Mathematical
Inequalities
& Applications
Volume 10, Number 3 (2007), 565–573

A FURTHER GENERALIZATION OF ACZÉL'S INEQUALITY AND POPOVICIU'S INEQUALITY

SHANHE WU

(communicated by Z. Pales)

Abstract. In this paper, a new generalization of Aczél's inequality is established, which contains as special case a sharpened version of Popoviciu's inequality:

$$\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{\frac{1}{p}}\left(b_1^q-\sum_{i=2}^n b_i^q\right)^{\frac{1}{q}}\leqslant a_1b_1-\left(\sum_{i=2}^n a_ib_i\right)-\frac{a_1b_1}{\max\{p,q\}}\left(\sum_{i=2}^n\left(\frac{a_i^p}{a_1^p}-\frac{b_i^q}{b_1^q}\right)\right)^2,$$

where p, q, a_i, b_i $(i=1,2,\cdots,n)$ are positive numbers, $p^{-1}+q^{-1}=1$, $a_1^p-\sum_{i=2}^n a_i^p>0$ and $b_1^q-\sum_{i=2}^n b_i^q>0$. Moreover, an integral inequality of Aczél-Popoviciu type is given.

1. Introduction

In 1956, Aczél [1] proved the following result:

$$\left(a_1^2-\sum_{i=2}^n a_i^2\right)\left(b_1^2-\sum_{i=2}^n b_i^2\right)\leqslant\left(a_1b_1-\sum_{i=2}^n a_ib_i\right)^2, \tag{1}$$

where a_i, b_i $(i=1,2,\cdots,n)$ are positive numbers such that $a_1^2-\sum_{i=2}^n a_i^2>0$ or $b_1^2-\sum_{i=2}^n b_i^2>0$. This inequality is called Aczél's inequality.

It is well-known that Aczél's inequality has important applications in the theory of functional equations in non-Euclidean geometry. In recent years, this inequality has attracted the interest of many mathematicians and has motivated a large number of research papers involving different proofs, various generalizations, improvements and applications (see [2–11] and references therein).

We state here a brief history on improvements of Aczél's inequality.

In [12], an exponential extension of Aczél's inequality was first presented by Popoviciu, i.e.

Mathematics subject classification (2000): 26D15, 26D20.

Key words and phrases: Aczél's inequality, Popoviciu's inequality, generalized Hölder's inequality, Bernoulli's inequality, generalization, sharpen.

THEOREM A. *Let* $p>0,\ q>0,\ p^{-1}+q^{-1}=1$, *and let* $a_i,\ b_i\ \ (i=1,2,\cdots,n)$ *be positive numbers such that* $a_1^p-\sum_{i=2}^{n}a_i^p>0$ *and* $b_1^q-\sum_{i=2}^{n}b_i^q>0$. *Then*

$$\left(a_1^p-\sum_{i=2}^{n}a_i^p\right)^{\frac{1}{p}}\left(b_1^q-\sum_{i=2}^{n}b_i^q\right)^{\frac{1}{q}}\leqslant a_1b_1-\sum_{i=2}^{n}a_ib_i\ . \tag{2}$$

Wu and Debnath [13] generalized inequality (2) to the following inequality:

THEOREM B. *Let* $p>0,\ q>0$, *and let* $a_i,\ b_i\ \ (i=1,2,\cdots,n)$ *be positive numbers such that* $a_1^p-\sum_{i=2}^{n}a_i^p>0$ *and* $b_1^q-\sum_{i=2}^{n}b_i^q>0$. *Then*

$$\left(a_1^p-\sum_{i=2}^{n}a_i^p\right)^{\frac{1}{p}}\left(b_1^q-\sum_{i=2}^{n}b_i^q\right)^{\frac{1}{q}}\leqslant n^{1-\min\{p^{-1}+q^{-1},\,1\}}\ a_1b_1-\sum_{i=2}^{n}a_ib_i. \tag{3}$$

