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ON THE SOLUTION OF VARIATIONAL INEQUALITY PROBLEMS BY USING CUTTING PLANE METHODS

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Abstract. In this paper, variational inequality problems (VIPs) defined by generalized monotone and pseudomonotone single-valued and multivalued mappings are considered. Some properties of generalized monotone and pseudomonotone mappings are established. The idea of cutting plane methods, developed originally for solving discrete optimization problems (in particular, integer linear programming problems), is applied for solving the considered VIPs.

1. Introduction

Let B be an open convex set in $\mathbb{R}^n$ and C be a closed convex subset of B. Let $F : B \to \mathbb{R}^n$ be a continuous mapping and VIP (F, C) denote the variational inequality problem associated with the mapping F and the set C.

Recall that a vector $\bar{x} \in C$ is a *solution* of VIP (F, C) if and only if

$$\langle F(\bar{x}), x - \bar{x} \rangle \geqslant 0 \quad \forall x \in C, \tag{1}$$

where $\langle \ , \rangle$ denotes the usual inner product of $\mathbb{R}^n$.

Denote by C^* the solution set of VIP (F, C).

If the constraint set C is the nonnegative orthant $\mathbb{R}^n_+ \equiv \{x \in \mathbb{R}^n : x \geqslant 0\}$ of $\mathbb{R}^n$, then the VIP reduces to the complementarity problem (CP).

Recall that the *nonlinear complementarity problem* NCP (F) is to find a point $x \in \mathbb{R}^n$ such that

$$x \geqslant 0, \quad F(x) \geqslant 0, \quad \langle x, F(x) \rangle = 0,$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$, and inequalities of the form $x \geqslant 0$ are componentwise.

Variational inequalities arise in different mathematical problems, for example, in nonlinear optimization; they are connected with operator theory, especially monotone operators, etc.

Variational inequalities have been studied in many works.

The monograph of Kinderlehrer and Stampacchia [11] is a complete introduction to this topic.

Equivalence of variational inequality problems to unconstrained optimization problems is studied in [Peng 16].

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Unconstrained optimization reformulations of variational inequality problems are proposed in [Yamashita, Taji, and Fukushima 24]. Reformulations of variational inequalities are also considered in [Andreani and Martínez 1].

Newton-type methods for solving variational inequalities are suggested, e.g., in [Marcotte and Dussault 14], [Qi 17], [Qi and Sun 18], [Taji, Fukushima, and Ibaraki 23], etc.

A hybrid projection-proximal point algorithm is proposed in [Solodov and Svaiter 19].

Nesterov and Vial ([15]) introduced a homogeneous analytic center cutting plane method (HACCPM) which solves monotone VIPs in a conic setting and pseudopolynomial-time complexity.

Analytic center cutting plane method (ACCPM) is considered, for example, in [Sonnevend 21]. An ACCPM for pseudomonotone variational inequalities and a complexity bound was derived in [Goffin, Marcotte, and Zhu 10]. An analytic center quadratic cut method for strongly monotone variational inequality problems is proposed in [Lüthi and Büeler 12].

Descent methods for asymmetric variational inequality problems are suggested in [Fukushima 8].

Lagrangian dual methods for variational inequality problems are considered, for example, in [Auslender and Teboulle 3], [Stefanov 22], etc.

Characterization of strong regularity for variational inequalities over polyhedral sets is considered in [Dontchev and Rockafellar 6].

Complementarity problems are considered, e.g., in [Facchinei and Kanzow 7], [Gabriel and Moré 9], [Mangasarian and Solodov 13], [Solodov and Svaiter 20], etc.

