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GENERALIZED VECTOR QUASI–VARIATIONAL–LIKE INEQUALITIES

JIAN-WEN PENG

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Abstract. In this paper, some existence theorems for the generalized vector quasi-variational-like inequalities without monotonicity are obtained.

1. Introduction

The Vector Variational Inequality (for short, VVI) in a finite dimensional Euclidean space has been introduced in [1] and applications have been given. Chen and Cheng [2] studied the VVI in infinite dimensional space and applied it to Vector Optimization Problem (for short, VOP). Since then, many authors [3-11] have intensively studied the VVI on different assumptions in infinite-dimensional spaces. Lee et al. [12, 13], Lin et al. [14], Konnov and Yao [15], Daniilidis and Hadiisavvas [16], Yang and Yao [17], and Oettli and Schlager [18] studied the generalized vector variational inequality and obtained some existence results. Chen et al. [19] and Lee et al. [20] introduced and studied the generalized vector quasi-variational inequality and established some existence theorems. Ansari [21, 22], Ding [23, 24] and Luo [25] studied the generalized vector variational-like inequalities. Ding [26] introduced and study a class of generalized vector quasi-variational-like inequality problem (in short, GVQVLIP), which generalizes and unifies generalized vector quasi-variational inequalities, generalized vector variational-like inequalities as well as various extensions of the classic variational inequalities in the literature. By employing the scalarization technique, Ding [26] established several existence results for (GVQVLIP) involving C_+ - η -monotone and weakly C_+ - η -monotone set-valued mappings.

In this paper, we shall use different method with that in [26] and present the existence of a solution for (GVQVLIP) without any monotone conditions.

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2. Preliminaries

Let Y be a real Hausdorff topological vector space and X be a nonempty convex subset in a real locally convex Hausdorff topological vector space E. we denote $L(E,Y)$ the space of all continuous linear operators from E into Y and by $\langle u, y\rangle$ the evaluation of $u \in L(E,Y)$ at $y \in E$. Let σ be the family of all bounded subsets of X whose union is total in E, i.e., the linear hull of $\cup\{S : S \in \sigma\}$ is dense in X. Let β be a neighbourhood base of 0 in Y. When S runs through σ, V through β, the family

$$M(S,V) = \{l \in L(E,Y) : \cup_{x \in S}\langle l, x\rangle \subset V\}$$

is a neighbourhood base of 0 in $L(E,Y)$ at $x \in E$ (see [27, pp. 79–80]). By the Corollary of Schaefer [27, pp. 80], $L(E,Y)$ becomes a locally convex topological vector space under σ-topology, where Y is assumed a locally convex topological space.

Let $\operatorname{int} A$ and $\operatorname{Co} A$ denote the interior and convex hull of a set A, respectively. Let $C : X \to 2^Y$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone with $\operatorname{int} C(x) \neq \emptyset$ for each $x \in X$. Let $\eta : X \times X \to E$ be a single-valued mapping, $D : X \to 2^X$ and $T : X \to 2^{L(E,Y)}$ be two set-valued mappings, Ding [26] introduced a generalized vector quasi-variational-like inequality problem (GVQVLIP), which is to find $\bar{x}$ in X such that $\bar{x} \in D(\bar{x})$, and

$$\forall y \in D(\bar{x}), \exists \hat{v} \in T(\bar{x}) : \ \langle \hat{v}, \eta(y,\bar{x})\rangle \notin -\operatorname{int} C(\bar{x}). \tag{1}$$

Then the point $\bar{x}$ is said to be a solution of the (GVQVLIP).

It is easy to see that $\bar{x}$ is a solution of the (GVQVLIP) is equivalent to $\bar{x}$ in X satisfying $\bar{x} \in D(\bar{x})$, and

$$\forall y \in D(\bar{x}), \langle T(\bar{x}), \eta(y,\bar{x})\rangle \not\subseteq -\operatorname{int} C(\bar{x}). \tag{2}$$

Where $\langle T(\bar{x}), \eta(y,\bar{x})\rangle = \cup_{v \in T(\bar{x})}\langle v, \eta(y,\bar{x})\rangle$.

