

POSITIVITY OF A CLASS OF COSINE SUMS, II

QINGHE YIN

(communicated by C. E. M. Pearce)

Abstract. Assume that $a_0 \geq a_1 \geq \dots \geq a_n \geq \dots > 0$. If $(3k-1)a_{2k-1} \geq 3ka_{2k}$ for $k \geq 1$, then, for all $n \geq 0$ and $0 \leq x < \pi$ one has

$$\sum_{k=0}^n a_k \cos kx > 0.$$

This generalizes a theorem of Vietoris, in which the condition on the coefficients is $(2k-1)a_{2k-1} \geq 2ka_{2k}$.

1. Introduction and main results

It is well known that for $0 < x < \pi$ one has

$$\sum_{k=1}^n \frac{\sin kx}{k} > 0 \tag{1}$$

and

$$1 + \sum_{k=1}^n \frac{\cos kx}{k} > 0 \tag{2}$$

(1) and (2) are known as Fejér ([4]) and Young ([7]) inequalities respectively. In 1958, Vietoris [6] gave a dramatic improvement of (1.1) and (1.2). He proved

THEOREM A. (Vietoris). *If $a_0 \geq a_1 \geq \dots \geq a_n > 0$ and $(2k-1)a_{2k-1} \geq 2ka_{2k}$ ($k \geq 1$), then for $0 < x < \pi$ one has*

$$\sum_{k=1}^n a_k \sin kx > 0 \quad \text{and} \quad \sum_{k=0}^n a_k \cos kx > 0.$$

Vietoris also showed that Theorem A is equivalent to the following extreme case:

THEOREM B. (Vietoris). *Let $a_0 = a_1 = 1$ and*

$$a_{2k} = a_{2k+1} = \frac{2k-1}{2k} a_{2k-1}$$

Mathematics subject classification (2000): 42A32, 42A05, 26D05.

Key words and phrases: trigonometric sum, positivity, Vietoris theorem.

for $k \geq 1$. Then

$$\sum_{k=1}^n a_k \sin kx > 0 \quad \text{and} \quad \sum_{k=0}^n a_k \cos kx > 0.$$

The result of Vietoris on sine sums is the best possible in the sense that the coefficient sequence is decreasing and as large as possible. But for cosine sums, it is not the case. In 1984, Brown and Hewitt showed in [1] that if $a_0 \geq a_1 \geq \dots > 0$ and $2ka_{2k-1} = (2k+1)a_{2k}$ then we have $\sum_{k=1}^n a_k \cos kx > 0$ for all $n \geq 0$ and $0 \leq x < \pi$. In 2001, Brown and the current author obtained further improvement in [3]. We changed the condition on coefficients into $a_0 \geq a_1 \geq \dots > 0$ and $(2k+1)a_{2k-1} = (2k+2)a_{2k}$.

Notice that the coefficients in Theorem B can be expressed as

$$a_0 = a_1 = 1 \quad \text{and} \quad a_{2k} = a_{2k+1} = \prod_{j=1}^k \left(1 - \frac{1}{2j}\right), \quad \text{for } k \geq 1.$$

If we define

$$a_0^{(\alpha)} = a_1^{(\alpha)} = 1 \quad \text{and} \quad a_{2k}^{(\alpha)} = a_{2k+1}^{(\alpha)} = \prod_{j=1}^k \left(1 - \frac{1}{2j + \alpha}\right), \quad \text{for } k \geq 1$$

and

$$U_n^{(\alpha)}(x) = \sum_{k=0}^n a_k^{(\alpha)} \cos kx$$

then Vietoris theorem B presents that, for all $n \geq 0$ and $0 \leq x < \pi$, we have $U_n^{(0)}(x) > 0$. The Brown-Hewitt result in [1] is correspond to $\alpha = 1$ and the Brown-Yin result in [3] is with $\alpha = 2$. It is not true for $\alpha = 3$, since, for example, we have $U_3^{(3)}\left(\frac{\pi}{3}\right) < 0$.

