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THE DUNKL–WILLIAMS INEQUALITY WITH n ELEMENTS IN NORMED LINEAR SPACES

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Abstract. In this paper we establish a generalization of the Dunkl-Williams inequality for finitely many elements in a normed linear space. As a consequence, we get some recently obtained results on the generalized triangle inequality and its reverse inequality. The case of equality for elements of a strictly convex normed linear space is also considered.

1. Introduction

The well-known Dunkl-Williams inequality [1] states that for any two nonzero elements x, y in a normed linear space

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leqslant \frac{4\|x-y\|}{\|x\|+\|y\|}. \tag{1}$$

Over the years, various refinements of the Dunkl-Williams inequality have been given, for example, see [4] or [3]. The refinement established by L. Maligranda in [3], that is,

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leqslant \frac{\|x-y\| + \big|\|x\|-\|y\|\big|}{\max\{\|x\|, \|y\|\}} \tag{2}$$

is the sharpest one. P. R. Mercer [5] has recently obtained the reverse inequality of (2) by showing that

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \geqslant \frac{\|x-y\| - \big|\|x\|-\|y\|\big|}{\min\{\|x\|, \|y\|\}} \tag{3}$$

for any pair of nonzero elements x and y in a normed linear space.

In this paper we generalize the inequalities (2) and (3) for an arbitrary number of finitely many nonzero elements of a normed linear space. We also characterize the case of equality for elements of a strictly convex normed linear space.

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2. The results

THEOREM 2.1. *Let X be a normed linear space and $x_1, \dots, x_n$ nonzero elements of X. Then we have*

$$\left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \leqslant \min_{i\in\{1,\dots,n\}} \left\{ \frac{1}{\|x_i\|} \left(\left\| \sum_{j=1}^{n} x_j \right\| + \sum_{j=1}^{n} \big| \|x_j\| - \|x_i\| \big| \right) \right\} \tag{4}$$

and

$$\left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \geqslant \max_{i\in\{1,\dots,n\}} \left\{ \frac{1}{\|x_i\|} \left(\left\| \sum_{j=1}^{n} x_j \right\| - \sum_{j=1}^{n} \big| \|x_j\| - \|x_i\| \big| \right) \right\}. \tag{5}$$

Proof. Let us fix $i \in \{1,\dots,n\}$. Then we compute

$$\begin{aligned}
\left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| &= \left\| \frac{x_i}{\|x_i\|} + \sum_{j\in\{1,\dots,n\}\setminus\{i\}} \frac{x_j}{\|x_j\|} \right\| \\
&= \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_i\|} - \sum_{j\in\{1,\dots,n\}\setminus\{i\}} \frac{x_j}{\|x_i\|} + \sum_{j\in\{1,\dots,n\}\setminus\{i\}} \frac{x_j}{\|x_j\|} \right\| \\
&= \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_i\|} - \sum_{j\in\{1,\dots,n\}\setminus\{i\}} \left(\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \right) x_j \right\| \\
&= \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_i\|} - \sum_{j=1}^{n} \left(\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \right) x_j \right\| \\
&\leqslant \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_i\|} \right\| + \sum_{j=1}^{n} \left| \frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \right| \|x_j\| \\
&= \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_i\|} \right\| + \sum_{j=1}^{n} \left| \frac{\|x_j\|}{\|x_i\|} - 1 \right| \\
&= \frac{1}{\|x_i\|} \left\| \sum_{j=1}^{n} x_j \right\| + \frac{1}{\|x_i\|} \sum_{j=1}^{n} \big| \|x_j\| - \|x_i\| \big| \\
&= \frac{1}{\|x_i\|} \left(\left\| \sum_{j=1}^{n} x_j \right\| + \sum_{j=1}^{n} \big| \|x_j\| - \|x_i\| \big| \right).
\end{aligned}$$