The following problems are the special cases of the (GVQVLIP).

(i) For all $x \in X$, if $D(x) \equiv X$, then the (GVQVLIP) reduces to the generalized vector variational-like inequality problem (in short, GVVLIP) which is to find $\bar{x}$ in X such that there exists an $\hat{v} \in T(\bar{x})$ satisfying

$$\langle \hat{v}, \eta(y,\bar{x})\rangle \notin -\operatorname{int} C(\bar{x}), \forall y \in X. \tag{3}$$

This problem was studied in [21-25].

(ii) If T is a single-valued mapping and $\eta(y,x) = y - g(x), \forall x, y \in X$, where $g : X \to E$ is a single-valued mapping, then the (GVQVLIP) reduces to finding $\bar{x}$ in X such that $\bar{x} \in D(\bar{x})$, satisfying

$$\langle T(\bar{x}), y - g(\bar{x})\rangle \notin -\operatorname{int} C(\bar{x}), \forall y \in D(\bar{x}). \tag{4}$$

This is a new problem. If for all $x \in X$, if $D(x) \equiv X$, then the problem (4) reduces to finding $\bar{x}$ in X such that

$$\langle T(\bar{x}), y - g(\bar{x})\rangle \notin -\operatorname{int} C(\bar{x}), \forall y \in X. \tag{5}$$

The problem (5) was considered by Siddiqi et al. [28].

(*iii*) If $\eta(y,x) = y - x$, $\forall x, y \in X$, then the (GVQVLIP) reduces to finding $\bar{x}$ in X such that $\bar{x} \in D(\bar{x})$, and

$$\forall y \in D(\bar{x}), \exists \hat{v} \in T(\bar{x}) : \ \langle \hat{v}, y - x \rangle \notin -\operatorname{int} C(\bar{x}). \tag{6}$$

Problem (6) is called the generalized vector quasi-variational inequality problem (GVQVIP) which is new. When $C(x) = C$, $\forall x \in X$ is a constant cone, problem (6) was studied by Chen and Li [19] and Lee et al. [20].

(*iv*) If $D(x) \equiv X$, $\forall x \in X$ and $\eta(y,x) = y - x$, $\forall x, y \in X$, then the (GVQVLIP) reduces to find $\bar{x}$ in X such that

$$\forall y \in X, \exists \hat{v} \in T(\bar{x}) : \ \langle \hat{v}, \eta(y, \bar{x}) \rangle \notin -\operatorname{int} C(\bar{x}). \tag{7}$$

Problem (7) and its special cases are called the generalized vector variational inequality (GVVIP) which was introduced and studied in [12-18].

(*v*) If T is single-valued function, then the (GVQVLIP) reduces to find $\bar{x}$ in X such that $\bar{x} \in D(\bar{x})$, and

$$\langle T(\bar{x}), \eta(y, \bar{x}) \rangle \notin -\operatorname{int} C(\bar{x}), \ \forall y \in D(\bar{x}). \tag{8}$$

When $D(x) \equiv X$, $\forall x \in X$, problem (8) and its special cases were studied by many authors, see [1-11].

(*vi*) If $Y = R$ and $C(x) = [0, \infty), \forall x \in X$, then $L(E, Y) = E^*$, where E^* is the dual space of E, and the (GVQVLIP) reduces to find $\bar{x}$ in X such that $\bar{x} \in D(\bar{x})$, and

$$\forall y \in D(\bar{x}), \exists \hat{v} \in T(\bar{x}) : \ \langle \hat{v}, \eta(y, \bar{x}) \rangle \geqslant 0. \tag{9}$$

Problem (9) includes many classes of scalar type generalized quasi-variational inequality and generalized quasi-variational-like inequality problems as special cases, see [29-35].