Now we improve the Vietoris result for cosine sums in another way. Define

$$b_0^{(\alpha)} = b_1^{(\alpha)} = 1 \quad \text{and} \quad b_{2k}^{(\alpha)} = b_{2k+1}^{(\alpha)} = \prod_{j=1}^k \left(1 - \frac{\alpha}{j}\right), \quad \text{for } k \geq 1.$$

We can also express $b_{2k}^{(\alpha)}$ and $b_{2k+1}^{(\alpha)}$ in the following form:

$$b_{2k}^{(\alpha)} = b_{2k+1}^{(\alpha)} = \frac{(1-\alpha)_k}{k!}$$

where $(s)_0 = 1$ and $(s)_k = s(s+1)\cdots(s+k-1)$ for $k \geq 1$. Define

$$V_n^\alpha(x) = \sum_{k=0}^n b_k^{(\alpha)} \cos kx.$$

Then Vietoris theorem B presents that $V_n^{(1/2)} > 0$ for all n and $0 \leq x < \pi$. In this paper, we consider $\alpha = \frac{1}{3}$. For the convenience of writing, we will use b_k to denote $b_k^{(1/3)}$.

THEOREM 1. Let b_k be defined as above. Then for all $n \geq 0$ and $0 \leq x < \pi$ we have

$$\sum_{k=0}^n b_k \cos kx > 0.$$

By an argument of summation by parts we obtain the following result:

THEOREM 2. Assume that $c_0 \geq c_1 \geq \dots \geq c_n \geq \dots$. If

$$\frac{c_{2k}}{c_{2k-1}} \leq \frac{3k-1}{3k}$$

then for all $n \geq 0$ and $0 \leq x < \pi$ we have

$$\sum_{k=0}^n c_k \cos kx \geq 0.$$

REMARK 1. Theorem 1 implies the Brown-Hewitt result for $U_n^{(1)}(x)$, since for all $k \geq 1$ we have

$$\frac{a_{2k}^{(1)}}{a_{2k-1}^{(1)}} = \frac{2k}{2k+1} \leq \frac{3k-1}{3k}.$$

REMARK 2. Neither of Theorem 1 and the positivity of $U_n^{(2)}$ contains the other. In fact,

$$\frac{a_2^{(2)}}{a_1^{(2)}} = \frac{3}{4} > \frac{2}{3} = \frac{3 \cdot 1 - 1}{3 \cdot 1}$$

and

$$\frac{b_{2k}}{b_{2k-1}} = \frac{3k-1}{3k} > \frac{2k+1}{2k-1}$$

for $k \geq 3$.

REMARK 3. It is known that the sum $\sum_{k=1}^n \frac{1}{n^\alpha} \cos kx$ is uniformly bounded below if and only if $\alpha \geq \alpha_0$, where $\alpha_0 = 0.308443 \dots$ is the root of the equation

$$\int_0^{3\pi/2} \frac{\cos t}{t^\alpha} dt = 0.$$

(Littlewood, Salem and Izumi, see [8]). By this and the fact that the limit $\lim_{n \rightarrow \infty} n^\alpha b_{2n}$ exists (see the proof of Lemma 3 in next section), we see that for $\alpha < \alpha_0$, $V_n^\alpha(x)$ is not uniformly bounded below. Hence it is possible to have $V_n^\alpha(x) < 0$ for some n and some $0 < x < \pi$. Brown et al proved in [2] that $1 + \sum_{k=1}^n \frac{1}{n^\alpha} \cos kx \geq 0$ for $0 \leq x < \pi$, when $\alpha \geq \alpha_0$. I wonder if it is also true for $V_n^{(\alpha)}$.

REMARK 4. All the numerical calculations involved in the proof of Theorem 1 have been done by using Maple V.