The VIP and the CP can be reformulated as equivalent unconstrained optimization problems by using the $D-$ gap function (for the VIP) and the implicit Lagrangian (for the CP). The implicit Lagrangian was proposed by Mangasarian and Solodov ([13]) for the CP, and Peng ([16]) extended the implicit Lagrangian approach to the VIP and showed that the implicit Lagrangian can be expressed as the difference of two regularized gap functions proposed by Fukushima ([8]). Yamashita, Taji and Fukushima ([24]) extended the results of Peng and studied properties of the *$D-$ gap function* $g_{\alpha\beta}(x) \stackrel{\text{def}}{=} f_\alpha(x) - f_\beta(x)$, where α and β are arbitrary parameters with $\beta > \alpha > 0$ and f_α is the following regularized gap function $f_\alpha(x) \stackrel{\text{def}}{=} \max_{y \in X} \langle F(x), x-y \rangle - \frac{\alpha}{2}\|y-x\|^2 = \langle F(x), x-y_\alpha(x) \rangle - \frac{\alpha}{2}\|y_\alpha(x)-x\|^2$, $y_\alpha(x) \stackrel{\text{def}}{=} \Pi_X\left(x - \frac{1}{\alpha}F(x)\right)$ and $\Pi_X(.)$ is the projection operator onto the constraint set X. The implicit Lagrangian is a particular case of the $D-$ gap function with $\beta = \frac{1}{\alpha}$.

In this paper, some new concepts of monotonicity and generalized monotonicity are considered (Section 2 and Section 4). They are used to obtain conditions ensuring the convergence of cutting plane methods that use cutting planes defined by the inequality constraint of the form

$$H_i = \{x : \langle F(x^i), x - x^i \rangle \leqslant 0\}, \tag{2}$$

where x^i denotes the current ith iterate. In Section 3, the cut property is considered. In Section 5, the cutting plane methods, developed originally for solving integer linear programming problems, are applied to variational inequality problems with generalized monotone and pseudomonotone mappings.

This paper is a continuation of author's research [22] on variational inequalities.

2. Some classes of monotonicity

2.1. Generalized monotone mappings

Recall that the mapping F is *monotone on* B if for all $x, y \in B$:

$$\langle F(x) - F(y), x - y \rangle \geqslant 0. \tag{3}$$

DEFINITION 1. Define the following monotonicity classes.

monotone$^+$: The mapping F is *monotone*$^+$ *on* B if it is monotone on B and for all $x, y \in B$,

$$\langle F(x) - F(y), x - y \rangle = 0 \quad \Rightarrow \quad F(x) = F(y). \tag{4}$$

monotone$^+_*$: The mapping F is *monotone*$^+_*$ *on* B if it is monotone on B and for all $x, y \in B$,

$$\langle F(x), x - y \rangle = 0 \quad \text{and} \quad \langle F(y), x - y \rangle = 0 \quad \Rightarrow \quad F(x) = F(y). \tag{5}$$

monotone$_*$: The mapping F is *monotone*$_*$ *on* B if it is monotone on B and

$$\begin{aligned} &x, y \in B,\ \langle F(x), x - y \rangle = 0 \quad \text{and} \quad \langle F(y), x - y \rangle = 0 \quad \Rightarrow \\ &\Rightarrow \quad \text{there exists} \quad k > 0 \quad \text{such that} \quad F(x) = kF(y). \end{aligned} \tag{6}$$

The relationship among these classes is:

F monotone$^+$ $\Rightarrow$ F monotone$^+_*$ $\Rightarrow$ F monotone$_*$ $\Rightarrow$ F monotone.

The following result holds true.

PROPOSITION 1. *Let f be convex and differentiable on the open convex set B. Then ∇f is monotone$^+$ on B.*

Proof. Let $a, b \in B$ and $\langle \nabla f(a) - \nabla f(b), b - a \rangle = 0$, that is, $\langle \nabla f(a), b - a \rangle = \langle \nabla f(b), b - a \rangle$. Using that f is convex, it follows that

$$\langle \nabla f(b), b - a \rangle \geqslant f(b) - f(a) \geqslant \langle \nabla f(a), b - a \rangle.$$

From the assumption we get

$$f(b) - f(a) = \langle \nabla f(a), b - a \rangle = \langle \nabla f(b), b - a \rangle. \tag{7}$$

Define the convex function g as follows

$$g(x) \stackrel{\text{def}}{=} f(x) - \langle \nabla f(a), x - a \rangle. \tag{8}$$