In a recent paper [14], Wu and Debnath established a further extension of inequality (3), as follows

THEOREM C. *Let* $p,\ q,\ a_i,\ b_i\ \ (i=1,2,\cdots,n)$ *be positive numbers, and let* k $(1\leqslant k<n)$ *be a positive integer such that* $\sum_{i=1}^{k}a_i^p-\sum_{i=k+1}^{n}a_i^p>0,\ \ \sum_{i=1}^{k}b_i^q-\sum_{i=k+1}^{n}b_i^q>0$ *and the sequences* $(a_1,a_2,\cdots,a_k)$ *and* $(b_1,b_2,\cdots,b_k)$ *are monotonic in the same direction. Then*

$$\begin{aligned}&\left(\sum_{i=1}^{k}a_i^p-\sum_{i=k+1}^{n}a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{k}b_i^q-\sum_{i=k+1}^{n}b_i^q\right)^{\frac{1}{q}}\\&\qquad\leqslant(n-k+1)^{1-\min\{p^{-1}+q^{-1},1\}}k^{p^{-1}+q^{-1}-\min\{(p+q)^{-1},1\}}\sum_{i=1}^{k}a_ib_i-\sum_{i=k+1}^{n}a_ib_i.\end{aligned} \tag{4}$$

In this paper, by using the generalized Hölder inequality we give a new generalization of Aczél's inequality, our result is a unified improvement of Aczél's inequality and Popoviciu's inequality. Moreover, a new Aczél-Popoviciu type integral inequality is established.

2. Lemmas

To prove our main results, the following lemmas are necessary.

LEMMA 1. (Generalized Hölder inequality [13][15]) *Let* $p_j>0,\ \ a_{ij}>0\ \ (i=1,2,\cdots,n,\ j=1,2,\cdots,m)$. *Then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m}a_{ij}^{p_j}\ \leqslant\ n^{\max\{1-p_1-p_2-\cdots-p_m,\,0\}}\prod_{j=1}^{m}\left(\sum_{i=1}^{n}a_{ij}\right)^{p_j}, \tag{5}$$

with equality holding if and only if $a_{1j} = a_{2j} = \cdots = a_{nj}$ $(j = 1,2,\cdots,m)$ *for* $p_1 + p_2 + \cdots + p_m < 1$, *or* $\frac{a_{i1}}{\sum_{i=1}^{n} a_{i1}} = \frac{a_{i2}}{\sum_{i=1}^{n} a_{i2}} = \cdots = \frac{a_{im}}{\sum_{i=1}^{n} a_{im}}$ $(i = 1,2,\cdots,n)$ *for* $p_1 + p_2 + \cdots + p_m = 1$.

LEMMA 2. *Let* $0 \leqslant x < 1$, $\alpha > 0$. *Then*

$$(1-x)^{\alpha} \leqslant 1 - \min\{\alpha, 1\}x, \tag{6}$$

with equality holding if and only if $\alpha = 1$ *or* $x = 0$.

Proof. When $\alpha = 1$ or $x = 0$, (6) is the identity. When $0 < \alpha < 1$, it follows from Bernoulli's inequality [16, p. 34] that $(1-x)^{\alpha} < 1 - \alpha x$ for all $x \in (0,1)$. When $\alpha > 1$, we have $(1-x)^{\alpha} < 1 - x$ for all $x \in (0,1)$, since the function $f(x) = a^x$ $(0 < a < 1)$ is strictly decreasing on $(-\infty, +\infty)$. Inequality (6) is proved.

LEMMA 3. *Let* $0 < x < 1$, $0 < y < 1$, $p > 0$, $q > 0$. *Then*

$$xy + (1-x^p)^{\frac{1}{p}}(1-y^q)^{\frac{1}{q}} \leqslant 2^{\max\{1-p^{-1}-q^{-1},0\}}\left(1-(x^p-y^q)^2\right)^{\min\{p^{-1},q^{-1}\}}, \tag{7}$$

with equality holding if and only if $x^p = y^q = \frac{1}{2}$ *for* $p^{-1} + q^{-1} < 1$, *or* $x^p = y^q$ *for* $p^{-1} + q^{-1} = 1$.