By Theorem 2 we can obtain the following result:

THEOREM 3. Assume that $d_0 \geq d_1 \geq \dots \geq d_n > 0$ and $3kd_{k-1} \geq (3k-1)d_k$, for $k \geq 1$. Then for $0 < x < 2\pi$, $0 \leq \delta \leq \frac{1}{4}$ and $\frac{1}{4} \leq \eta \leq \frac{1}{2}$, we have

$$\sum_{k=0}^n d_k \cos(k + \delta)x > 0 \quad \text{and} \quad \sum_{k=0}^n d_k \sin(k + \eta)x > 0.$$

In fact, the result for the cosine sum with $\delta = \frac{1}{4}$ follows from Theorem 2 and

$$\cos 2kx + \cos(2k + 1)x = 2 \cos(2k + \frac{1}{2})x \cos \frac{x}{2}.$$

Replacing x by $2\pi - x$, we obtain the result for the sine sum with $\eta = \frac{1}{4}$. As $\delta < \frac{1}{4}$ and $\eta > \frac{1}{4}$, the results follow from

$$\cos(k + \delta)x = \cos(k + \frac{1}{4})x \cos(\frac{1}{4} - \delta)x + \sin(k + \frac{1}{4})x \sin(\frac{1}{4} - \delta)x$$

and

$$\sin(k + \eta)x = \sin(k + \frac{1}{4})x \cos(\eta - \frac{1}{4})x + \cos(k + \frac{1}{4})x \sin((\eta - \frac{1}{4})x).$$

We will prove Theorem 1 in following sections. In section 2 we first calculate the infinite sum $\sum_{n=0}^{\infty} b_n \cos kx$, and then estimate the “tail” and the coefficients of the series. These results appear as three Lemmas. From section 3 to 5 we consider a subinterval in each section.

2. Lemmas

From now on, we use $C_n(x)$ to denote the sum $\sum_{k=0}^n b_k \cos kx$. First we calculate the infinite sum $C_{\infty}(x)$:

LEMMA 1. For $0 < x < \pi$ we have

$$C_{\infty}(x) = 2^{1/3} \cos \frac{x}{2} (\sin x)^{-2/3} \sin \frac{\pi + x}{6}.$$

Proof. For $|z| \leq 1, z \neq 1$, we have

$$(1 - z)^{-2/3} = \sum_{n=0}^{\infty} \frac{(2/3)_n}{n!} z^n = \sum_{n=0}^{\infty} b_{2n} z^n$$

where $(s)_n$ has the same meaning as in section 1. Furthermore,

$$\begin{aligned} \sum_{n=0}^{\infty} b_n z^n &= \sum_{n=0}^{\infty} (b_{2n} z^{2n} + b_{2n+1} z^{2n+1}) = \sum_{n=0}^{\infty} b_{2n} (z^{2n} + z^{2n+1}) \\ &= (1 + z) \sum_{n=0}^{\infty} b_{2n} z^{2n} = (1 + z)(1 - z^2)^{-2/3}. \end{aligned}$$

Set $z = \cos x + i \sin x$. Then

$$(1 + z)(1 - z^2)^{-2/3} = \sum_{n=0}^{\infty} b_n (\cos nx + i \sin nx) = C_{\infty}(x) + iS_{\infty}(x). \tag{3}$$

We have

$$\begin{aligned} 1 - z^2 &= 1 - \cos 2x - i \sin 2x = 2 \sin^2 x - 2i \sin x \cos x \\ &= 2 \sin x (\sin x - i \cos x) = 2 \sin x \left(\cos(\frac{\pi}{2} - x) - i \sin(\frac{\pi}{2} - x) \right) \\ &= 2 \sin x e^{-i \frac{\pi - 2x}{2}}. \end{aligned}$$

Hence,

$$\begin{aligned} (1 - z^2)^{-2/3} &= (2 \sin x)^{-2/3} \left(e^{i(\pi-2x)} \right)^{1/3} = (2 \sin x)^{-2/3} e^{i\frac{\pi-2x+2j\pi}{3}} \\ &= (2 \sin x)^{-2/3} \left(\cos \frac{\pi - 2x + 2j\pi}{3} + i \sin \frac{\pi - 2x + 2j\pi}{3} \right) \end{aligned} \tag{4}$$

where $j = 0, 1$ or 2 . We need to determine the value of j . Now we have

$$\begin{cases} \sum_{n=0}^{\infty} b_{2n} \cos 2nx = (2 \sin x)^{-2/3} \cos \frac{\pi - 2x + 2j\pi}{3} \\ \sum_{n=0}^{\infty} b_{2n} \sin 2nx = (2 \sin x)^{-2/3} \sin \frac{\pi - 2x + 2j\pi}{3}. \end{cases}$$