In order to prove the main results, we need the following definitions and lemmas.

DEFINITION 2.1 [23] Let E, Y be two real topological vector spaces, X be a nonempty and convex subset of E, $C : X \to 2^Y$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone for each $x \in X$. Let $\eta : X \times X \to E$ be a single-valued mapping. $T : X \to 2^{L(E,Y)}$ is said to satisfy the generalized L-η-condition iff for any finite set $\{y_1, y_2, ..., y_n\}$ in X, $\bar{x} = \sum_{j=1}^{n} \alpha_j y_j$ with $\alpha_j \geqslant 0$ and $\sum_{j=1}^{n} \alpha_j = 1$, there exists $\bar{v} \in T(\bar{x})$, such that

$$\left\langle \bar{v}, \sum_{j=1}^{n} \alpha_j \eta(y_j, \bar{x}) \right\rangle \notin -\operatorname{int} C(\bar{x}).$$

REMARK 2.1. If $\eta(y,x)$ is affine in the first argument and $\forall x \in X$, $\exists v \in T(x)$, such that

$$\langle \bar{v}, \eta(x,x) \rangle \notin -\operatorname{int} C(x),$$

Then T satisfy the generalized L-η-condition.

If $\eta(y,x) = y - x$, $\forall x, y \in X$, then we have that

$$\left\langle \bar{v}, \sum_{j=1}^{n} \alpha_j (y_j - \bar{x}) \right\rangle = \langle \bar{v}, \bar{x} - \bar{x}) \rangle = 0 \notin -\operatorname{int} C(\bar{x}), \forall v \in T(\bar{x}).$$

And hence T satisfy the generalized L-η-condition trivially.

DEFINITION 2.2 [36] Let X and Y be two topological spaces and $T : X \to 2^Y$ be a set-valued mapping. Then

(1) T is said to be upper semicontinuous if, for any $x_0 \in X$ and for each open set U in Y containing $T(x_0)$, there is a nerghborhood V of x_0 in X such that $T(x) \subseteq U$, for all $x \in V$.

(2) T is said to have open lower sections if the set $T^{-1}(y) = \{x \in X : y \in T(x)\}$ is open in X for each $y \in Y$.

(3) T is said to be closed, if the set $\{(x,y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$.

LEMMA 2.1. *(see [37]) Let X be a paracompact Hausdorff space and Y be a linear topological space. Suppose $T : X \to 2^Y$ is a set-valued mapping such that*

(i) for each $x \in X$, $T(x)$ is nonempty,

(ii) for each $x \in X$, $T(x)$ is convex, and

(iii) T has open lower sections.

Then there exists a continuous function $f : X \to Y$ such that $f(x) \in T(x)$ for all $x \in X$.

LEMMA 2.2. *(see [36]) Let X and Y be topological spaces. If $T : X \to 2^Y$ is a upper semicontinuous set-valued mapping with closed values, then T is closed.*

LEMMA 2.3. *(see [38]) Let X and Y be topological spaces and $T : X \to 2^Y$ is a upper semicontinuous set-valued mapping with compact values. Suppose $\{x_\alpha\}$ is a net in X such that $x_\alpha \to x_0$. If $y_\alpha \in T(x_\alpha)$ for each α, then there is a $y_0 \in T(x_0)$ and a subset $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.*

LEMMA 2.4. *(see [37]) Let X and Y be two topological spaces. Suppose $T : X \to 2^Y$ and $K : X \to 2^Y$ are set-valued mappings having open lower sections, then*

(i) the set-valued mapping $F : X \to 2^Y$ defined by, for each $x \in X$, $F(x) = \operatorname{Co}(T(x))$ has open lower sections.

(ii) the set-valued mapping $\theta : X \to 2^Y$ defined by, for each $x \in X$, $\theta(x) = T(x) \cap K(x)$ has open lower sections.