Using again the following property of convex functions $f(x) - f(a) \geqslant \langle \nabla f(a), x - a \rangle$, that is, $f(x) - \langle \nabla f(a), x - a \rangle \geqslant f(a)$, we get $g(x) \geqslant f(a)$ for all x. Since $g(a) = f(a)$ according to definition of $g(x)$, then

$$g(b) \stackrel{(8)}{=} f(b) - \langle \nabla f(a), b - a \rangle \stackrel{(7)}{=} f(a) \stackrel{(8)}{=} g(a) = \inf_{x \in B} g(x)$$

where for the last equality we have used that $g(x) \geqslant f(a) = g(a)$ for all $x \in B$. Since B is an open set and $g(b) = \inf_{x \in B} g(x)$, we have $0 = \nabla g(b) \overset{(8)}{=} \nabla f(b) - \nabla f(a)$. Hence, $\nabla f(a) = \nabla f(b)$, that is, ∇f is monotone$^+$ on B according to Definition 1. □

PROPOSITION 2. *The sum of monotone$^+$ mappings is a monotone$^+$ mapping.*

Proof. Let $F_i, i = 1, \ldots, m$, be monotone$^+$ mappings, that is,

$$\langle F_i(x) - F_i(y), x - y \rangle \geqslant 0, \quad i = 1, \ldots, m \quad \forall x, y \in B$$

and

$$\langle F_i(x) - F_i(y), x - y \rangle = 0 \quad \Rightarrow \quad F_i(x) = F_i(y), \quad i = 1, \ldots, m \quad \forall x, y \in B.$$

Denote $F = F_1 + \cdots + F_m$. Then

$$\begin{aligned} \langle F(x) - F(y), x - y \rangle &= \langle (F_1 + \cdots + F_m)(x) - (F_1 + \cdots + F_m)(y), x - y \rangle = \\ &= \langle F_1(x) - F_1(y), x - y \rangle + \cdots + \langle F_m(x) - F_m(y), x - y \rangle \\ &\geqslant 0 \end{aligned}$$

where the last inequality is satisfied according to the hypothesis of Proposition 2. Then F is monotone by definition.

From

$$\begin{aligned} 0 &= \langle F(x) - F(y), x - y \rangle = \\ &= \langle (F_1 + \cdots + F_m)(x) - (F_1 + \cdots + F_m)(y), x - y \rangle = \\ &= \langle F_1(x) - F_1(y), x - y \rangle + \cdots + \langle F_m(x) - F_m(y), x - y \rangle \end{aligned}$$

and

$$\langle F_i(x) - F_i(y), x - y \rangle \geqslant 0, \quad i = 1, \ldots, m \quad \forall x, y \in B$$

it follows that

$$\langle F_i(x) - F_i(y), x - y \rangle = 0, \quad i = 1, \ldots, m.$$

However, this implies $F_i(x) = F_i(y), \quad i = 1, \ldots, m$ by hypothesis of Proposition 2 (F_i are monotone$^+$). Therefore, $F_1(x) + \cdots + F_m(x) = F_1(y) + \cdots + F_m(y)$, that is, $F(x) = F(y)$. Hence, F is monotone$^+$ according to Definition 1. □

2.2. Generalized pseudomonotone mappings

DEFINITION 2. A mapping F is called *pseudomonotone on* B if for all $x, y \in B$,

$$\langle F(x), y - x \rangle \geqslant 0 \quad \Rightarrow \quad \langle F(y), y - x \rangle \geqslant 0. \tag{9}$$

The following definitions introduce stronger forms of pseudomonotonicity that correspond to the classes of monotone mappings introduced above.

DEFINITION 3. Define the pseudomonotonicity classes as follows.

pseudomonotone$^+$: The mapping F is *pseudomonotone$^+$ on* B if it is pseudomonotone on B and for all $x, y \in B$,

$$\langle F(x) - F(y), x - y \rangle = 0 \quad \Rightarrow \quad F(x) = F(y). \tag{10}$$

pseudomonotone$_*^+$: The mapping F is *pseudomonotone*$_*^+$ *on* B if it is pseudomonotone on B and for all $x, y \in B$,

$$\langle F(x), x-y \rangle = 0 \quad \text{and} \quad \langle F(y), x-y \rangle = 0 \quad \Rightarrow \quad F(x) = F(y). \tag{11}$$

pseudomonotone$_*$: The mapping F is *pseudomonotone*$_*$ *on* B if it is pseudomonotone on B and

$$\begin{aligned} &x, y \in B, \langle F(x), x-y \rangle = 0 \text{ and } \langle F(y), x-y \rangle = 0 \quad \Rightarrow \\ &\Rightarrow \quad \text{there exists} \quad k > 0 \quad \text{such that} \quad F(x) = kF(y). \end{aligned} \tag{12}$$

The relationship among these four pseudomonotone classes is:

F pseudomonotone$^+$ $\Rightarrow$ F pseudomonotone$_*^+$ $\Rightarrow$ F pseudomonotone$_*$ $\Rightarrow$ $\Rightarrow$ F pseudomonotone.