From this it follows that

$$\left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \leqslant \min_{i\in\{1,\dots,n\}} \left\{ \frac{1}{\|x_i\|} \left(\left\| \sum_{j=1}^{n} x_j \right\| + \sum_{j=1}^{n} \big| \|x_j\| - \|x_i\| \big| \right) \right\},$$

which is the inequality (4). We proceed in a similar way to obtain the inequality (5). As in the first part of the proof, for a fixed $i \in \{1,\dots,n\}$ we have

$$\left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| = \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_i\|} - \sum_{j=1}^{n} \left(\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \right) x_j \right\|.$$

Thus, we get

$$\begin{aligned}\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| &\geqslant \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_i\|}\right\| - \left\|\sum_{j=1}^{n}\left(\frac{1}{\|x_i\|}-\frac{1}{\|x_j\|}\right)x_j\right\| \\ &\geqslant \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_i\|}\right\| - \sum_{j=1}^{n}\left|\frac{1}{\|x_i\|}-\frac{1}{\|x_j\|}\right|\|x_j\| \\ &= \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_i\|}\right\| - \sum_{j=1}^{n}\left|\frac{\|x_j\|}{\|x_i\|}-1\right| \\ &= \frac{1}{\|x_i\|}\left\|\sum_{j=1}^{n}x_j\right\| - \frac{1}{\|x_i\|}\sum_{j=1}^{n}\left|\|x_j\|-\|x_i\|\right| \\ &= \frac{1}{\|x_i\|}\left(\left\|\sum_{j=1}^{n}x_j\right\| - \sum_{j=1}^{n}\left|\|x_j\|-\|x_i\|\right|\right).\end{aligned}$$

Therefore,

$$\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| \geqslant \max_{i\in\{1,\dots,n\}}\left\{\frac{1}{\|x_i\|}\left(\left\|\sum_{j=1}^{n}x_j\right\| - \sum_{j=1}^{n}\left|\|x_j\|-\|x_i\|\right|\right)\right\},$$

which completes the proof. □

REMARK 2.2. Note that in the case when $n=2$ by putting $x_1 := x$ and $x_2 := -y$ in Theorem 2.1 we get the inequalities (2) and (3) obtained by L. Maligranda and P. R. Mercer, respectively.

In [2] M. Kato, K. S. Saito and T. Tamura sharpened the triangle inequality in normed linear spaces. More precisely, in Theorem 1 of [2] they estimated the difference $\sum_{j=1}^{n}\|x_j\| - \|\sum_{j=1}^{n}x_j\|$, where $x_1,\dots,x_n$ are nonzero elements of a normed linear space. Let us say here that, in order to obtain our result, we followed their model of proof, but made some modifications to it. Now, their result can also be derived from our Theorem 2.1, as shown in the following corollary.

COROLLARY 2.3. *Let X be a normed linear space and $x_1,\dots,x_n$ nonzero elements of X. Then we have*

$$\sum_{j=1}^{n}\|x_j\| \leqslant \left\|\sum_{j=1}^{n}x_j\right\| + \left(n - \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right)\max_{j\in\{1,\dots,n\}}\|x_j\| \tag{6}$$

and

$$\sum_{j=1}^{n}\|x_j\| \geqslant \left\|\sum_{j=1}^{n}x_j\right\| + \left(n - \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right)\min_{j\in\{1,\dots,n\}}\|x_j\|. \tag{7}$$

Proof. Let $\|x_i\| = \max\{\|x_j\| : j = 1, \dots, n\}$. Using (4) from Theorem 2.1 we obtain

$$\Big\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \Big\| \leqslant \frac{1}{\|x_i\|} \Big(\Big\| \sum_{j=1}^{n} x_j \Big\| + \sum_{j=1}^{n} \big| \|x_j\| - \|x_i\| \big| \Big) = \frac{1}{\|x_i\|} \Big(\Big\| \sum_{j=1}^{n} x_j \Big\| + n\|x_i\| - \sum_{j=1}^{n} \|x_j\| \Big).$$