Proof. When $p > q > 0$, it implies that $\frac{1}{p} > 0$, $\frac{1}{q} - \frac{1}{p} > 0$. Using Lemma 1 gives

$$\begin{aligned}
&xy + (1-x^p)^{\frac{1}{p}}(1-y^q)^{\frac{1}{q}} \\
&= (y^q)^{\frac{1}{p}}(x^p)^{\frac{1}{p}}(y^q)^{\frac{1}{q}-\frac{1}{p}} + (1-x^p)^{\frac{1}{p}}(1-y^q)^{\frac{1}{p}}(1-y^q)^{\frac{1}{q}-\frac{1}{p}} \\
&\leqslant 2^{\max\{1-p^{-1}-q^{-1},0\}}(y^q + (1-x^p))^{\frac{1}{p}}(x^p + (1-y^q))^{\frac{1}{p}}(y^q + (1-y^q))^{\frac{1}{q}-\frac{1}{p}} \\
&= 2^{\max\{1-p^{-1}-q^{-1},0\}}\left(1-(x^p-y^q)^2\right)^{\frac{1}{p}}.
\end{aligned}$$

When $0 < p < q$, it implies that $\frac{1}{p} > 0$, $\frac{1}{p} - \frac{1}{q} > 0$. Using Lemma 1, we have

$$\begin{aligned}
&xy + (1-x^p)^{\frac{1}{p}}(1-y^q)^{\frac{1}{q}} \\
&= (x^p)^{\frac{1}{q}}(y^q)^{\frac{1}{q}}(x^p)^{\frac{1}{p}-\frac{1}{q}} + (1-y^q)^{\frac{1}{q}}(1-x^p)^{\frac{1}{q}}(1-x^p)^{\frac{1}{p}-\frac{1}{q}} \\
&\leqslant 2^{\max\{1-p^{-1}-q^{-1},0\}}(x^p + (1-y^q))^{\frac{1}{q}}(y^q + (1-x^p))^{\frac{1}{q}}(x^p + (1-x^p))^{\frac{1}{p}-\frac{1}{q}} \\
&= 2^{\max\{1-p^{-1}-q^{-1},0\}}\left(1-(x^p-y^q)^2\right)^{\frac{1}{q}}.
\end{aligned}$$

When $p = q$, $p > 0$, $q > 0$. It follows from Lemma 1 that

$$\begin{aligned}
xy + (1-x^p)^{\frac{1}{p}}(1-y^q)^{\frac{1}{q}} &= xy + (1-x^p)^{\frac{1}{p}}(1-y^p)^{\frac{1}{p}} \\
&\leqslant 2^{\max\{1-\frac{2}{p},0\}}(y^p + (1-x^p))^{\frac{1}{p}}(x^p + (1-y^p))^{\frac{1}{p}} \\
&= 2^{\max\{1-\frac{2}{p},0\}}\left(1-(x^p-y^p)^2\right)^{\frac{1}{p}} \\
&= 2^{\max\{1-p^{-1}-q^{-1},0\}}\left(1-(x^p-y^q)^2\right)^{\frac{1}{p}}.
\end{aligned}$$

In addition, the condition of equality for inequality (7) can easily be deduced from Lemma 1. This completes the proof of Lemma 3.

A combination of Lemma 2 and Lemma 3 leads to the following

LEMMA 4. *Let* $0<x<1,\ 0<y<1,\ p>0,\ q>0$. *Then*

$$xy+(1-x^p)^{\frac{1}{p}}(1-y^q)^{\frac{1}{q}} \leqslant 2^{\max\{1-p^{-1}-q^{-1},0\}}\left(1-\min\{p^{-1},q^{-1},1\}(x^p-y^q)^2\right), \quad (8)$$

with equality holding if and only if $x^p=y^q=\frac{1}{2}$ *for* $p^{-1}+q^{-1}<1$, *or* $x^p=y^q$ *for* $p^{-1}+q^{-1}=1$.

