A^{-1}.$$

This is a consequence of (1.1); here is a simple proof (see also [2, Lemma 3]):

Proof. Note that $A^{\frac{1}{2}}$ is defined by the Dunford integral and it is invertible J-selfadjoint. The assumption and Lemma 2.2 yield

$$I = A^{-\frac{1}{2}} A A^{-\frac{1}{2}} \geqq^J A^{-\frac{1}{2}} B A^{-\frac{1}{2}} = (B^{\frac{1}{2}} A^{-\frac{1}{2}})^\sharp (B^{\frac{1}{2}} A^{-\frac{1}{2}}).$$

Hence, by a theorem of Potapov-Ginzburg (see [3, Chapter 2, Section 4]),

$$I \geqq^J (B^{\frac{1}{2}} A^{-\frac{1}{2}})(B^{\frac{1}{2}} A^{-\frac{1}{2}})^\sharp = B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}}.$$

Therefore, by Lemma 2.2,

$$B^{-1} = B^{-\frac{1}{2}} I B^{-\frac{1}{2}} \geqq^J B^{-\frac{1}{2}} B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}} B^{-\frac{1}{2}} = A^{-1},$$

and the proof is complete. □

THEOREM 2.4. *Let A, B be J-selfadjoint matrices with non-negative eigenvalues and $0 < \alpha < 1$. If*

$$I \geqq^J A \geqq^J B,$$

then J-selfadjoint powers A^α, B^α are well defined and

$$I \geqq^J A^\alpha \geqq^J B^\alpha.$$

Proof. When all eigenvalues of A and B are positive, the assertion is a consequence of (1.1) with the operator monotone function $f(t) = t^\alpha$ on $(0, \infty)$.

When A or B has 0 as its eigenvalue, the powers can be defined by Lemma 2.1. The assumption $A \geqq^J B$ and Lemma 2.3 imply that

$$(\lambda I + B)^{-1} \geqq^J (\lambda I + A)^{-1} \quad (\lambda > 0),$$

or

$$A(\lambda I + A)^{-1} \geqq^J B(\lambda I + B)^{-1} \quad (\lambda > 0).$$

Therefore, $A^\alpha \geqq^J B^\alpha$ and the proof is complete. □

REMARK 2.5. We comment on an alternative proof of the inequality $A^\alpha \geqq^J B^\alpha$. For simplicity, assume that A, B are invertible. Then similar argument as in the proof of [4, Theorem V.1.9] and (1.2) yield $A^r \geqq^J B^r$ for all dyadic rationals $r \in [0, 1]$. By continuity, the assertion follows.

3. Furuta inequality of indefinite type

LEMMA 3.1. *Let A, B be J-selfadjoint matrices with non-negative eigenvalues and $I \geqq^J A, I \geqq^J B$. Then the eigenvalues of ABA are non-negative and*

$$I \geqq^J A^\lambda$$

for $\lambda > 0$.

Proof. By assumption, A and $B^{\frac{1}{2}}$ are J-contractive, so is $B^{\frac{1}{2}} A$. Since

$$ABA = (B^{\frac{1}{2}} A)^\sharp (B^{\frac{1}{2}} A) (\leqq^J I),$$

it follows from a theorem of Potapov-Ginzburg that

$$\sigma(ABA) \subseteqq [0, \infty).$$

The second assertion is easy to see; the detail is left to the reader, and the proof is complete. □

LEMMA 3.2. (cf.,[6, Lemma 1]) *Let A, B be J-selfadjoint matrices with positive eigenvalues and $I \geqq^J A, I \geqq^J B$. Then*

$$(ABA)^{\lambda} = AB^{\frac{1}{2}}(B^{\frac{1}{2}}A^2B^{\frac{1}{2}})^{\lambda-1}B^{\frac{1}{2}}A$$

holds for $\lambda \in \mathbb{R}$.

Proof. Let us consider a J-polar decomposition of $AB^{\frac{1}{2}}$:

$$AB^{\frac{1}{2}} = UH,$$

where U is J-unitary; $U^{\sharp}U = I$ and H is the J-modulus $\{(AB^{\frac{1}{2}})^{\sharp}(AB^{\frac{1}{2}})\}^{\frac{1}{2}} = (B^{\frac{1}{2}}A^2B^{\frac{1}{2}})^{\frac{1}{2}}$. Then it follows that

$$\begin{aligned}(ABA)^{\lambda} &= (AB^{\frac{1}{2}}(AB^{\frac{1}{2}})^{\sharp})^{\lambda} = (UH^2U^{\sharp})^{\lambda}\\ &= UH^{2\lambda}U^{\sharp} = AB^{\frac{1}{2}}H^{-1}H^{2\lambda}H^{-1}B^{\frac{1}{2}}A\\ &= AB^{\frac{1}{2}}(H^2)^{\lambda-1}B^{\frac{1}{2}}A = AB^{\frac{1}{2}}(B^{\frac{1}{2}}A^2B^{\frac{1}{2}})^{\lambda-1}B^{\frac{1}{2}}A\end{aligned}$$

and the proof is complete. □

PROPOSITION 3.3. *Let $A \in M_n(\mathbb{C})$ be J-selfadjoint with non-negative eigenvalues and $I \geqq^J A$, and A_n $(n \in \mathbb{N})$ J-selfadjoint with positive eigenvalues. Suppose that*

$$A_n \to A$$

as $n \to \infty$. Then for $0 < \alpha < 1$,

$$A_n^{\alpha} \to A^{\alpha}$$

as $n \to \infty$.