LEMMA 2.5. *(see [40]) Let E be a locally convex topological linear space and X be a compact convex subset in E. Suppose $T : X \to 2^X$ is a set-valued mapping such that*

(i) for each $x \in X$, $T(x)$ is nonempty,

(ii) for each $x \in X$, $T(x)$ is convex and closed,

(iii) T is upper semicontinuous.

Then there exists a $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$.

3. Existence results

In this section, we shall present some existence result of the (GVQVLIP) without any monotone conditions.

THEOREM 3.1. *Let Y be a real Hausdorff topological vector space, X be a nonempty, compact, convex and metrizable set in a real locally convex Hausdorff topological vector space E, and $L(E,Y)$ be equipped with the σ-topology. Let $D : X \to 2^X$ be an upper semicontinuous set-valued mapping with nonempty convex closed values and open lower sections, $C : X \to 2^Y$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone with $\operatorname{int} C(x) \neq \emptyset$ for each $x \in X$, and the set-valued mapping $M = Y \backslash (-\operatorname{int} C) : X \to 2^Y$ be upper semicontinuous on X. Let $T : X \to 2^{L(E,Y)}$ be upper semicontinous on X with compact values and $\eta : X \times X \longrightarrow E$ be continuous with respect to the second argument, such that T satisfies the generalized L-η-condition. Then, the (GVQVLIP) has a solution $\bar{x} \in X$.*

Proof. Define a set-valued mapping $P : X \to 2^X$ by

$$\begin{aligned} P(x) &= \{y \in X : \langle T(x), \eta(y,x) \rangle \subseteq -\operatorname{int} C(x)\} \\ &= \{y \in X : \langle v, \eta(y,x) \rangle \in -\operatorname{int} C(x), \forall v \in T(x)\}, \forall x \in X. \end{aligned}$$

Thus, to show the conclusion of the theorem, it is equivalent to showing that there exists $\bar{x} \in X$ such that $\bar{x} \in D(\bar{x})$ and $D(\bar{x}) \cap P(\bar{x}) = \emptyset$.

We first prove that $x \notin \operatorname{Co}(P(x))$ for all $x \in X$. To see this, suppose, by way of contradiction, that there exists some point $\bar{x} \in X$ such that $\bar{x} \in \operatorname{Co}(P(\bar{x}))$. Then there exists finite points $y_1, y_2, ..., y_n$ in X, and $\alpha_j \geqslant 0$ with $\sum_{j=1}^n \alpha_j = 1$ such that $\bar{x} = \sum_{j=1}^n \alpha_j y_j$ and $y_j \in P(\bar{x})$ for all $j = 1, 2, ..., n$. That is,

$$\langle v, \eta(y_j, \bar{x}) \rangle \in -\operatorname{int} C(\bar{x}), \quad \forall v \in T(x) \text{ and } j = 1, 2, ..., n.$$

Since $\operatorname{int} C(\bar{x})$ is convex, we obtain

$$\langle v, \sum_{j=1}^n \alpha_j \eta(y_j, \bar{x}) \rangle \in -\operatorname{int} C(\bar{x}), \forall v \in T(x),$$

which contradicts the fact that T satisfies the generalized L-η-condition. Therefore $x \notin \operatorname{Co}(P(x))$ for all $x \in X$.

Now we prove that the set

$$\begin{aligned} P^{-1}(y) &= \{x \in X : \langle T(x), \eta(y,x) \rangle \subseteq -\operatorname{int} C(x)\} \\ &= \{x \in X : \langle v, \eta(y,x) \rangle \in -\operatorname{int} C(x), \forall v \in T(x)\} \end{aligned}$$

is open for each $y \in X$. That is, P has open lower sections in X. Consider the set-valued mapping $S : X \to 2^Y$ defined by

$$\begin{aligned} S(y) &= \{x \in X : \langle T(x), \eta(y,x) \rangle \not\subseteq -\operatorname{int} C(x)\} \\ &= \{x \in X : \exists\, v \in T(x) \end{aligned}$$

such that $\langle v, \eta(y,x)\rangle \notin -\operatorname{int} C(x)\}$.

