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SHARP TRIANGLE INEQUALITY AND ITS REVERSE IN BANACH SPACES

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Dedicated to Professor Yasuji Takahashi
on the occasion of his 60th birthday

(*communicated by S. Saitoh*)

Abstract. We shall present a sharp triangle inequality and its reverse inequality with n elements in a Banach space X, or equivalently we shall estimate the difference $\sum_{j=1}^{n}\|x_j\|-\|\sum_{j=1}^{n}x_j\|$ for given $x_1,x_2,\ldots,x_n$ in X, where equality attainedness will be characterized. Several applications will be given.

1. Introduction

The triangle inequality is undoubtedly one of the most fundamental inequalities in analysis. Several authors have been treating its generalizations and reverse inequalities, etc. (see e.g., [10, 2, 9]). In this paper we shall present an inequality which is sharper than the triangle inequality and its reverse inequality with n elements in a Banach space X, or equivalently, we shall estimate the difference $\sum_{j=1}^{n}\|x_j\|-\|\sum_{j=1}^{n}x_j\|$ for given $x_1,x_2,\ldots,x_n$ in X (cf. H. Hudzik [4], L. Maligranda [6] for the two element case). As a straightforward consequence it will be derived that for nonzero $x_1,x_2,\ldots,x_n$ in X, $\|\sum_{j=1}^{n}x_j\|=\sum_{j=1}^{n}\|x_j\|$ if and only if $\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\|=n$. Each of these conditions implies that the sharp triangle inequality and the reverse one attain equality *at the same time*; the converse is true unless all the norms of $x_1,x_2,\ldots,x_n$ are the same. These inequalities will be powerful especially for treating geometric properties of Banach spaces; indeed we shall discuss the uniform non ℓ_1^n-ness of X as such an example. The reverse triangle inequality with two elements immediately yields an inequality by J. L. Massera and J. J. Schaeffer ([7]; see also Dunkl-Williams [3], [9, p. 516]).

The rest of this paper will be devoted to characterizing equality attainedness for *each* of our inequalities under the condition that X is strictly convex. We shall also obtain that in a strictly convex space X these inequalities attain equality at the same time if and only if all the norms of $x_1,x_2,\ldots,x_n$ are the same or $\frac{x_1}{\|x_1\|}=\frac{x_2}{\|x_2\|}=\cdots=\frac{x_n}{\|x_n\|}$.

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2. Sharp triangle inequality and its reverse

In a Banach space X the triangle inequality (with n elements) is sharpened as follows, where we shall obtain its reverse inequality as well.

THEOREM 1. *For all nonzero elements* $x_1, x_2, \ldots, x_n$ *in a Banach space* X

$$\left\|\sum_{j=1}^{n} x_j\right\| + \left(n - \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\|\right) \min_{1\leqslant j\leqslant n} \|x_j\| \leqslant \sum_{j=1}^{n} \|x_j\| \tag{1}$$

$$\leqslant \left\|\sum_{j=1}^{n} x_j\right\| + \left(n - \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\|\right) \max_{1\leqslant j\leqslant n} \|x_j\|. \tag{2}$$

Proof. If $\|x_1\| = \|x_2\| = \cdots = \|x_n\|$, both inequalities (1) and (2) hold with equality. Therefore we may assume this is not the case. Let us see the first inequality. Let $\|x_{j_0}\| = \min\{\|x_j\| : 1 \leqslant j \leqslant n\}$ and $J_0 = \{j : \|x_j\| = \|x_{j_0}\|,\ 1 \leqslant j \leqslant n\}$. Then for any nonzero $x_1, \cdots, x_n \in X$ we have