Recall that a function $f : S \to \mathbb{R}$, where S is a nonempty open convex set in $\mathbb{R}^n$, is called *pseudoconvex* if it is differentiable on S and for each $x, y \in S$ with $\langle \nabla f(x), y-x \rangle \geqslant 0$ we have $f(y) \geqslant f(x)$; or, equivalently, if $f(y) < f(x)$ implies $\langle \nabla f(x), y-x \rangle < 0$.

The following result holds true.

PROPOSITION 3. *Let* $f : [a,b] \to \mathbb{R}$ *be differentiable on the interval* $[a,b]$. *If* f *is pseudoconvex on* $[a,b]$ *then its derivative* f' *is pseudomonotone*$^+$ *on* $[a,b]$.

Proof. Since f is pseudoconvex on $[a,b]$, by definition f is differentiable on $[a,b]$ and $f(x) < f(y)$ implies $f'(y)(x-y) < 0$.

First of all, we have to prove that f' is pseudomonotone. If we assume the contrary, that $f'(x)(y-x) \geqslant 0$ implies $f'(y)(y-x) < 0$, then the inequality $f'(y)(y-x) < 0$, which is equivalent to $f'(y)(x-y) > 0$, would imply $f(x) > f(y)$ according to pseudoconvexity, which – again using pseudoconvexity – would imply $f'(x)(y-x) < 0$, a contradiction with the assumption.

Therefore $f'(x)(y-x) \geqslant 0$ implies $f'(y)(y-x) \geqslant 0$, that is, f' is pseudomonotone according to Definition 2.

Further, let $(f'(x) - f'(y))(x-y) = 0$. We want to prove that $f'(x) = f'(y)$.

If we assume the contrary, that $f'(x) \neq f'(y)$ we would have $x - y = 0$, that is, $x = y$, which contradicts the assumption $f'(x) \neq f'(y)$.

Therefore f' is pseudomonotone$^+$ on $[a,b]$ in accordance with Definition 3. □

PROPOSITION 4. *Let* f *be pseudoconvex and differentiable on the open convex set* B. *Then* ∇f *is pseudomonotone*$_*$ *on* B.

Proof. Let a and b be two arbitrary distinct points in B such that $\nabla f(a) \neq 0, \nabla f(b) \neq 0$ and

$$\langle \nabla f(a), b-a \rangle = \langle \nabla f(b), b-a \rangle = 0. \tag{13}$$

Since f is pseudoconvex by assumption, then $\langle \nabla f(a), b-a \rangle \geqslant 0$ implies $f(b) \geqslant f(a)$ and $\langle \nabla f(b), a-b \rangle \geqslant 0$ implies $f(a) \geqslant f(b)$. Using (13), we get $f(a) = f(b)$. Let $w \in \mathbb{R}^n$ with $\langle \nabla f(a), w \rangle < 0$. Since B is open, then there exists a positive number

p such that $a+pw \in B$ and $f(a+pw) < f(a) = f(b)$. Because f is pseudoconvex, $\langle \nabla f(b), a+pw-b \rangle < 0$ by definition. However, since $\langle \nabla f(b), a-b \rangle = 0$ by assumption, then $\langle \nabla f(b), w \rangle < 0$. Similarly, replacing a by b and b by a, we get $\langle \nabla f(b), w \rangle < 0$ implies $\langle \nabla f(a), w \rangle < 0$. Both implications give $\langle \nabla f(a), w \rangle < 0$ if and only if $\langle \nabla f(b), w \rangle < 0$. Therefore there exists a nonnegative number k such that $\nabla f(b) = k \nabla f(a)$. Thus, ∇f is pseudomonotone$_*$ on B according to Definition 3. □

3. The cut property

If a mapping is pseudomonotone on B, then the solution set C^* of VIP (F,C), that might be empty, is convex and for every $y \in C$,

$$C^* \subseteq \{x \in C : \langle F(y), y-x \rangle \geqslant 0\}.$$

We need this separation property in order to prove the convergence of a cutting plane method based on the information about the vector $F(y)$, where y is usually taken as the current iterate. However, in general this condition is not sufficient.