We only need to prove that $S(y)$ is closed for all $y \in X$. In fact, consider a net $x_t \in S(y)$ such that $x_t \to x \in X$. Since $x_t \in S(y)$, there exists $s_t \in T(x_t)$ such that

$$\langle s_t, \eta(y,x_t)\rangle \notin -\operatorname{int} C(x_t).$$

From the upper semicontinuity and compact values of T and Lemma 2.3, it suffices to find a subset $\{s_{t_j}\}$ which converges to some $s \in T(x)$. By Lemma 1 in [23, pp. 114], we know that $\langle . \rangle$ is continuous, and hence

$$\langle s_{t_j}, \eta(y,x_{t_j})\rangle \to \langle s, \eta(y,x)\rangle .$$

By Lemma 2.2 and upper semicontinuity of M, we have $\langle s, \eta(y,x)\rangle \notin -\operatorname{int} C(x)$, and hence $x \in S(y)$, $S(y)$ is closed. Therefore, P has open lower sections in X.

Also define another set-valued mapping, $G : X \to 2^X$ by $G(x) = D(x) \cap \mathrm{Co}(P(x))$, $\forall x \in X$. Let the set $W = \{x \in X : G(x) \neq \emptyset\}$. Since D and P has open lower sections in X, and by Lemma 2.4, we know that $\mathrm{Co}(P)$ and G also has open lower sections in X. Hence, $W = \cup_{y\in X} G^{-1}(y)$ is an open set in X. Then, the set-valued mapping $G|_W : W \to 2^X$ has open lower sections in W, and for all $x \in W$, $G(x)$ is nonempty and convex. Also, since X is a metrizable space, W is paracompact [39, P.831]. Hence, by Lemma 2.1, there is a continuous function $f : W \to X$ such that $f(x) \in G(x) \subset D(x)$ for all $x \in W$. Define $Q : X \to 2^X$ by

$$Q(x) = \begin{cases} f(x) & \text{if} \quad x \in W, \\ D(x) & \text{if} \quad x \notin W. \end{cases}$$

Now, we prove that Q is upper semicontinuous. In fact, for each open set V in X, the set

$$\begin{aligned} \{x \in X : Q(x) \subset V\} &= \{x \in W : f(x) \in V\} \cup \{x \in X \setminus W : D(x) \subset V\} \\ &\subset \{x \in W : f(x) \in V\} \cup \{x \in X : D(x) \subset V\}. \end{aligned}$$

On the other hand, when $x \in W$, and $f(x) \in V$, we have $Q(x) = f(x) \in V$. when $x \in X$ and $D(x) \subset V$, since $f(x) \in D(x)$ if $x \in W$, we know that $Q(x) \subset V$ and so

$$\{x \in W : f(x) \in V\} \cup \{x \in X : D(x) \subset V\} \subset \{x \in X : Q(x) \subset V\}.$$

Therefore,

$$\{x \in X : Q(x) \subset V\} = \{x \in W : f(x) \in V\} \cup \{x \in X : D(x) \subset V\}.$$

Since f is continuous and D is upper semicontinuous, the sets $\{x \in W : f(x) \in V\}$ and $\{x \in X : D(x) \subset V\}$ are open. It follows that $\{x \in X : Q(x) \subset V\}$ is open and so the mapping $Q : X \to 2^X$ is upper semicontinuous. Since for each $x \in X$, $Q(x)$ is convex, closed, and nonempty, by Lemma 2.5, there is $\bar{x} \in X$ such that $\bar{x} \in Q(\bar{x})$. Note that $\bar{x} \notin W$. Otherwise, $\bar{x} = f(\bar{x}) \in P(\bar{x}) \subseteq \mathrm{Co}(P(\bar{x}))$, which contradicts to $x \notin \mathrm{Co}(P(x)$ for all $x \in X$.