Hence,

$$\|x_i\| \Big\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \Big\| \leqslant \Big\| \sum_{j=1}^{n} x_j \Big\| + n\|x_i\| - \sum_{j=1}^{n} \|x_j\|.$$

Then it follows that

$$\sum_{j=1}^{n} \|x_j\| \leqslant \Big\| \sum_{j=1}^{n} x_j \Big\| + \Big(n - \Big\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \Big\| \Big) \|x_i\| = \Big\| \sum_{j=1}^{n} x_j \Big\| + \Big(n - \Big\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \Big\| \Big) \max_{j \in \{1,\dots,n\}} \|x_j\|,$$

and (6) is proved.

To prove the inequality (7) let us denote $\|x_k\| = \min\{\|x_j\| : j = 1, \dots, n\}$. By (5) it holds

$$\Big\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \Big\| \geqslant \frac{1}{\|x_k\|} \Big(\Big\| \sum_{j=1}^{n} x_j \Big\| - \sum_{j=1}^{n} \big| \|x_j\| - \|x_k\| \big| \Big) = \frac{1}{\|x_k\|} \Big(\Big\| \sum_{j=1}^{n} x_j \Big\| + n\|x_k\| - \sum_{j=1}^{n} \|x_j\| \Big).$$

Hence,

$$\|x_k\| \Big\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \Big\| \geqslant \Big\| \sum_{j=1}^{n} x_j \Big\| + n\|x_k\| - \sum_{j=1}^{n} \|x_j\|,$$

from which we get

$$\sum_{j=1}^{n} \|x_j\| \geqslant \Big\| \sum_{j=1}^{n} x_j \Big\| + \Big(n - \Big\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \Big\| \Big) \|x_k\| = \Big\| \sum_{j=1}^{n} x_j \Big\| + \Big(n - \Big\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \Big\| \Big) \min_{j \in \{1,\dots,n\}} \|x_j\|.$$

This completes the proof. □

REMARK 2.4. It is evident from the proof of Corollary 2.3 that, in the case $\|x_i\| = \max\{\|x_j\| : j = 1, \dots, n\}$, the inequality (6) is equivalent to

$$\Big\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \Big\| \leqslant \frac{1}{\|x_i\|} \Big(\Big\| \sum_{j=1}^{n} x_j \Big\| + \sum_{j=1}^{n} \big| \|x_j\| - \|x_i\| \big| \Big) \tag{8}$$

while, in the case $\|x_i\| = \min\{\|x_j\| : j = 1,\ldots,n\}$, the inequality (7) is equivalent to

$$\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| \geqslant \frac{1}{\|x_i\|}\left(\left\|\sum_{j=1}^{n}x_j\right\| - \sum_{j=1}^{n}\big|\|x_j\| - \|x_i\|\big|\right). \tag{9}$$

Note that for $n = 2$ the best estimations in the inequalities (4) and (5) are achieved when $i \in \{1,2\}$ is chosen such that $\|x_i\| = \max\{\|x_j\| : j = 1,2\}$ and $\|x_i\| = \min\{\|x_j\| : j = 1,2\}$, respectively. Thus, the inequalities (4) and (5) are precisely the inequalities (6) and (7) obtained by M. Kato, K. S. Saito and T. Tamura in [2].

However, in the case when $n > 2$, we can find nonzero vectors $x_1,\ldots,x_n \in X$ such that the inequality (4) is the sharpest for some $i \in \{1,\ldots,n\}$ for which $\|x_i\| \neq \max\{\|x_j\| : j = 1,\ldots,n\}$. Also, the best estimation in the inequality (5) can be obtained for some $i \in \{1,\ldots,n\}$ such that $\|x_i\| \neq \min\{\|x_j\| : j = 1,\ldots,n\}$. This shows that our inequalities (4) and (5) give better estimations than the inequalities (6) and (7) obtained by M. Kato, K. S. Saito and T. Tamura. We ilustrate this in the following example.