3. Generalizations of Aczél's inequality and Popoviciu's inequality

THEOREM 1. *Let* $p>0,\ q>0,\ a_i>0,\ b_i>0\ (i=1,2,\cdots,n)$, *and let* k $(1\leqslant k<n)$ *be a positive integer such that* $\sum_{i=1}^{k}a_i^p-\sum_{i=k+1}^{n}a_i^p>0$ *and* $\sum_{i=1}^{k}b_i^q-\sum_{i=k+1}^{n}b_i^q>0$. *Then*

$$\begin{aligned}&\left(\sum_{i=1}^{k}a_i^p-\sum_{i=k+1}^{n}a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{k}b_i^q-\sum_{i=k+1}^{n}b_i^q\right)^{\frac{1}{q}}\\ &\quad\leqslant 2^{\max\{1-p^{-1}-q^{-1},0\}}\left(\sum_{i=1}^{k}a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{k}b_i^q\right)^{\frac{1}{q}}-\left(\sum_{i=k+1}^{n}a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=k+1}^{n}b_i^q\right)^{\frac{1}{q}}\\ &\qquad-\frac{2^{\max\{1-p^{-1}-q^{-1},0\}}}{\max\{p,q,1\}}\left(\sum_{i=1}^{k}a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{k}b_i^q\right)^{\frac{1}{q}}\left(\frac{\sum_{i=k+1}^{n}a_i^p}{\sum_{i=1}^{k}a_i^p}-\frac{\sum_{i=k+1}^{n}b_i^q}{\sum_{i=1}^{k}b_i^q}\right)^2.\end{aligned} \quad (9)$$

Equality holds if and only if

$$\left(\sum_{i=1}^{k}a_i^p\right)\Big/\left(\sum_{i=k+1}^{n}a_i^p\right)=\left(\sum_{i=1}^{k}b_i^q\right)\Big/\left(\sum_{i=k+1}^{n}b_i^q\right)=2 \quad \text{for} \quad p^{-1}+q^{-1}<1,$$

or

$$\left(\sum_{i=1}^{k}a_i^p\right)\Big/\left(\sum_{i=k+1}^{n}a_i^p\right)=\left(\sum_{i=1}^{k}b_i^q\right)\Big/\left(\sum_{i=k+1}^{n}b_i^q\right) \quad \text{for} \quad p^{-1}+q^{-1}=1.$$

Proof. By the hypotheses in Theorem 1, we find

$$0<\left(\sum_{i=1}^{k}a_i^p-\sum_{i=k+1}^{n}a_i^p\right)^{\frac{1}{p}}\Big/\left(\sum_{i=1}^{k}a_i^p\right)^{\frac{1}{p}}<1,$$

$$0<\left(\sum_{i=1}^{k}b_i^q-\sum_{i=k+1}^{n}b_i^q\right)^{\frac{1}{q}}\Big/\left(\sum_{i=1}^{k}b_i^q\right)^{\frac{1}{q}}<1.$$

Using Lemma 4 with a substitution:

$$\begin{aligned} x &\longmapsto \left(\sum_{i=1}^{k} a_i^p - \sum_{i=k+1}^{n} a_i^p\right)^{\frac{1}{p}} \Bigg/ \left(\sum_{i=1}^{k} a_i^p\right)^{\frac{1}{p}}, \\ y &\longmapsto \left(\sum_{i=1}^{k} b_i^q - \sum_{i=k+1}^{n} b_i^q\right)^{\frac{1}{q}} \Bigg/ \left(\sum_{i=1}^{k} b_i^q\right)^{\frac{1}{q}}, \end{aligned} \tag{10}$$