$$\begin{aligned}
\left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\| &= \left\|\sum_{j\in J_0} \frac{x_j}{\|x_j\|} + \sum_{j\in J_0^c} \frac{x_j}{\|x_j\|}\right\| \\
&= \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_{j_0}\|} - \sum_{j\in J_0^c} \frac{x_j}{\|x_{j_0}\|} + \sum_{j\in J_0^c} \frac{x_j}{\|x_j\|}\right\| \\
&= \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_{j_0}\|} - \sum_{j\in J_0^c} \left(\frac{1}{\|x_{j_0}\|} - \frac{1}{\|x_j\|}\right) x_j\right\| \\
&\geqslant \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_{j_0}\|}\right\| - \sum_{j\in J_0^c} \left(\frac{1}{\|x_{j_0}\|} - \frac{1}{\|x_j\|}\right) \|x_j\| \qquad (3) \\
&= \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_{j_0}\|}\right\| - \sum_{j=1}^{n} \left(\frac{1}{\|x_{j_0}\|} - \frac{1}{\|x_j\|}\right) \|x_j\| \\
&= \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_{j_0}\|}\right\| - \left(\sum_{j=1}^{n} \frac{\|x_j\|}{\|x_{j_0}\|} - n\right),
\end{aligned}$$

from which it follows that

$$\sum_{j=1}^{n} \frac{\|x_j\|}{\|x_{j_0}\|} \geqslant \frac{\|\sum_{j=1}^{n} x_j\|}{\|x_{j_0}\|} + \left(n - \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\|\right).$$

Hence we obtain

$$\sum_{j=1}^{n}\|x_j\| \geqslant \|\sum_{j=1}^{n}x_j\| + \left(n - \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right)\|x_{j_0}\|,$$

or the inequality (1).

For the second inequality let $\|x_{j_1}\| = \max\{\|x_j\| : 1 \leqslant j \leqslant n\}$ and $J_1 = \{j : \|x_j\| = \|x_{j_1}\|,\ 1 \leqslant j \leqslant n\}$. Then we have

$$\begin{aligned}
\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| &= \left\|\sum_{j\in J_1}\frac{x_j}{\|x_j\|} + \sum_{j\in J_1^c}\frac{x_j}{\|x_j\|}\right\| \\
&= \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_{j_1}\|} - \sum_{j\in J_1^c}\frac{x_j}{\|x_{j_1}\|} + \sum_{j\in J_1^c}\frac{x_j}{\|x_j\|}\right\| \\
&= \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_{j_1}\|} + \sum_{j\in J_1^c}\left(\frac{1}{\|x_j\|} - \frac{1}{\|x_{j_1}\|}\right)x_j\right\| \\
&\leqslant \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_{j_1}\|}\right\| + \sum_{j\in J_1^c}\left(\frac{1}{\|x_j\|} - \frac{1}{\|x_{j_1}\|}\right)\|x_j\| \qquad (4) \\
&= \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_{j_1}\|}\right\| + \sum_{j=1}^{n}\left(\frac{1}{\|x_j\|} - \frac{1}{\|x_{j_1}\|}\right)\|x_j\| \\
&= \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_{j_1}\|}\right\| + n - \left(\sum_{j=1}^{n}\frac{\|x_j\|}{\|x_{j_1}\|}\right),
\end{aligned}$$

and hence

$$\sum_{j=1}^{n}\|x_j\| \leqslant \left\|\sum_{j=1}^{n}x_j\right\| + \left(n - \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right)\|x_{j_1}\|.$$

Thus we have the conclusion.

Theorem 1 is reformulated as follows, which estimates the difference of the two terms in the triangle inequality.

COROLLARY 1. *For all nonzero elements $x_1, x_2, \ldots, x_n$ in a Banach space X*

$$\begin{aligned}
\left(n - \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right)\min_{1\leqslant j\leqslant n}\|x_j\| &\leqslant \sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n}x_j\right\| \\
&\leqslant \left(n - \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right)\max_{1\leqslant j\leqslant n}\|x_j\|.
\end{aligned} \qquad (5)$$

In particular for all nonzero elements x, y *in* X

$$\begin{aligned}\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)\min\{\|x\|,\|y\|\} &\leqslant \|x\|+\|y\|-\|x+y\| \\ &\leqslant \left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)\max\{\|x\|,\|y\|\}\end{aligned} \tag{6}$$

(cf. Hudzik [4], Maligranda [6]).