When $x = \pi/2$, both sine sum and cosine sum are convergent and the sine sum is equal to 0. Hence we need to set $j = 0$. Therefore

$$(1 - z^2)^{-2/3} = (2 \sin x)^{-2/3} \left(\cos \frac{\pi - 2x}{3} + i \sin \frac{\pi - 2x}{3} \right). \tag{5}$$

We also have

$$\begin{aligned} 1 + z &= 1 + \cos x + i \sin x = 2 \cos^2 \frac{x}{2} + 2i \sin \frac{x}{2} \cos \frac{x}{2} \\ &= 2 \cos \frac{x}{2} \left(\cos \frac{x}{2} + i \sin \frac{x}{2} \right). \end{aligned} \tag{6}$$

By (5) and (6) we can get

$$(1 + z)(1 - z^2)^{-2/3} = 2 \cos \frac{x}{2} (2 \sin x)^{-2/3} e^{i\frac{2\pi-x}{6}}. \tag{7}$$

Then

$$\begin{aligned} C_{\infty}(x) &= \mathbf{Re} \left((1 + z)(1 - z^2)^{-2/3} \right) \\ &= 2 \cos \frac{x}{2} (2 \sin x)^{-2/3} \cos \frac{2\pi - x}{6} = 2 \cos \frac{x}{2} (2 \sin x)^{-2/3} \sin \frac{\pi + x}{6}. \end{aligned}$$

This completes the proof of Lemma 1.

Next we estimate the ‘‘tail’’ of the infinite series. We have

LEMMA 2. Let $m = \lceil \frac{n+1}{2} \rceil$. Then

$$\begin{aligned} C_n(x) \geq C_{\infty}(x) &- \frac{b_{2m}}{2 \sin \frac{x}{2}} \left[-\sin(n + \frac{1}{2})x \right. \\ &\left. + \frac{1}{6(m+1) \sin x} \left(\cos(2m + \frac{1}{2})x + \frac{2(1 + \sin(2m + \frac{3}{2})x)}{3(m+2) \sin x} \right) \right]. \end{aligned}$$

Proof. For fixed ℓ , consider $\sum_{k=n+1}^{n+j} b_k \cos kx$. Using summation by parts we get

$$\sum_{k=n+1}^{n+\ell} b_k \cos kx = \sum_{k=n+1}^{n+\ell-1} (b_k - b_{k+1}) \sum_{j=n+1}^k \cos kx + b_{n+\ell} \sum_{j=n+1}^{n+\ell} \cos jx.$$

We have

$$\sum_{j=n+1}^{n+k} \cos jx = \frac{\sin(k + \frac{1}{2})x - \sin(n + \frac{1}{2})x}{2 \sin \frac{x}{2}}.$$

Hence, when $\ell \rightarrow \infty$,

$$\sum_{k=n+1}^{\infty} b_k \cos kx = \sum_{k=n+1}^{\infty} (b_k - b_{k+1}) \frac{\sin(k + \frac{1}{2})x - \sin(n + \frac{1}{2})x}{2 \sin \frac{x}{2}}.$$

Recall that $b_{2k} = b_{2k+1}$. Let $m = \lfloor \frac{n+1}{2} \rfloor$. Then

$$\begin{aligned} \sum_{k=n+1}^{\infty} b_k \cos kx &= \frac{1}{2 \sin \frac{x}{2}} \sum_{k=m}^{\infty} (b_{2k} - b_{2k+2}) \left(\sin(2k + 1 + \frac{1}{2})x - \sin(n + \frac{1}{2})x \right) \\ &= -b_{2m} \frac{\sin(n + \frac{1}{2})x}{2 \sin \frac{x}{2}} + \frac{1}{2 \sin \frac{x}{2}} \sum_{k=m}^{\infty} (b_{2k} - b_{2k+2}) \sin(2k + \frac{3}{2})x \quad (8) \\ &= -b_{2m} \frac{\sin(n + \frac{1}{2})x}{2 \sin \frac{x}{2}} + \frac{1}{2 \sin \frac{x}{2}} \sum_{k=m}^{\infty} \frac{b_{2k}}{3(k+1)} \sin(2k + \frac{3}{2})x. \end{aligned}$$