Thus $\bar{x} \in D(\bar{x})$ and $G(\bar{x}) = \emptyset$. That is, $\bar{x} \in D(\bar{x})$ and $D(\bar{x}) \cap \mathrm{Co}(P(\bar{x})) = \emptyset$, which implies $\bar{x} \in D(\bar{x})$ and $D(\bar{x}) \cap P(\bar{x}) = \emptyset$. Consequently, there exists $\bar{x} \in X$, such that $\bar{x} \in D(\bar{x})$, and $\forall y \in D(\bar{x})$, $\exists v \in T(\bar{x})$ satisfying $\langle v, \eta(y, \bar{x}) \rangle \notin -\operatorname{int} C(\bar{x})$. That is, the (GVQVLIP) has a solution $\bar{x} \in X$.

By Theorem 3.1 and Remark 2.1, we have

COROLLARY 3.2. *Let Y be a real Hausdorff topological vector space, X be a nonempty, compact, convex and metrizable set in a real locally convex Hausdorff topological vector space E, and $L(E,Y)$ be equipped with the σ-topology. Let $D : X \to 2^X$ be an upper semicontinuous set-valued mapping with nonempty convex closed values and open lower sections, $C : X \to 2^Y$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone with $\operatorname{int} C(x) \neq \emptyset$ for each $x \in X$, and the set-valued mapping $M = Y \backslash (-\operatorname{int} C) : X \to 2^Y$ be upper semicontinuous on X. Let $T : X \to 2^{L(E,Y)}$ be upper semicontinous on X with compact values and $\eta : X \times X \longrightarrow E$ be continuous with respect to the second argument and affine with respect to the first argument such that $\forall x \in X$, $\exists v \in T(x)$, satisfying*

$$\langle \bar{v}, \eta(x,x) \rangle \notin -\operatorname{int} C(x),$$

Then, the (GVQVLIP) has a solution $\bar{x} \in X$.

REMARK 3.1. Let $D(x) = X$, by Corollary 3.2, we recover Theorem 1 in [25] from the (GVVLIP) cases to the (GVQVLIP) cases with additional condition that X is metrizable, so Theorem 3.1 and Corollary 3.2 are generalizations of Theorem 1 in [25]. It is also easy to see that Theorem 3.1 and Corollary 3.2, respectively, are generalization of Theorem 1 and Corollary 1 in [23]. Consequently, Theorem 3.1 and Corollary 3.2 are also generalizations of Theorem 2.1 in [7].

THEOREM 3.3. *Let Y be a Hausdorff topological vector space, X be a nonempty, compact, convex and metrizable set in a locally convex Hausdorff topological vector space E, and $L(E,Y)$ be equipped with the σ-topology. Let $D : X \to 2^X$ be an upper semicontinuous set-valued mapping with nonempty convex closed values and open lower sections, $C : X \to 2^Y$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone with $\operatorname{int} C(x) \neq \emptyset$ for each $x \in X$, and the set-valued mapping $M = Y \backslash (-\operatorname{int} C) : X \to 2^Y$ be upper semicontinuous on X. Let $T : X \to 2^{L(E,Y)}$ be upper semicontinous on X with compact values and $\eta : X \times X \longrightarrow E$ be continuous with respect to the second argument. Suppose that there exists a mapping $h : X \times X \to Y$, such that:*

(i) $\forall x, y \in X$, $\exists v \in T(x)$, such that

$$h(x,y) - \langle v, \eta(y,x) \rangle \in -\operatorname{int} C(x);$$

(ii) for any finite set $\{y_1, y_2, ..., y_n\} \subseteq X$ and $\bar{x} = \sum_{j=1}^{n} \alpha_j y_j$ with $\alpha_j \geqslant 0$ and $\sum_{j=1}^{n} \alpha_j = 1$, there is a $j \in \{1, 2, ..., n\}$, such that $h(\bar{x}, y_j) \notin -\operatorname{int} C(\bar{x})$. Then, the (GVQVLIP) has a solution $\bar{x} \in X$.