From (6) an inequality by Massera and Schaeffer is derived ([7]; see also Dunkl-Williams [3], [9, p. 516]):

COROLLARY 2. (J. L. Massera and J. J. Schaeffer [7]) *For all nonzero elements* x, y *in a Banach space* X

$$\|x-y\| \;\geqslant\; \frac{1}{2}\max\{\|x\|,\|y\|\}\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|. \tag{7}$$

Indeed, assume $\|x\| \leqslant \|y\|$. Then by the second inequality of (6)

$$\|x-y\|+2\|y\|-\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\|y\| \geqslant \|x\|+\|y\|.$$

Hence

$$\|x-y\|-\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\|y\| \geqslant \|x\|-\|y\| \geqslant -\|x-y\|.$$

Therefore we have

$$2\|x-y\| \geqslant \left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\|y\|,$$

as desired.

By Theorem 1 we immediately have the following.

COROLLARY 3. *Let* $x_1, x_2, \ldots, x_n$ *be arbitrary nonzero elements in a Banach space* X. *Then the following are equivalent.*

(*i*) $\left\|\sum_{j=1}^n x_j\right\| = \sum_{j=1}^n \|x_j\|$.

(*ii*) $\left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\| = n$.

REMARK 1. The above condition (ii), or equivalently (i), assures equality attainedness in the both inequalities (1) and (2) *at the same time*. The converse is true unless all the norms of x_j are the same. We shall treat the equality attainedness for *each* inequality of (1) and (2) in the next section under the assumption that X is strictly convex.

At the end of this section we mention a direct application of Theorem 1 to a geometric property of Banach spaces. Recall that a Banach space X is called *uniformly non-*ℓ_1^n ([5]; see also [1]) provided there exists ε $(0<\varepsilon<1)$ such that for any $x_1, \cdots, x_n$ in the *unit sphere* of X there exists $\theta = (\theta_j)$ of n signs ± 1 for which

$$\left\|\sum_{j=1}^n \theta_j x_j\right\| \leqslant n(1-\varepsilon). \tag{8}$$

When $n = 2$, X is called *uniformly non-square* ([5]; cf. [1]). By virtue of Theorem 1 we immediately have the following fact.

COROLLARY 4. *For a Banach space X the following are equivalent.*

(i) X is uniformly non-ℓ_1^n.

(ii) There exists ε $(0 < \varepsilon < 1)$ such that for any $x_1, \cdots, x_n$ in the unit ball of X there exists $\theta = (\theta_j)$ of n signs ± 1 for which (8) holds true.

Indeed, assume that there exists ε $(0 < \varepsilon < 1)$ such that for any $x_1, \cdots, x_n$ in the *unit sphere* of X there exist $\theta = (\theta_j)$ of n signs ± 1 for which (8) is valid. Take $x_1, \cdots, x_n$ from the *unit ball* of X. If $\|x_{j_0}\| := \min\{\|x_1,\| \ldots, \|x_n\|\} \leqslant 1/2$, we have

$$\left\| \sum_{j=1}^{n} \theta_j x_j \right\| \leqslant \sum_{j \neq j_0} \|x_j\| + \|x_{j_0}\| \leqslant (n-1) + \frac{1}{2} \leqslant n(1 - \frac{1}{2n}).$$

Let $\|x_{j_0}\| \geqslant \frac{1}{2}$. According to our assumption there exists n signs (θ_j) for which (8) is valid for $x_1/\|x_1\|, \cdots, x_n/\|x_n\|$. Therefore by the first inequality of Theorem 1

$$\begin{aligned} \left\| \sum_{j=1}^{n} \theta_j x_j \right\| &\leqslant \sum_{j=1}^{n} \|x_j\| - \left(n - \left\| \sum_{j=1}^{n} \theta_j \frac{x_j}{\|x_j\|} \right\| \right) \|x_{j_0}\| \\ &\leqslant n - \frac{n\varepsilon}{2} = n\left(1 - \frac{\varepsilon}{2}\right). \end{aligned}$$

Consequently by letting $\varepsilon_0 = \min\{\frac{\varepsilon}{2}, \frac{1}{2n}\}$ we have the conclusion.