Using summation by parts on the sum $\sum_{k=m}^{\infty} \frac{b_{2k}}{3(k+1)} \sin(2k + \frac{3}{2})x$ and noting that

$$\frac{b_{2k}}{k+1} - \frac{b_{2k+2}}{k+2} = \frac{4}{3(k+1)(k+2)}$$

we get

$$\begin{aligned} \sum_{k=m}^{\infty} \frac{b_{2k}}{k+1} \sin(2k + \frac{3}{2})x &= \sum_{k=m}^{\infty} \left(\frac{b_{2k}}{k+1} - \frac{b_{2k+2}}{k+2} \right) \sum_{j=m}^k \sin(2j + \frac{3}{2})x \\ &= \sum_{k=m}^{\infty} \left(\frac{b_{2k}}{k+1} - \frac{b_{2k+2}}{k+2} \right) \frac{\cos(2m + \frac{1}{2})x - \cos(2k + \frac{5}{2})x}{2 \sin x} \quad (9) \\ &= \frac{b_{2m} \cos(2m + \frac{1}{2})x}{2(m+1) \sin x} \\ &\quad - \frac{2}{3 \sin x} \sum_{k=m}^{\infty} \frac{b_{2k}}{(k+1)(k+2)} \cos(2k + \frac{5}{2})x. \end{aligned}$$

Substituting (9) into (8) and applying summation by parts on the sum $\sum_{k=m}^{\infty} \frac{b_{2k}}{(k+1)(k+2)} \cos(2k + \frac{5}{2})x$, we obtain

$$\begin{aligned} \sum_{k=0}^n b_k \cos kx &= \frac{b_{2m}}{2 \sin \frac{x}{2}} \left(-\sin(n + \frac{1}{2})x + \frac{\cos(2m + \frac{1}{2})x}{6(m+1) \sin x} \right) \\ &\quad - \frac{1}{9 \sin \frac{x}{2} \sin x} \sum_{k=m}^{\infty} \left(\frac{b_{2k}}{(k+1)(k+2)} - \frac{b_{2k+2}}{(k+2)(k+3)} \right) \sum_{j=m}^k \cos(2j + \frac{5}{2})x \\ &\leq \frac{b_{2m}}{2 \sin \frac{x}{2}} \left[-\sin(n + \frac{1}{2})x \right. \\ &\quad \left. + \frac{1}{6(m+1) \sin x} \left(\cos(2m + \frac{1}{2})x + \frac{2(1 + \sin(2m + \frac{3}{2})x)}{3(m+2) \sin x} \right) \right]. \end{aligned}$$

Next we estimate b_{2m} in terms of $\frac{1}{(2m + \frac{1}{2})^{1/3}}$.

LEMMA 3. For all $m \geq 1$ we have

$$b_{2m} \leq \frac{1}{(2m + \frac{1}{2})^{1/3}} \cdot \frac{2^{1/3}}{\Gamma(2/3)} < \frac{0.93044}{(2m + \frac{1}{2})^{1/3}}.$$

Proof. It is known that

$$\Gamma(s) = \lim_{n \rightarrow \infty} \frac{n^{s-1}n!}{(s)_n} \tag{10}$$

(see G. Pólya & G. Szegő [4], II31, p. 54). Letting $s = 2/3$ we have

$$\lim_{m \rightarrow \infty} m^{1/3} b_{2m} = \frac{1}{\Gamma(2/3)}.$$

Thus

$$\lim_{m \rightarrow \infty} \left(2m + \frac{1}{2}\right)^{1/3} b_{2m} = 2^{1/3} \lim_{m \rightarrow \infty} m^{1/3} b_{2m} \left(1 + \frac{1}{4m}\right)^{1/3} = \frac{2^{1/3}}{\Gamma(2/3)} < 0.93044.$$