Proof. Define two set-valued mappings $P : X \to 2^X$, $P_1 : X \to 2^X$ by

$$P(x) = \{y \in X : \langle v, \eta(y,x)\rangle \in -\operatorname{int} C(x), \forall v \in T(x)\}, \forall x \in X.$$

$$P_1(x) = \{y \in X : h(x,y) \in -\operatorname{int} C(x)\}, \forall x \in X.$$

We first prove that $x \notin \mathrm{Co}(P_1(x))$ for all $x \in X$. To see this, suppose, by way of contradiction, that there exists some point $\bar{x} \in X$ such that $\bar{x} \in \mathrm{Co}(P_1(\bar{x}))$. Then there exists finite points $y_1, y_2, ..., y_n$ in X, and $\alpha_j \geqslant 0$ with $\sum_{j=1}^n \alpha_j = 1$ such that $\bar{x} = \sum_{j=1}^n \alpha_j y_j$ and $y_j \in P_1(\bar{x})$ for all $j = 1, 2, ..., n$. That is,

$$h(\bar{x}, y_j) \in -\operatorname{int} C(\bar{x}), \quad j = 1, 2, ..., n.$$

This contradicts to the condition (ii). Therefore $x \notin \mathrm{Co}(P_1(x))$ for all $x \in X$.

The condition (i) implies that $P_1(x) \supseteq P(x)$ for all $x \in X$. Hence, $x \notin \mathrm{Co}(P(x))$, $\forall x \in X$.

The rest of the proof is the same as in the proof of Theorem 3.1.

COROLLARY 3.4. *Let Y be a real Hausdorff topological vector space, X be a nonempty, compact, convex and metrizable set in a real locally convex Hausdorff topological vector space E, and $L(E,Y)$ be equipped with the σ-topology. Let $D : X \to 2^X$ be an upper semicontinuous set-valued mapping with nonempty convex closed values and open lower sections, $C : X \to 2^Y$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone with $\operatorname{int} C(x) \neq \emptyset$ for each $x \in X$, and the set-valued mapping $M = Y \backslash (-\operatorname{int} C) : X \to 2^Y$ be upper semicontinuous on X. Let $T : X \to L(E,Y)$ be continuous on X and $\eta : X \times X \longrightarrow E$ be continuous with respect to the second argument. Suppose that there exists a mapping $h : X \times X \to Y$, such that:*

(i) $\forall x, y \in X$, $h(x,y) - \langle T(x), \eta(y,x)\rangle \in -\operatorname{int} C(x)$;

(ii) the set $\{y \in X : h(x,y) \in -\operatorname{int} C(x)\}$ is convex for all $x \in X$;

(iii) $h(x,x) \notin -\operatorname{int} C(x), \forall x \in X$.

Then, there exists $\bar{x} \in X$, such that $\bar{x} \in D(\bar{x})$ and $\langle T(\bar{x}), \eta(y,\bar{x})\rangle \notin -\operatorname{int} C(\bar{x})$, $\forall y \in D(\bar{x})$.

Proof. Following the same argument of the proof of Corollary 3 in [23], by the condition (ii) and (iii), we know that the condition (ii) of Theorem 3.3 holds. By Theorem 3.3, we know that the conclusion is correct.

REMARK 3.2. Theorem 3.3 and Corollary 3.4, respectively, generalize the Theorem 2 and Corollary 3 in [23] and Theorem 2.2 in [7] with additional conditions of compactness and metrizablity of X.

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College of Mathematics and Computer Science
Chongqing Normal University
Chongqing 400047
P. R. China
e-mail: jwpeng6@yahoo.com.cn