3. Equality attainedness in a strictly convex Banach space

In what follows we shall consider equality attainedness for each of our inequalities in a strictly convex Banach space. The following lemma is quite powerful in our subsequent discussions.

LEMMA 1. *Let X be a strictly convex Banach space. Let $x_1, x_2, \ldots, x_n$ be nonzero elements in X. Then the following are equivalent.*

(i) $\left\| \sum_{j=1}^{n} \alpha_j x_j \right\| = \sum_{j=1}^{n} \alpha_j \|x_j\|$ with any positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$.

(ii) $\left\| \sum_{j=1}^{n} \alpha_j x_j \right\| = \sum_{j=1}^{n} \alpha_j \|x_j\|$ with some positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$.

(iii) $\frac{x_1}{\|x_1\|} = \frac{x_2}{\|x_2\|} = \cdots = \frac{x_n}{\|x_n\|}$.

Proof. We see the implications $(ii) \Rightarrow (iii) \Rightarrow (i)$. Assume that $\left\|\sum_{j=1}^n \alpha_j x_j\right\| = \sum_{j=1}^n \alpha_j \|x_j\|$ with some positive $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then for any $1 < k \leqslant n$

$$\begin{aligned}\|\alpha_1 x_1 + \alpha_k x_k\| &\geqslant \left\|\sum_{j=1}^n \alpha_j x_j\right\| - \left\|\sum_{j\neq 1,k} \alpha_j x_j\right\| \\ &\geqslant \left\|\sum_{j=1}^n \alpha_j x_j\right\| - \sum_{j\neq 1,k} \alpha_j \|x_j\| \\ &= \sum_{j=1}^n \alpha_j \|x_j\| - \sum_{j\neq 1,k} \alpha_j \|x_j\| \\ &= \alpha_1\|x_1\| + \alpha_k\|x_k\|,\end{aligned}$$

whence $\|\alpha_1 x_1 + \alpha_k x_k\| = \alpha_1\|x_1\| + \alpha_k\|x_k\|$. As X is strictly convex, we have $\frac{x_1}{\|x_1\|} = \frac{x_k}{\|x_k\|}$. Next assume (iii) and let $\frac{x_1}{\|x_1\|} = \frac{x_2}{\|x_2\|} = \cdots = \frac{x_n}{\|x_n\|} = y$. Then for any positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ we have

$$\left\|\sum_{j=1}^n \alpha_j x_j\right\| = \left\|\sum_{j=1}^n \alpha_j \|x_j\| y\right\| = \left(\sum_{j=1}^n \alpha_j \|x_j\|\right)\|y\| = \sum_{j=1}^n \alpha_j \|x_j\|,$$

or (i). The rest implication $(i) \Rightarrow (ii)$ is trivial.