Goffin, Marcotte and Zhu ([10]) introduced the notions of monotone$^+$ and pseudomonotone$^+$ mappings, considered in Definition 1 and Definition 3, respectively.

These notions are sufficient conditions to ensure that the mappings possess the *cut property*:

$$x^* \in C^*, \quad \bar{x} \in C, \quad \bar{x} \neq x^*, \quad \langle F(\bar{x}), \bar{x} - x^* \rangle = 0 \quad \Rightarrow \quad \bar{x} \in C^*. \qquad (14)$$

The larger class of pseudomonotone$_*$ mappings, which corresponds to gradients of pseudoconvex functions, also satisfies the cut property.

Pseudomonotonicity$_*$ can be considered in some way as *the minimal condition* ensuring that the cut property holds independently of the subset C of the open set B.

Indeed, let $a, b \in \operatorname{int} B$ be such that $a \in C^*$ and $\langle F(a), b-a \rangle = \langle F(b), b-a \rangle = 0$ without any number k satisfying $F(b) = kF(a)$. Set

$$H = \{x \in B : \langle F(a), x-a \rangle = 0\}. \qquad (15)$$

By definition of H and the assumption, $a, b \in H, a \in C^*$ but $b \notin C^*$ since a vector h can be constructed such that $b+h \in \operatorname{int} B, \langle F(a), h \rangle = 0$ and $\langle F(b), h \rangle < 0$. Hence, the cut property is not satisfied.

PROPOSITION 5. *If F is pseudomonotone$_*$ on C then F satisfies the cut property.*

Proof. Let $x^* \in C^*$ and $\tilde{x}$ be such that $\langle F(\tilde{x}), x^* - \tilde{x} \rangle = 0$. Since x^* is a solution of VIP (F,C) and F is pseudomonotone$_*$ on C by assumption, then $F(\tilde{x}) = kF(x^*)$ for some $k > 0$ according to Definition 3, and for any $x \in C$ we have

$$\begin{aligned} \langle F(\tilde{x}), x-\tilde{x} \rangle &\equiv \langle F(\tilde{x}), x-x^* \rangle + \langle F(\tilde{x}), x^*-\tilde{x} \rangle \\ &= k\langle F(x^*), x-x^* \rangle + \langle F(\tilde{x}), x^*-\tilde{x} \rangle \\ &\equiv k\langle F(x^*), x-x^* \rangle \geqslant 0. \end{aligned}$$

This means that $\tilde{x}$ is a solution of VIP (F,C), that is, F satisfies the cut property. □

4. Generalized multivalued monotone and pseudomonotone mappings

DEFINITION 4. A multivalued mapping F is called *monotone on* B if for all $x, y \in B, t_x \in F(x), t_y \in F(y)$:

$$\langle t_x - t_y, y - x\rangle \geqslant 0. \quad (16)$$

DEFINITION 5. Define the multivalued monotonicity classes as follows.

monotone$^+$: A multivalued mapping F is *monotone*$^+$ *on* B if it is monotone on B and for all $x, y \in B, t_x \in F(x), t_y \in F(y)$:

$$\langle t_x - t_y, x - y\rangle = 0 \quad \Rightarrow \quad t_y \in F(x). \quad (17)$$

monotone$_*^+$: F is *monotone*$_*^+$ *on* B if it is monotone on B and for all $x, y \in B, t_x \in F(x), t_y \in F(y)$:

$$\langle t_x, x - y\rangle = 0 \quad \text{and} \quad \langle t_y, x - y\rangle = 0 \quad \Rightarrow \quad t_y \in F(x). \quad (18)$$

monotone$_*$: F is *monotone*$_*$ *on* B if it is monotone on B and there exists a positive constant k such that for $x, y \in B, t_x \in F(x), t_y \in F(y)$:

$$\langle t_x, x - y\rangle = 0 \quad \text{and} \quad \langle t_y, x - y\rangle = 0 \quad \Rightarrow \quad kt_y \in F(x). \quad (19)$$

The following relationship, similar to that for single-valued mappings, holds true:

F monotone$^+$ $\Rightarrow$ F monotone$_*^+$ $\Rightarrow$ F monotone$_*$ $\Rightarrow$ F monotone.