EXAMPLE 2.5. (a) Let X be an inner-product space whose dimension is greater than one. Let x_1, x_2 be two orthogonal unit elements of X. Let us put $x_3 := -x_1 - x_2$. Then, obviously $\|x_3\| = \sqrt{2}$. One can easily verify that the right hand side in the inequality (8) is equal to $\sqrt{2} - 1$ when $i \in \{1,2\}$ and $\sqrt{2}(\sqrt{2} - 1)$ when $i = 3$. Hence, the sharpest estimation in (4) is obtained for $i \in \{1,2\}$, but $\|x_1\| = \|x_2\| = 1 \neq \sqrt{2} = \max\{\|x_1\|, \|x_2\|, \|x_3\|\}$.

(b) Let X be a normed linear space. Let us take $x_1, x_3 \in X$ such that $\|x_1\| = 1$ and $\|x_3\| = 0.8$. Let us put $x_2 := -x_1$. An easy computation shows that the right hand side in the inequality (9) is equal to 0.6 when $i \in \{1,2\}$ and 0.5 when $i = 3$. Therefore, the sharpest estimation in (5) is obtained for $i \in \{1,2\}$, but $\|x_1\| = \|x_2\| = 1 \neq 0.8 = \min\{\|x_1\|, \|x_2\|, \|x_3\|\}$.

The case of equality in (6) and (7) was studied by M. Kato, K. S. Saito and T. Tamura [2] for elements of a strictly convex Banach space X. Their consideration is based on the characterization of strictly convex normed linear spaces given in [2, Lemma 1] (see also [2, Remark 2]). We note here that Lemma 1 of [2] and its Remark 2 are also valid without assumption that X is complete.

In what follows we shall consider the case of equality in (4) and (5) when X is a strictly convex normed linear space. For our proofs we use Lemma 1 and Remark 2 of [2].

THEOREM 2.6. *Let X be a strictly convex normed linear space and $x_1,\ldots,x_n$ nonzero elements of X such that $\|x_1\| = \cdots = \|x_n\|$ does not hold. Then there is $i \in \{1,\ldots,n\}$ such that the following two statements are mutually equivalent.*

(i) $$\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| = \frac{1}{\|x_i\|}\left(\left\|\sum_{j=1}^{n}x_j\right\| + \sum_{j=1}^{n}\big|\|x_j\| - \|x_i\|\big|\right).$$

(ii) There exists $v \in X$ *satisfying* $\operatorname{sgn}(\|x_i\| - \|x_j\|)\frac{x_j}{\|x_j\|} = v$ *for all* $j \in \{1, \dots, n\}$ *such that* $\|x_j\| \neq \|x_i\|$ *and* $\sum_{j=1}^{n} x_j = \left\| \sum_{j=1}^{n} x_j \right\| v.$

Proof. Denote $J = \{j \in \{1, \dots, n\} : \|x_j\| \neq \|x_i\|\}$. Let us put $x_j' = \operatorname{sgn}(\|x_i\| - \|x_j\|)x_j,\ j \in J$. Since

$$\begin{aligned}\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} &= \left|\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right| \operatorname{sgn}\left(\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right) \\ &= -\left|\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right| \operatorname{sgn}(\|x_i\| - \|x_j\|),\end{aligned}$$

we can write

$$\sum_{j=1}^{n}\left(\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right)x_j = -\sum_{j\in J}\left|\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right| x_j'. \tag{10}$$

$(i) \Rightarrow (ii)$ From the proof of the inequality (4) we can see that (i) holds precisely when

$$\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_i\|} - \sum_{j=1}^{n}\left(\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right)x_j\right\| = \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_i\|}\right\| + \sum_{j=1}^{n}\left|\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right| \|x_j\|. \tag{11}$$