one obtain

$$\left(\frac{\sum_{i=1}^{k} a_i^p - \sum_{i=k+1}^{n} a_i^p}{\sum_{i=1}^{k} a_i^p}\right)^{\frac{1}{p}} \left(\frac{\sum_{i=1}^{k} b_i^q - \sum_{i=k+1}^{n} b_i^q}{\sum_{i=1}^{k} b_i^q}\right)^{\frac{1}{q}} + \left(\frac{\sum_{i=k+1}^{n} a_i^p}{\sum_{i=1}^{k} a_i^p}\right)^{\frac{1}{p}} \left(\frac{\sum_{i=k+1}^{n} b_i^q}{\sum_{i=1}^{k} b_i^q}\right)^{\frac{1}{q}}$$

$$\leqslant 2^{\max\{1-p^{-1}-q^{-1},0\}} \left(1-\min\{p^{-1},q^{-1},1\}\left(\frac{\sum_{i=1}^{k} a_i^p - \sum_{i=k+1}^{n} a_i^p}{\sum_{i=1}^{k} a_i^p} - \frac{\sum_{i=1}^{k} b_i^q - \sum_{i=k+1}^{n} b_i^q}{\sum_{i=1}^{k} b_i^q}\right)^2\right),$$

this is

$$\begin{aligned} &\left(\sum_{i=1}^{k} a_i^p - \sum_{i=k+1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{k} b_i^q - \sum_{i=k+1}^{n} b_i^q\right)^{\frac{1}{q}} + \left(\sum_{i=k+1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=k+1}^{n} b_i^q\right)^{\frac{1}{q}} \\ &\qquad \leqslant 2^{\max\{1-p^{-1}-q^{-1},0\}} \left(\sum_{i=1}^{k} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{k} b_i^q\right)^{\frac{1}{q}} \times \\ &\qquad\qquad \times \left(1-\min\{p^{-1},q^{-1},1\}\left(\frac{\sum_{i=k+1}^{n} a_i^p}{\sum_{i=1}^{k} a_i^p} - \frac{\sum_{i=k+1}^{n} b_i^q}{\sum_{i=1}^{k} b_i^q}\right)^2\right), \end{aligned}$$

hence

$$\begin{aligned} &\left(\sum_{i=1}^{k} a_i^p - \sum_{i=k+1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{k} b_i^q - \sum_{i=k+1}^{n} b_i^q\right)^{\frac{1}{q}} + \left(\sum_{i=k+1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=k+1}^{n} b_i^q\right)^{\frac{1}{q}} \\ &\qquad \leqslant 2^{\max\{1-p^{-1}-q^{-1},0\}} \left(\sum_{i=1}^{k} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{k} b_i^q\right)^{\frac{1}{q}} \\ &\qquad\qquad - \frac{2^{\max\{1-p^{-1}-q^{-1},0\}}}{\max\{p,\,q,1\}} \left(\sum_{i=1}^{k} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{k} b_i^q\right)^{\frac{1}{q}} \left(\frac{\sum_{i=k+1}^{n} a_i^p}{\sum_{i=1}^{k} a_i^p} - \frac{\sum_{i=k+1}^{n} b_i^q}{\sum_{i=1}^{k} b_i^q}\right)^2, \end{aligned}$$

which is the desired inequality (9). In addition, from Lemma 4, we obtain immediately the condition of equality for inequality (9). The proof of Theorem 1 is complete.

THEOREM 2. *Let* $p > 0$, $q > 0$, *and let* a_i, b_i $(i = 1, 2, \cdots, n)$ *be positive numbers such that* $a_1^p - \sum_{i=2}^n a_i^p > 0$ *and* $b_1^q - \sum_{i=2}^n b_i^q > 0$. *Then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} \leqslant 2^{\max\{1-p^{-1}-q^{-1},0\}} a_1 b_1 - \left(\frac{1}{n-1}\right)^{\max\{1-p^{-1}-q^{-1},0\}} \left(\sum_{i=2}^n a_i b_i\right) - \frac{2^{\max\{1-p^{-1}-q^{-1},0\}} a_1 b_1}{\max\{p, q, 1\}} \left(\sum_{i=2}^n \left(\frac{a_i^p}{a_1^p} - \frac{b_i^q}{b_1^q}\right)\right)^2. \tag{11}$$