Multivalued monotone$^+$ mappings arise, for example, as subdifferentials of lower-semicontinuous convex functions.

DEFINITION 6. Define the multivalued generalized pseudomonotonicity classes as follows.

pseudomonotone: A multivalued mapping F is *pseudomonotone on* B if for all $x, y \in B, t_x \in F(x), t_y \in F(y)$ we have

$$\langle t_x, y - x\rangle \geqslant 0 \quad \Rightarrow \quad \langle t_y, y - x\rangle \geqslant 0. \quad (20)$$

pseudomonotone$^+$: The mapping F is *pseudomonotone*$^+$ *on* B if it is pseudomonotone on B and for all $x, y \in B, t_x \in F(x), t_y \in F(y)$ we have

$$\langle t_x - t_y, x - y\rangle = 0 \quad \Rightarrow \quad t_y \in F(x). \quad (21)$$

pseudomonotone$_*^+$: The mapping F is *pseudomonotone*$_*^+$ *on* B if it is pseudomonotone on B and for all $x, y \in B, t_x \in F(x), t_y \in F(y)$ we have

$$\langle t_x, x - y\rangle = 0 \quad \text{and} \quad \langle t_y, x - y\rangle = 0 \quad \Rightarrow \quad t_y \in F(x). \quad (22)$$

pseudomonotone$_*$: The mapping F is *pseudomonotone*$_*$ *on* B if it is pseudomonotone on B and there exists a positive constant k such that for $x, y \in B, t_x \in F(x), t_y \in F(y)$ we have

$$\langle t_x, x - y\rangle = 0 \quad \text{and} \quad \langle t_y, x - y\rangle = 0 \quad \Rightarrow \quad kt_y \in F(x). \quad (23)$$

5. Some applications to variational inequalities

Consider a variational inequality problem VIP (F, C) defined by a single-valued continuous mapping F and a convex compact set C in $\mathbb{R}^n$:

find a point $x^* \in C$ such that $\langle F(x^*), x - x^* \rangle \geqslant 0 \quad \forall x \in C.$

There exists at least one solution to VIP (F, C).

PROPOSITION 6. *Let F be pseudomonotone on C and $x^* \in C^*$. Then every solution $\tilde{x}$ of VIP (F, C) lies on the hyperplane*

$$H^* = \{y : \langle F(x^*), y - x^* \rangle = 0\}.$$

Proof. Let $\tilde{x} \in C^*$. Using the definition of C^*, we get

$$\langle F(x^*), \tilde{x} - x^* \rangle \geqslant 0 \quad \forall \tilde{x} \in C \tag{24}$$

and

$$\langle F(\tilde{x}), x^* - \tilde{x} \rangle \geqslant 0 \quad \forall x^* \in C.$$

Since F is pseudomonotone on C by assumption,

$$\langle F(\tilde{x}), x^* - \tilde{x} \rangle \geqslant 0 \quad \Rightarrow \langle F(x^*), x^* - \tilde{x} \rangle \geqslant 0. \tag{25}$$

Then (24) and (25) imply

$$\langle F(x^*), \tilde{x} - x^* \rangle = 0,$$

that is, $\tilde{x}$ lies on the hyperplane H^* defined above. □

PROPOSITION 7. *Let $F : C \to \mathbb{R}^n$ be pseudomonotone$_*^+$ mapping. Then F is constant on the solution set C^* of VIP (F, C). If F is pseudomonotone$_*$ and $x, y \in C^*$ then there exists a $k > 0$ such that $F(y) = kF(x)$.*

Proof. Let $x, y \in C^*$. From Proposition 6 it follows that $\langle F(y), y - x \rangle = 0$ and $\langle F(x), y - x \rangle = 0$. If F is a pseudomonotone$_*^+$ mapping then $F(y) = F(x) =$ const.; if F is pseudomonotone$_*$ then $F(y) = kF(x)$ with $k > 0$ according to Definition 3. □

If F is continuous and pseudomonotone on C, then solving VIP (F, C) is equivalent to finding a point $x^* \in C$ satisfying the system

$$\langle F(x), y - x \rangle \geqslant 0 \quad \forall y \in C \tag{26}$$

or, equivalently, x^* is a global minimizer of the gap function

$$d(x) = \sup_{y \in C} \langle F(y), x - y \rangle. \tag{27}$$

Function d is convex as a supremum of affine functions.