Using (10) we can write (11) as

$$\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_i\|} + \sum_{j\in J}\left|\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right| x_j'\right\| = \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_i\|}\right\| + \sum_{j\in J}\left|\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right| \|x_j'\|. \tag{12}$$

First, let us consider the case when $\sum_{j=1}^{n} x_j = 0$. By (12) we have

$$\left\|\sum_{j\in J}\left|\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right| x_j'\right\| = \sum_{j\in J}\left|\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right| \|x_j'\|.$$

Then by Lemma 1 of [2] there is $v \in X$ such that $\frac{x_j'}{\|x_j'\|} = v$ for all $j \in J$, that is, $\operatorname{sgn}(\|x_i\| - \|x_j\|)\frac{x_j}{\|x_j\|} = v$ for all $j \in \{1, \dots, n\}$ such that $\|x_j\| \neq \|x_i\|$. Obviously, $\sum_{j=1}^{n} x_j = \|\sum_{j=1}^{n} x_j\| v$.

It remains to consider the case $\sum_{j=1}^{n} x_j \neq 0$. Using (12) and Lemma 1 of [2] we deduce that there exists $v \in X$ such that

$$\frac{\sum_{j=1}^{n} x_j}{\left\|\sum_{j=1}^{n} x_j\right\|} = \frac{x_j'}{\|x_j'\|} = v, \quad j \in J. \tag{13}$$

Hence, (ii) is proved.

$(ii) \Rightarrow (i)$ Suppose first that $\sum_{j=1}^{n} x_j = 0$. By the assumption we have $\frac{x_j'}{\|x_j'\|} = v$, so

$$\left\| \sum_{j\in J} \left| \frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \right| x_j' \right\| = \left\| \sum_{j\in J} \left| \frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \right| \|x_j'\| v \right\|$$
$$= \sum_{j\in J} \left| \frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \right| \|x_j'\|$$

which is the equality (12). This proves (i).

Let us suppose now that $\sum_{j=1}^{n} x_j \neq 0$. By the assumption we have (13). Then, according to Lemma 1 of [2], the equality (12) also holds. This implies (i) and the theorem is proved. □

COROLLARY 2.7. *Let X be a strictly convex normed linear space and $x_1, \ldots, x_n$ nonzero elements of X. Then the following two statements are mutually equivalent.*

(i) $\left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| = \min_{i\in\{1,\ldots,n\}} \left\{ \frac{1}{\|x_i\|} \left(\left\| \sum_{j=1}^{n} x_j \right\| + \sum_{j=1}^{n} \big| \|x_j\| - \|x_i\| \big| \right) \right\}$.

(ii) $\|x_1\| = \cdots = \|x_n\|$ *or there exist* $i \in \{1, \ldots, n\}$ *and* $v \in X$ *satisfying* $\operatorname{sgn}(\|x_i\| - \|x_j\|) \frac{x_j}{\|x_j\|} = v$ *for all* $j \in \{1, \ldots, n\}$ *such that* $\|x_j\| \neq \|x_i\|$ *and* $\sum_{j=1}^{n} x_j = \left\| \sum_{j=1}^{n} x_j \right\| v$.

Proof. If $\|x_1\| = \cdots = \|x_n\|$ we are done. So, suppose that this is not the case. Then our corollary follows immediately from Theorem 2.6 and the inequality (4) of Theorem 2.1. □

THEOREM 2.8. *Let X be a strictly convex normed linear space and $x_1, \ldots, x_n$ nonzero elements of X such that $\|x_1\| = \cdots = \|x_n\|$ does not hold. Then there is $i \in \{1, \ldots, n\}$ such that the following two statements are mutually equivalent.*

(i) $\left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| = \frac{1}{\|x_i\|} \left(\left\| \sum_{j=1}^{n} x_j \right\| - \sum_{j=1}^{n} \big| \|x_j\| - \|x_i\| \big| \right)$.