Equality holds if and only if $(2n-2)^{-\frac{1}{p}} a_1 = a_2 = \cdots = a_n$ *and* $(2n-2)^{-\frac{1}{q}} b_1 = b_2 = \cdots = b_n$ *for* $p^{-1} + q^{-1} < 1$, *or* $\frac{a_1^p}{b_1^q} = \frac{a_2^p}{b_2^q} = \cdots = \frac{a_n^p}{b_n^q}$ *for* $p^{-1} + q^{-1} = 1$.

Proof. Putting $k = 1$ in Theorem 1 gives

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} \leqslant 2^{\max\{1-p^{-1}-q^{-1},0\}} a_1 b_1 - \left(\sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} - \frac{2^{\max\{1-p^{-1}-q^{-1},0\}} a_1 b_1}{\max\{p, q, 1\}} \left(\sum_{i=2}^n \left(\frac{a_i^p}{a_1^p} - \frac{b_i^q}{b_1^q}\right)\right)^2, \tag{12}$$

where the equality holds if and only if $a_1^p \Big/ \left(\sum_{i=2}^n a_i^p\right) = b_1^q \Big/ \left(\sum_{i=2}^n b_i^q\right) = 2$ for $p^{-1} + q^{-1} < 1$, or $a_1^p \Big/ \left(\sum_{i=2}^n a_i^p\right) = b_1^q \Big/ \left(\sum_{i=2}^n b_i^q\right)$ for $p^{-1} + q^{-1} = 1$.

On the other hand, it follows from Lemma 1 that

$$\left(\sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} \geqslant \left(\frac{1}{n-1}\right)^{\max\{1-p^{-1}-q^{-1},0\}} \left(\sum_{i=2}^n a_i b_i\right), \tag{13}$$

where the equality holds if and only if $a_2 = a_3 = \cdots = a_n$ and $b_2 = b_3 = \cdots = b_n$ for $p^{-1} + q^{-1} < 1$, or $a_i^p \Big/ \left(\sum_{i=2}^n a_i^p\right) = b_i^q \Big/ \left(\sum_{i=2}^n b_i^q\right)$ $(i = 2, 3, \cdots, n)$ for $p^{-1} + q^{-1} = 1$.

Combining inequalities (12) and (13) yields inequality (11). This completes the proof of Theorem 2.

Choosing $p^{-1}+q^{-1} \geqslant 1$ in Theorem 2, we obtain

COROLLARY 1. *Let* $p>0,\ q>0,\ p^{-1}+q^{-1} \geqslant 1$*, and let* $a_i,\ b_i\ (i=1,2,\cdots,n)$ *be positive numbers such that* $a_1^p-\sum_{i=2}^n a_i^p>0$ *and* $b_1^q-\sum_{i=2}^n b_i^q>0$*. Then*

$$\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{\frac{1}{p}}\left(b_1^q-\sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} \leqslant a_1b_1-\left(\sum_{i=2}^n a_ib_i\right)-\frac{a_1b_1}{\max\{p,\ q,\ 1\}}\left(\sum_{i=2}^n\left(\frac{a_i^p}{a_1^p}-\frac{b_i^q}{b_1^q}\right)\right)^2, \tag{14}$$

with equality holding if and only if $\frac{a_1^p}{b_1^q}=\frac{a_2^p}{b_2^q}=\cdots=\frac{a_n^p}{b_n^q}$ *and* $p^{-1}+q^{-1}=1$.

In particular, putting $p^{-1}+q^{-1}=1$ in Corollary 1, the following sharpened version of Popoviciu's inequality is derived.