REMARK 2. Let $x_1, x_2, \ldots, x_n$ be nonzero elements in a general Banach space X. If $\frac{x_1}{\|x_1\|} = \frac{x_2}{\|x_2\|} = \cdots = \frac{x_n}{\|x_n\|}$, then we have

$$\frac{\sum_{n=1}^n x_j}{\|\sum_{n=1}^n x_j\|} = \frac{x_1}{\|x_1\|}. \tag{9}$$

Indeed, since $\sum_{n=1}^n x_j = \sum_{n=1}^n \|x_j\| \frac{x_1}{\|x_1\|}$, we have $\|\sum_{n=1}^n x_j\| = \sum_{n=1}^n \|x_j\|$ and hence (9). Therefore, if X is strictly convex and if $\|\sum_{j=1}^n \alpha_j x_j\| = \sum_{j=1}^n \alpha_j\|x_j\|$ with some positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, we have

$$\frac{\sum_{n=1}^n \alpha_j x_j}{\|\sum_{n=1}^n \alpha_j x_j\|} = \frac{x_1}{\|x_1\|}. \tag{10}$$

THEOREM 2. *Let X be a strictly convex Banach space and $x_1, x_2, \ldots, x_n$ nonzero elements in X. Let $\|x_{j_0}\| = \min\{\|x_j\| : 1 \leqslant j \leqslant n\}$ and $\|x_{j_1}\| = \max\{\|x_j\| : 1 \leqslant j \leqslant n\}$. Let $J_0 = \{j : \|x_j\| = \|x_{j_0}\|,\ 1 \leqslant j \leqslant n\}$. Then*

$$\left\|\sum_{j=1}^n x_j\right\| + \left(n - \left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\|\right) \min_{1\leqslant j\leqslant n} \|x_j\| = \sum_{j=1}^n \|x_j\| \tag{11}$$

if and only if either

(a) $\|x_1\| = \|x_2\| = \cdots = \|x_n\|$

or

(b) $\frac{x_j}{\|x_j\|} = \frac{x_{j_1}}{\|x_{j_1}\|}$ *for all* $j \in J_0^c$ *and* $\sum_{j=1}^n \frac{x_j}{\|x_j\|} = \|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\| \frac{x_{j_1}}{\|x_{j_1}\|}$.

Proof. We first note that, according to (3) in the proof of Theorem 1, the identity (11) is equivalent to

$$\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_{j_0}\|}-\sum_{j\in J_0^c}\left(\frac{1}{\|x_{j_0}\|}-\frac{1}{\|x_j\|}\right)x_j\right\|=\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_{j_0}\|}\right\|-\sum_{j\in J_0^c}\left(\frac{1}{\|x_{j_0}\|}-\frac{1}{\|x_j\|}\right)\|x_j\|. \quad (12)$$

Let (11) hold true and assume that the assertion (a) is not the case. Then $J_0^c\neq\emptyset$. Put

$$y=\sum_{j=1}^{n}\frac{x_j}{\|x_{j_0}\|}\quad\text{and}\quad z=\sum_{j\in J_0^c}\left(\frac{1}{\|x_{j_0}\|}-\frac{1}{\|x_j\|}\right)x_j. \quad (13)$$

Then

$$y-z=\sum_{j=1}^{n}\frac{x_j}{\|x_j\|} \quad (14)$$

(recall the proof of Theorem 1). By (12) we have

$$\begin{aligned}\|y\|&=\|(y-z)+z\|\\&\leqslant\|y-z\|+\|z\|\\&\leqslant\|y-z\|+\sum_{j\in J_0^c}\left(\frac{1}{\|x_{j_0}\|}-\frac{1}{\|x_j\|}\right)\|x_j\|=\|y\|,\end{aligned}$$

from which it follows that

$$\|(y-z)+z\|=\|y-z\|+\|z\| \quad (15)$$

and

$$\|z\|=\left\|\sum_{j\in J_0^c}\left(\frac{1}{\|x_{j_0}\|}-\frac{1}{\|x_j\|}\right)x_j\right\|=\sum_{j\in J_0^c}\left(\frac{1}{\|x_{j_0}\|}-\frac{1}{\|x_j\|}\right)\|x_j\|. \quad (16)$$