(ii) *There exists* $v \in X$ *satisfying* $\operatorname{sgn}(\|x_j\| - \|x_i\|) \frac{x_j}{\|x_j\|} = v$ *for all* $j \in \{1, \ldots, n\}$ *such that* $\|x_j\| \neq \|x_i\|$ *and* $\sum_{j=1}^{n} \frac{x_j}{\|x_j\|} = \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| v$.

Proof. Let us denote $J = \{j \in \{1, \ldots, n\} : \|x_j\| \neq \|x_i\|\}$. Put

$$y = \sum_{j=1}^{n} \frac{x_j}{\|x_i\|} \quad \text{and} \quad z = \sum_{j=1}^{n} \left(\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} \right) x_j.$$

Let us denote $x_j' = \operatorname{sgn}(\|x_j\| - \|x_i\|)x_j,\ j \in J.$ Since

$$\begin{aligned}\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|} &= \left|\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right| \operatorname{sgn}\left(\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right)\\ &= \left|\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right| \operatorname{sgn}(\|x_j\| - \|x_i\|),\end{aligned}$$

we can also write

$$z = \sum_{j\in J} \left|\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right| x_j'.$$

$(i) \Rightarrow (ii)$ Passing the proof of the inequality (5) we deduce that (i) holds if and only if the following two conditions are satisfied:

$$\left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\| = \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_i\|}\right\| - \left\|\sum_{j=1}^{n} \left(\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right)x_j\right\| \tag{14}$$

and

$$\left\|\sum_{j=1}^{n} \left(\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right)x_j\right\| = \sum_{j=1}^{n} \left|\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right| \|x_j\|. \tag{15}$$

Note that (15) can be written as

$$\left\|\sum_{j\in J} \left|\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right| x_j'\right\| = \sum_{j\in J} \left|\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right| \|x_j'\|. \tag{16}$$

Therefore, by Lemma 1 of [2] there exists $v \in X$ such that

$$\frac{x_j'}{\|x_j'\|} = v, \quad j \in J, \tag{17}$$

that is, $\operatorname{sgn}(\|x_j\| - \|x_i\|)\frac{x_j}{\|x_j\|} = v$ for all $j \in \{1,\dots,n\}$ such that $\|x_j\| \neq \|x_i\|$.

Note that (14) is equivalent to $\|y - z\| = \|y\| - \|z\|$, i.e.,

$$\|(y - z) + z\| = \|y - z\| + \|z\|. \tag{18}$$

From (16) it follows that $z \neq 0$, as $J \neq \emptyset$ and $x_j' \neq 0$ for all $j \in J$. If $y - z = \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} = 0$ we are done, since $\sum_{j=1}^{n} \frac{x_j}{\|x_j\|} = \|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\| v$. So, assume that $y - z \neq 0$. Then by (18) and Lemma 1 of [2] we get

$$\frac{y - z}{\|y - z\|} = \frac{z}{\|z\|}. \tag{19}$$

Using (16), (17) and Remark 2 of [2] we have

$$\frac{z}{\|z\|} = \frac{\sum_{j\in J} \left|\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right| x_j'}{\left\|\sum_{j\in J} \left|\frac{1}{\|x_i\|} - \frac{1}{\|x_j\|}\right| x_j'\right\|} = v. \tag{20}$$

From this and (19) we conclude that

$$\frac{\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}}{\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|}=v,$$

as desired.

$(ii) \Rightarrow (i)$ Note that $(i) \Leftrightarrow ((14)$ and $(15))$, $(14) \Leftrightarrow (18)$ and $(15) \Leftrightarrow (16)$. Thus, to prove (i) we must show that (16) and (18) hold. Since $\frac{x_j'}{\|x_j'\|}=v$ for all $j \in J$, (16) follows from Lemma 1 of [2]. Now, by using Remark 2 of [2] we get $\frac{z}{\|z\|}=v$. By the assumption we have $y-z=\|y-z\|v$. Thus, $y=z+\|y-z\|v=\|z\|v+\|y-z\|v$ from which it follows that $\|y\|=\|z\|+\|y-z\|$, which is the equality (18). This completes the proof. □