We complete the proof by showing that that $\left\{ \left(2m + \frac{1}{2}\right)^{1/3} b_{2m} \right\}$ is increasing with respect to m :

$$\begin{aligned} &\left(2m + 2 + \frac{1}{2}\right)^{1/3} b_{2m+2} - \left(2m + \frac{1}{2}\right)^{1/3} b_{2m} \\ &= \left(\left(2m + 1 + \frac{1}{2}\right)^{1/3} b_{2m} \right) \left[\left(1 - \frac{1}{3(m+1)}\right) - \left(1 - \frac{1}{m + 1 + \frac{1}{4}}\right)^{1/3} \right]. \end{aligned}$$

For $0 < x < 1$ we have

$$(1 - x)^{1/3} < 1 - \frac{1}{3}x - \frac{1}{9}x^2.$$

This is because the third derivative of $(1 - x)^{1/3}$ is negative. Hence

$$\begin{aligned} \left(1 - \frac{1}{m + 1 + \frac{1}{4}}\right)^{1/3} &< 1 - \frac{1}{3} \cdot \frac{1}{m + 1 + \frac{1}{4}} - \frac{1}{9} \left(\frac{1}{m + 1 + \frac{1}{4}}\right)^2 \\ &= 1 - \frac{1}{3(m + 1)} + \frac{1}{12(m + 1)(m + 1 + \frac{1}{4})} - \frac{1}{9(m + 1 + \frac{1}{4})^2} \\ &= 1 - \frac{1}{3(m + 1)} + \frac{1}{3(m + 1 + \frac{1}{4})} \left(\frac{1}{4(m + 1)} - \frac{1}{3(m + 1 + \frac{1}{4})}\right) \\ &< 1 - \frac{1}{3(m + 1)}. \end{aligned}$$

This complete the proof.

3. Proof of Theorem 1: $0 \leq x \leq \frac{\pi}{n}$

The method to prove Theorem 1 for $0 \leq x \leq \frac{\pi}{n}$ is standard. We have

$$C'_n(x) = - \sum_{k=1}^n kb_k \sin kx \leq 0.$$

We need to check that $C_n(\frac{\pi}{n}) > 0$:

$$\begin{aligned} C_n\left(\frac{\pi}{n}\right) &= \sum_{k=0}^n b_k \cos \frac{k\pi}{n} = \sum_{k=0}^{\lfloor n/2 \rfloor} \left(b_k \cos \frac{k\pi}{n} + b_{n-k} \cos \frac{(n-k)\pi}{n}\right) \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} (b_k - b_{n-k}) \cos \frac{k\pi}{n} > 0. \end{aligned}$$

4. Proof of Theorem 1: $\frac{\pi}{2m+\frac{1}{2}} \leq x \leq \frac{2\pi}{2m+\frac{1}{2}}$

Next we consider $\frac{\pi}{2m+\frac{1}{2}} \leq x \leq \frac{2\pi}{2m+\frac{1}{2}}$, where $m = \lfloor \frac{n+1}{2} \rfloor$, because by Lemma 2, it is more convenient to consider this interval than to consider $\frac{\pi}{n} \leq x \leq \frac{2\pi}{n}$. Notice that we have $\frac{\pi}{2m+\frac{1}{2}} < \frac{\pi}{n}$. Let $x = \frac{t+\pi}{2m+\frac{1}{2}}$. We rewrite Lemma 2 as follows.

$$\begin{aligned} C_n(x) &\geq C_\infty(x) - \frac{b_{2m}}{2 \sin \frac{x}{2}} \left[-\sin\left(n + \frac{1}{2}\right)x \right. \\ &\quad \left. + \frac{1}{6(m+1) \sin x} \left(-\cos t + \frac{2\left(1 - \sin\left(t + \frac{t+\pi}{2m+\frac{1}{2}}\right)\right)}{3(m+2) \sin x} \right) \right]. \end{aligned} \tag{11}$$

Notice that for $0 \leq t \leq \pi$ we have

$$\begin{aligned}
 2(m+1) \sin x &= 2(m+1) \sin \frac{t+\pi}{2m+\frac{1}{2}} \\
 &> 2(m+1) \left(\frac{t+\pi}{2m+\frac{1}{2}} - \frac{1}{6} \left(\frac{t+\pi}{2m+\frac{1}{2}} \right)^3 \right) \geq t+\pi, \text{ for } m \geq 1.
 \end{aligned}
 \tag{12}$$

We first assume $0 \leq t \leq \frac{\pi}{2}$ and then consider $\frac{\pi}{2} < t \leq \pi$.