One should note here that (15) and (16) conversely imply (12) or the identity (11). Now, by (16) and Lemma 1 we have $\frac{x_j}{\|x_j\|}=\frac{x_{j_1}}{\|x_{j_1}\|}$ for all $j\in J_0^c$, or the first assertion of (b). If $y-z=\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}=0$, the latter assertion of (b) is trivial. So we assume that this is not the case. Then by (15) and Lemma 1

$$\frac{y-z}{\|y-z\|}=\frac{z}{\|z\|} \quad (17)$$

(note that $z\neq 0$ by (12)). Put $\alpha_j=\frac{1}{\|x_{j_0}\|}-\frac{1}{\|x_j\|}>0$ $(j\in J_0^c)$. Then by Remark 2

$$\frac{z}{\|z\|}=\frac{\sum_{j\in J_0^c}\alpha_j x_j}{\|\sum_{j\in J_0^c}\alpha_j x_j\|}=\frac{x_{j_1}}{\|x_{j_1}\|}. \quad (18)$$

Combining (17) and (18) we obtain

$$\frac{\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}}{\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\|}=\frac{x_{j_1}}{\|x_{j_1}\|},$$

as desired.

Conversely, if $\|x_{j_0}\| = \|x_{j_1}\|$, the identity (11) clearly holds true as mentioned before. Therefore we assume the case (b), that is,

$$\frac{x_j}{\|x_j\|} = \frac{x_{j_1}}{\|x_{j_1}\|} \text{ for all } j \in J_0^c \tag{19}$$

and

$$\sum_{j=1}^{n} \frac{x_j}{\|x_j\|} = \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \frac{x_{j_1}}{\|x_{j_1}\|}. \tag{20}$$

We may merely show (15) and (16). Note first that $z \neq 0$. Indeed

$$z = \sum_{j \in J_0^c} \left(\frac{1}{\|x_{j_0}\|} - \frac{1}{\|x_j\|} \right) x_j = \sum_{j \in J_0^c} \left(\frac{1}{\|x_{j_0}\|} - \frac{1}{\|x_j\|} \right) \|x_j\| \frac{x_{j_1}}{\|x_{j_1}\|} \neq 0.$$

By (19) and Lemma 1 we have (16). Hence (18) is valid by Remark 2, which, combined with (20), yields that

$$y - z = \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} = \|y - z\| \frac{z}{\|z\|}.$$

Consequently we obtain $\|(y-z) + z\| = \|y - z\| + \|z\|$, or (15). Thus we have the identity (11), which completes the proof.

THEOREM 3. *Let X be a strictly convex Banach space and $x_1, x_2, \ldots, x_n$ nonzero elements in X. Let $\|x_{j_0}\| = \min\{\|x_j\| : 1 \leqslant j \leqslant n\}$ and $\|x_{j_1}\| = \max\{\|x_j\| : 1 \leqslant j \leqslant n\}$. Let $J_1 = \{j : \|x_j\| = \|x_{j_1}\|, 1 \leqslant j \leqslant n\}$. Then*

$$\sum_{j=1}^{n} \|x_j\| = \left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \max_{1 \leqslant j \leqslant n} \|x_j\| \tag{21}$$

if and only if either

(a) $\|x_1\| = \|x_2\| = \cdots = \|x_n\|$

or

(b) $\frac{x_j}{\|x_j\|} = \frac{x_{j_0}}{\|x_{j_0}\|}$ *for all* $j \in J_1^c$ *and* $\sum_{j=1}^{n} x_j = \| \sum_{j=1}^{n} x_j \| \frac{x_{j_0}}{\|x_{j_0}\|}$.