COROLLARY 2.9. *Let X be a strictly convex normed linear space and $x_1,\dots,x_n$ nonzero elements of X. Then the following two statements are mutually equivalent.*

(i) $\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| = \max_{i\in\{1,\dots,n\}}\left\{\frac{1}{\|x_i\|}\left(\left\|\sum_{j=1}^{n}x_j\right\|-\sum_{j=1}^{n}\big|\|x_j\|-\|x_i\|\big|\right)\right\}.$

(ii) $\|x_1\|=\cdots=\|x_n\|$ *or there exist* $i\in\{1,\dots,n\}$ *and* $v\in X$ *satisfying* $\operatorname{sgn}(\|x_j\|-\|x_i\|)\frac{x_j}{\|x_j\|}=v$ *for all* $j\in\{1,\dots,n\}$ *such that* $\|x_j\|\neq\|x_i\|$ *and* $\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}=\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|v.$

Proof. If $\|x_1\|=\cdots=\|x_n\|$ we are done. If this is not the case, our corollary follows from Theorem 2.8 and the inequality (5) of Theorem 2.1. □

Finally, as immediate consequences of Theorem 2.6 and Theorem 2.8, we state the results obtained in [2] which determine when the equalities in (6) and (7) hold.

COROLLARY 2.10. *Let X be a strictly convex normed linear space and $x_1,\dots,x_n$ nonzero elements of X. Then the following two statements are mutually equivalent.*

(i) $\sum_{j=1}^{n}\|x_j\|=\left\|\sum_{j=1}^{n}x_j\right\|+\left(n-\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right)\max_{j\in\{1,\dots,n\}}\|x_j\|.$

(ii) $\|x_1\|=\cdots=\|x_n\|$ *or there exists* $v\in X$ *satisfying* $\frac{x_j}{\|x_j\|}=v$ *for all* $j\in\{1,\dots,n\}$ *such that* $\|x_j\|\neq\max_{k\in\{1,\dots,n\}}\|x_k\|$ *and* $\sum_{j=1}^{n}x_j=\left\|\sum_{j=1}^{n}x_j\right\|v.$

Proof. If $\|x_1\|=\cdots=\|x_n\|$ we are done. So, suppose that this is not the case. Let us choose $i\in\{1,\dots,n\}$ such that $\|x_i\|=\max\{\|x_j\| : j=1,\dots,n\}$. Then (i) is equivalent to (i) of Theorem 2.6, while (ii) is equivalent to (ii) of Theorem 2.6. Therefore, corollary follows from Theorem 2.6. □

COROLLARY 2.11. *Let X be a strictly convex normed linear space and $x_1,\dots,x_n$ nonzero elements of X. Then the following two statements are mutually equivalent.*

(i) $\sum_{j=1}^{n}\|x_j\| = \left\|\sum_{j=1}^{n}x_j\right\| + \left(n - \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right)\min_{j\in\{1,\dots,n\}}\|x_j\|.$

(ii) *$\|x_1\| = \cdots = \|x_n\|$ or there exists $v\in X$ satisfying $\frac{x_j}{\|x_j\|} = v$ for all $j\in\{1,\dots,n\}$ such that $\|x_j\| \neq \min_{k\in\{1,\dots,n\}}\|x_k\|$ and $\sum_{j=1}^{n}\frac{x_j}{\|x_j\|} = \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| v$.*

Proof. If $\|x_1\| = \cdots = \|x_n\|$ we are done. So, suppose that this is not the case. Let us choose $i\in\{1,\dots,n\}$ such that $\|x_i\| = \min\{\|x_j\| : j = 1,\dots,n\}$. Then (i) is equivalent to (i) of Theorem 2.8, while (ii) is equivalent to (ii) of Theorem 2.8. Therefore, corollary follows from Theorem 2.8. □

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