COROLLARY 2. *Let* $p>0,\ q>0,\ p^{-1}+q^{-1}=1$*, and let* $a_i,\ b_i\ (i=1,2,\cdots,n)$ *be positive numbers such that* $a_1^p-\sum_{i=2}^n a_i^p>0$ *and* $b_1^q-\sum_{i=2}^n b_i^q>0$*. Then*

$$\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{\frac{1}{p}}\left(b_1^q-\sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} \leqslant a_1b_1-\left(\sum_{i=2}^n a_ib_i\right)-\frac{a_1b_1}{\max\{p,q\}}\left(\sum_{i=2}^n\left(\frac{a_i^p}{a_1^p}-\frac{b_i^q}{b_1^q}\right)\right)^2, \tag{15}$$

with equality holding if and only if $\frac{a_1^p}{b_1^q}=\frac{a_2^p}{b_2^q}=\cdots=\frac{a_n^p}{b_n^q}$.

4. Integral version of Aczél-Popoviciu type inequality

THEOREM 3. *Let* $p>0,\ q>0,\ p^{-1}+q^{-1}=1,\ A>0,\ B>0$*, and let* $f,\ g$ *be positive Riemann integrable functions on* $[a,b]$ *such that* $A^p-\int_a^b f^p(x)dx>0$ *and* $B^q-\int_a^b g^q(x)dx>0$*. Then*

$$\left(A^p-\int_a^b f^p(x)dx\right)^{\frac{1}{p}}\left(B^q-\int_a^b g^q(x)dx\right)^{\frac{1}{q}} \leqslant AB-\int_a^b f(x)g(x)dx-\frac{AB}{\max\{p,q\}}\left(\int_a^b\left(\frac{f^p(x)}{A^p}-\frac{g^q(x)}{B^q}\right)dx\right)^2. \tag{16}$$

Proof. For any positive integer n, we choose an equidistant partition of $[a,b]$ as

$$a < a+\frac{b-a}{n} < \cdots < a+\frac{b-a}{n}i < \cdots < a+\frac{b-a}{n}(n-1) < b,$$
$$\Delta x_i = \frac{b-a}{n}, \quad i=1,2,\cdots,n.$$

Note that the hypotheses $A^p - \int_a^b f^p(x)dx > 0$ and $B^q - \int_a^b g^q(x)dx > 0$, which implies

$$A^p - \lim_{n\to\infty}\sum_{i=1}^{n} f^p\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n} > 0$$

and

$$B^q - \lim_{n\to\infty}\sum_{i=1}^{n} g^q\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n} > 0.$$

So there exists a positive integer N such that

$$A^p - \sum_{i=1}^{n} f^p\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n} > 0$$

and

$$B^q - \sum_{i=1}^{n} g^q\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n} > 0 \text{ for all } n > N.$$

Applying Corollary 2, one obtain for any $n > N$ the inequality:

$$\left[A^p - \sum_{i=1}^{n} f^p\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n}\right]^{\frac{1}{p}}\left[B^q - \sum_{i=1}^{n} g^q\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n}\right]^{\frac{1}{q}}$$
$$\leqslant AB - \sum_{i=1}^{n} f\left(a+\frac{i(b-a)}{n}\right)g\left(a+\frac{i(b-a)}{n}\right)\left(\frac{b-a}{n}\right)^{\frac{1}{p}+\frac{1}{q}}$$
$$-\frac{AB}{\max\{p,q\}}\left[\sum_{i=1}^{n}\left(\frac{1}{A^p}f^p\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n} - \frac{1}{B^q}g^q\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n}\right)\right]^2.$$

Since $p^{-1}+q^{-1}=1$, the above inequality can be transformed to

$$\left[A^p - \sum_{i=1}^{n} f^p\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n}\right]^{\frac{1}{p}}\left[B^q - \sum_{i=1}^{n} g^q\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n}\right]^{\frac{1}{q}}$$
$$\leqslant AB - \sum_{i=1}^{n} f\left(a+\frac{i(b-a)}{n}\right)g\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n} \tag{17}$$
$$-\frac{AB}{\max\{p,q\}}\left[\sum_{i=1}^{n}\left(\frac{1}{A^p}f^p\left(a+\frac{i(b-a)}{n}\right) - \frac{1}{B^q}g^q\left(a+\frac{i(b-a)}{n}\right)\right)\frac{b-a}{n}\right]^2,$$

where the equality holds if and only if $f^p\left(a+\frac{i(b-a)}{n}\right)/A^p = g^q\left(a+\frac{i(b-a)}{n}\right)/B^q$, $i=1,2,\cdots,n$.