Proof. According to (4) in the proof of Theorem 1 the identity (21) is equivalent to

$$\left\| \sum_{j=1}^{n} \frac{x_j}{\|x_{j_1}\|} + \sum_{j \in J_1^c} \left(\frac{1}{\|x_j\|} - \frac{1}{\|x_{j_1}\|} \right) x_j \right\| = \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_{j_1}\|} \right\| + \sum_{j \in J_1^c} \left(\frac{1}{\|x_j\|} - \frac{1}{\|x_{j_1}\|} \right) \|x_j\|. \tag{22}$$

Assume that (21) is true and (a) is not the case. Then $J_1^c \neq \emptyset$. Let first $\sum_{j=1}^{n} x_j = 0$. Then by (22)

$$\left\| \sum_{j \in J_1^c} \left(\frac{1}{\|x_j\|} - \frac{1}{\|x_{j_1}\|} \right) x_j \right\| = \sum_{j \in J_1^c} \left(\frac{1}{\|x_j\|} - \frac{1}{\|x_{j_1}\|} \right) \|x_j\|. \tag{23}$$

Therefore by Lemma 1 we have $\frac{x_j}{\|x_j\|} = \frac{x_{j_0}}{\|x_{j_0}\|}$ for all $j \in J_1^c$. The latter assertion of (b) is trivial. Let $\sum_{j=1}^n x_j \neq 0$. Then by (22) and Lemma 1

$$\frac{\sum_{n=1}^n x_j}{\|\sum_{n=1}^n x_j\|} = \frac{x_j}{\|x_j\|} = \frac{x_{j_0}}{\|x_{j_0}\|} \quad \text{for all } j \in J_1^c. \tag{24}$$

Thus we obtain (b).

Conversely, in the case (a) the identity (21) is trivial. We assume (b). Let first $\sum_{j=1}^n x_j = 0$. By the first assertion of (b) we have

$$\begin{aligned}\left\|\sum_{j\in J_1^c}\left(\frac{1}{\|x_j\|}-\frac{1}{\|x_{j_1}\|}\right)x_j\right\| &= \left\|\sum_{j\in J_1^c}\left(\frac{1}{\|x_j\|}-\frac{1}{\|x_{j_1}\|}\right)\|x_j\|\frac{x_{j_0}}{\|x_{j_0}\|}\right\| \\ &= \sum_{j\in J_1^c}\left(\frac{1}{\|x_j\|}-\frac{1}{\|x_{j_1}\|}\right)\|x_j\|,\end{aligned}$$

or (23), which is none other than the identity (22). In case of $\sum_{j=1}^n x_j \neq 0$, we have (24) and hence (22) by Lemma 1. This completes the proof.

Finally we mention the case where both of the inequalities (1) and (2) attain equality at the same time under the condition that X is strictly convex.

THEOREM 4. *Let X be a strictly convex Banach space and $x_1, x_2, \dots, x_n$ nonzero elements in X. Then the equalities*

$$\begin{aligned}\left\|\sum_{j=1}^n x_j\right\| + \left(n - \left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\|\right)\min_{1\leqslant j\leqslant n}\|x_j\| &= \sum_{j=1}^n \|x_j\| \\ &= \left\|\sum_{j=1}^n x_j\right\| + \left(n - \left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\|\right)\max_{1\leqslant j\leqslant n}\|x_j\|\end{aligned} \tag{25}$$

hold if and only if

(a) $\|x_1\| = \|x_2\| = \cdots = \|x_n\|$

or

(b) $\dfrac{x_1}{\|x_1\|} = \dfrac{x_2}{\|x_2\|} = \cdots = \dfrac{x_n}{\|x_n\|}$.

Proof. Assume that the identities (25) hold and (a) is not valid. Then

$$\left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\| = n \tag{26}$$

and hence we have (b) by Lemma 1. Conversely, in case of (a), (25) is trivial. Assume (b). Then by Lemma 1 we have (26), which implies (25).

REMARK 3. Let X be a strictly convex Banach space. Take nonzero x and y in X with $\|x\| = \|y\|$ which are not colinear, that is, $x \neq \alpha y$ for any $\alpha > 0$. Then $\|x+y\| < \|x\| + \|y\|$, whereas in the inequality (1) with two elements equality holds with these x and y. This asserts that the equality condition of the sharp triangle inequality is different from that of the triangle inequality.

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