Denote by $D(x)$ the set of optimal solutions to (27). Since C is compact and F is continuous, then $D(x)$ is compact and nonempty.

The directional derivative of d at x with respect to v is

$$d'(x;v) = \max_{y \in D(x)} \langle F(y), v \rangle .$$

Hence, if F is pseudomonotone$_*^+$, then d is differentiable at any solution x^* of VIP (F,C) according to Proposition 7.

COROLLARY 1. *Let F be pseudomonotone$_*$ on C. If $\overline{x} \in C$ is not a solution of VIP (F,C), then its solution set C^* lies entirely within the open half-space $\{x : \langle F(\overline{x}), x - \overline{x} \rangle < 0\}$.*

Consider the variational inequality problem MVIP (F,C) involving multivalued mappings:

$$\text{find } x^* \in C \quad \text{such that} \quad \exists t^* \in F(x^*) : \langle t^*, x - x^* \rangle \geqslant 0 \ \ \forall x \in C. \tag{28}$$

The results obtained for VIP (F,C) can be extended to MVIP (F,C).

Let $\{x_k\}$ be a sequence of C such that $t_k \in F(x_k)$ and

$$\lim_{k \to \infty} \langle t_k, x_k - x^* \rangle = 0.$$

Then any limit point of the sequence $\{x_k\}$ is a solution of MVIP (F,C) provided that F is pseudomonotone$_*$ and closed on the convex set C.

Recall that a multivalued mapping $F : C \to \mathbb{R}^n$ with $C \subset \mathbb{R}^n$ is *closed* at x_0 if for any sequence $\{t_k\}$,

$$x_k \to x_0, t_k \in F(x_k), \lim_{k \to \infty} t_k = t_0 \quad \Rightarrow \quad t_0 \in F(x_0).$$

PROPOSITION 8. *Let C be compact and F be bounded, pseudomonotone$_*$ and closed on C. Let $\{x_k\}$ be a sequence of C and $\{t_k\}$ a sequence in $\mathbb{R}^n$ such that $t_k \in F(x_k)$ and $\lim_{k \to \infty} \langle t_k, x_k - x^* \rangle = 0$ for some solution x^* of MVIP (F,C). Then any limit point $\overline{x}$ of $\{x_k\}$ is a solution of MVIP (F,C).*

Proof. Let $\{x_{k_i}\}$ be a convergent subsequence of the sequence $\{x_k\}$ and $\overline{x}$ be its limit point. Let $t_{k_i} \in F(x_{k_i})$. Since F is bounded by assumption, there exists a subsequence $\{x_{k_j}\}$ of $\{x_{k_i}\}$ with $t_{k_j} \in F(x_{k_j})$ and $t_{k_j} \to t^*$ for some t^*. However, F is closed on C by assumption. Therefore

$$t^* \in F(\overline{x}) \quad \text{and} \quad \langle t^*, \overline{x} - x^* \rangle = 0. \tag{29}$$

Using pseudomonotonicity of F, which is implied by pseudomonotonicity$_*$, we conclude that $\langle -t^*, \overline{x} - x^* \rangle \geqslant 0$. However, x^* is a solution of MVIP (F,C) and therefore $\langle t^*, \overline{x} - x^* \rangle \geqslant 0$. Hence,

$$\langle t^*, \overline{x} - x^* \rangle = 0, t^* \in F(\overline{x}). \tag{30}$$

Since F is pseudomonotone$_*$, (29) and (30) imply $kt^* \in F(\overline{x})$ for some $k > 0$. Then for any $x \in C$ we have

$$k\langle t^*, x - \overline{x} \rangle \equiv k\langle t^*, x - x^* \rangle + k\langle t^*, x^* - \overline{x} \rangle = k\langle t^*, x - x^* \rangle \geqslant 0$$

with $kt^* \in F(\overline{x})$. Therefore $\overline{x}$ is a solution of MVIP (F,C). □

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