In view of the hypotheses that f, g are positive Riemann integrable functions on $[a,b]$ and $p>0$, $q>0$, we conclude that fg, f^p, g^q are also integrable on $[a,b]$. Passing the limit as $n\to\infty$ in both sides of inequality (17) together with the definition of definite integral, which yields the inequality (16). The proof of Theorem 3 is complete.

Acknowledgments. The author would like to express hearty thanks to the anonymous referee for his valuable comments and suggestions on this article. The author also expresses thanks to Project Foundation of Fujian Province Education Department (No. JA05324) and Natural Science Foundation of Fujian province of China (No. S0650003) for support.

REFERENCES

[1] J. ACZÉL, *Some general methods in the theory of functional equations in one variable, New applications of functional equations*, Uspehi. Mat. Nauk (N.S.), **11**, 69, (3) (1956), 3–68 (in Russian).
[2] Y. J. CHO, M. MATIĆ AND J. PEČARIĆ, *Improvements of some inequalities of Aczél's type*, J. Math. Anal. Appl., **259**, (2001), 226–240 .
[3] X. H. SUN, *Aczél-Chebyshev type inequality for positive linear functions*, J. Math. Anal. Appl., **245**, (2000), 393–403.
[4] L. LOSONCZI, Z. PÁLES, *Inequalities for indefinite forms*, J. Math. Anal. Appl., **205**, (1997), 148–156.
[5] A. M. MERCER, *Extensions of Popoviciu's inequality using a general method*, J. Inequal. Pure Appl. Math., **4**, (1) (2003), Article 11.
[6] V. MASCIONI, *A note on Aczél type inequalities*, J. Inequal. Pure Appl. Math., **3**, (5) (2002), Article 69.
[7] S. S. DRAGOMIR, B. MOND, *Some inequalities of Aczél type for gramians in inner product spaces*, Nonlinear Funct. Anal. Appl., **6**, (2001), 411–424.
[8] R. BELLMAN, *On an inequality concerning an indefinite form*, Amer. Math. Monthly., **63**, (1956), 108–109.
[9] P. M. VASIĆ, J. E. PEČARIĆ, *On the Jensen inequality for monotone functions*, An. Univ. Timisoara. Ser. St. Mathematice, **17**, (1) (1979), 95–104.
[10] J. C. KUANG, *Applied Inequalities*, second ed., Hunan Education Press, Changsha, China, 1993, 180–181 (in Chinese).
[11] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993, 117–120.
[12] T. POPOVICIU, *On an inequality*, Gaz. Mat. Fiz. Ser. A, **11**, (64) (1959), 451–461 (in Romanian).
[13] S. WU, L. DEBNATH, *Generalizations of Aczel's inequality and Popoviciu's inequality*, Indian J. Pure Appl. Math., **36**, (2) (2005), 49–62.
[14] S. WU, L. DEBNATH, *A new generalization of Aczél's inequality and its applications to the improvement of Bellman's inequality*, J. Math. Anal. Appl., (2007), to appear.
[15] Z. PÁLES, *On Hölder-type inequalities*, J. Math. Anal. Appl., **95**, (1983), 457–466.
[16] D. S. MITRINOVIĆ, P. M. VASIĆ, *Analytic Inequalities*, Springer-Verlag, New York, 1970, 34–35.

(Received October 23, 2006)

Department of Mathematics
Longyan College
Longyan Fujian 364012
People's Republic of China
e-mail: wushanhe@yahoo.com.cn