Proof. Note that $\mathbb{C}^n$ is the algebraic direct sum of $\mathcal{M} := \ker A$ and its J-orthocomplement $\mathcal{N}$ as in the proof of Lemma 2.1. Since A_n^{α}, A^{α} on $\mathcal{N}$ are given by the Dunford integral, the assertion on $\mathcal{N}$ is clear. Hence, it suffices to see that for $y \in \ker A$

$$\int_0^{\infty} \lambda^{\alpha-1}A_n(\lambda I + A_n)^{-1}y \ d\lambda$$

tends to

$$\int_0^{\infty} \lambda^{\alpha-1}A(\lambda I + A)^{-1}y \ d\lambda = 0$$

as $n \to \infty$. But this follows from Lebesgue's dominated convergence theorem, and the proof is complete. □

Here is a generalization of Furuta inequality:

THEOREM 3.4. (cf.,[5, Theorem]) *Let A, B be J-selfadjoint matrices with non-negative eigenvalues and $I \geqq^J A \geqq^J B$. For each $r \geqq 0$,*

$$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geqq^J (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

holds for $p \geqq 0, q \geqq 1$ with $(1+r)q \geqq p + r$.

Proof. Since a proof given here is just the same as that of [5, Theorem], we have a brief sketch; the detail is left to the reader.

Thanks to Proposition 3.3, we may assume that A, B are invertible; if necessary, by considering invertible

$$A_n := A + \frac{1}{n} I, B_n := B + \frac{1}{n} I,$$

and after having that

$$(A_n^{\frac{r}{2}} A_n^p A_n^{\frac{r}{2}})^{\frac{1}{q}} \geqq^J (A_n^{\frac{r}{2}} B_n^p A_n^{\frac{r}{2}})^{\frac{1}{q}},$$

then take n as $n \to \infty$ to get the conclusion.

For $1 \geqq p \geqq 0$, by Theorem 2.4,

$$A^p \geqq^J B^p$$

holds and the assertion in this case follows from Lemma 2.2.

Since the inequality for $p \geqq 1, q = \dfrac{p+r}{1+r}$ implies the inequality for $p \geqq 1, q > \dfrac{p+r}{1+r}$ because of Theorem 2.4, it suffices to consider the case $p \geqq 1, q = \dfrac{p+r}{1+r}$. For $0 \leqq r \leqq 1$, Lemmas 2.2, 2.3, and 3.2 yield that

$$\begin{aligned}
A^{1+r} &= A^{\frac{r}{2}} A A^{\frac{r}{2}} \\
&\geqq^J A^{\frac{r}{2}} B A^{\frac{r}{2}} \\
&= A^{\frac{r}{2}} B^{\frac{p}{2}} (B^{-\frac{p}{2}} B^{-r} B^{-\frac{p}{2}})^{\frac{p-1}{p+r}} B^{\frac{p}{2}} A^{\frac{r}{2}} \\
&\geqq^J A^{\frac{r}{2}} B^{\frac{p}{2}} (B^{-\frac{p}{2}} A^{-r} B^{-\frac{p}{2}})^{\frac{p-1}{p+r}} B^{\frac{p}{2}} A^{\frac{r}{2}} \\
&= A^{\frac{r}{2}} B^{\frac{p}{2}} (B^{\frac{p}{2}} A^{\frac{r}{2}} A^{\frac{r}{2}} B^{\frac{p}{2}})^{\frac{1-p}{p+r}} B^{\frac{p}{2}} A^{\frac{r}{2}} \\
&= (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}.
\end{aligned}$$

Note that by Lemma 3.1 eigenvalues of $B^{\frac{p}{2}} A^r B^{\frac{p}{2}}$ are positive, so are those of $B^{-\frac{p}{2}} A^{-r} B^{-\frac{p}{2}}$.

Let

$$A_1 := A^{1+r}, B_1 := (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}},$$

then by the above calculation, $A_1 \geqq^J B_1$ and

$$A_1^{1+r_1} \geqq^J (A_1^{\frac{r_1}{2}} B_1^{p_1} A_1^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}}$$

for $p_1 \geqq 1, 1 \geqq r_1 \geqq 0$.

Putting p_1, r_1 as

$$p_1 = \frac{p+r}{1+r} \geqq 1, r_1 = 1$$

implies

$$A^{2(1+r)} \geqq^J \{A^{\frac{1+r}{2}} (A^{\frac{r}{2}} B^p A^{\frac{r}{2}}) A^{\frac{1+r}{2}}\}^{\frac{2(1+r)}{p+2r+1}} = \{A^{r+\frac{1}{2}} B^p A^{r+\frac{1}{2}}\}^{\frac{2(1+r)}{p+2r+1}}.$$

And taking r as $1+s = 2(1+r)$ yields that $s \in [1,3]$ and

$$A^{1+s} \geqq^J \{A^{\frac{s}{2}} B^p A^{\frac{s}{2}}\}^{\frac{1+s}{p+s}}.$$

Repeating this manner, we can get the conclusion. □

REMARK 3.5. In Theorem 3.4, the assumption that $I \geqq^J A \geqq^J B$ can be replaced by that

$$\alpha I \geqq^J A \geqq^J B$$

for some $\alpha > 0$.

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