4.1. $0 \leq t \leq \frac{\pi}{2}$

We go by two steps. First we further simplify (11) into

$$C_n(x) \geq C_\infty(x) - \frac{b_{2m}}{2 \sin \frac{x}{2}} \left[\sin t + \frac{2}{9(2m+\frac{1}{2})^2} \right].
 \tag{13}$$

Then we use (13) to prove Theorem 1 for this interval.

A. proof of (13). When $0 \leq t \leq \frac{\pi}{2}$ we have

$$-\sin\left(n + \frac{1}{2}\right)x \leq -\sin\left(2m + \frac{1}{2}\right)x = \sin t.$$

By (12) we have

$$\frac{2 \left(1 - \sin\left(t + \frac{t+\pi}{2m+\frac{1}{2}}\right) \right)}{3(m+2) \sin x} \leq \frac{4}{3(t+\pi)} \left(1 - \sin\left(t + \frac{t+\pi}{2m+\frac{1}{2}}\right) \right).$$

If $t + \frac{t+\pi}{2m+\frac{1}{2}} \leq \frac{\pi}{2}$ then

$$-\cos t + \frac{4}{3(t+\pi)} \left(1 - \sin\left(t + \frac{t+\pi}{2m+\frac{1}{2}}\right) \right) \leq \frac{4}{3\pi} (1 - \sin t) - \cos t < 0.
 \tag{14}$$

The “ < 0 ” is by the convexity of $\frac{4}{3\pi}(1 - \sin t) - \cos t$ for $0 \leq t \leq \frac{\pi}{2}$. When $t + \frac{t+\pi}{2m+\frac{1}{2}} > \frac{\pi}{2}$ then

$$\begin{aligned}
 -\cos t + \frac{4}{3(t+\pi)} \left(1 - \sin\left(t + \frac{t+\pi}{2m+\frac{1}{2}}\right) \right) &\leq \frac{4}{3(t+\pi)} \left(1 - \sin\left(\frac{\pi}{2} + \frac{t+\pi}{2m+\frac{1}{2}}\right) \right) \\
 &= \frac{4}{3(t+\pi)} \left(1 - \cos \frac{t+\pi}{2m+\frac{1}{2}} \right) \\
 &\leq \frac{4}{3(t+\pi)} \cdot \frac{1}{2} \left(\frac{t+\pi}{2m+\frac{1}{2}} \right)^2 = \frac{2(t+\pi)}{3(2m+\frac{1}{2})^2}.
 \end{aligned}
 \tag{15}$$

By (14) and (15) we get (13).

B. proof of Theorem 1 with $0 \leq t \leq \frac{\pi}{2}$. By (13), Lemma 1 and Lemma 3,

$$\begin{aligned} & 2 \sin \frac{x}{2} C_n(x) \\ & \geq (2 \sin x)^{1/3} \sin \frac{\pi+x}{6} - b_{2m} \left(\sin t + \frac{2}{9(2m+\frac{1}{2})^2} \right) \\ & \geq \left(2 \sin \frac{t+\pi}{2m+\frac{1}{2}} \right)^{1/3} \sin \frac{\pi}{6} - \frac{0.93044}{(2m+\frac{1}{2})^{1/3}} \left(\sin t + \frac{2}{9(2m+\frac{1}{2})^2} \right) \\ & \geq 2^{-2/3} \left(\frac{t+\pi}{2m+\frac{1}{2}} - \frac{1}{6} \left(\frac{t+\pi}{2m+\frac{1}{2}} \right)^3 \right)^{1/3} - \frac{0.93044}{(2m+\frac{1}{2})^{1/3}} \left(\sin t + \frac{2}{9(2m+\frac{1}{2})^2} \right) \\ & \geq \left[\frac{(t+\pi)^{1/3}}{2^{2/3}} \left(1 - \frac{1}{6} \left(\frac{t+\pi}{2m+\frac{1}{2}} \right)^2 \right)^{1/3} - 0.93044 \left(\sin t + \frac{2}{9(2m+\frac{1}{2})^2} \right) \right] \frac{1}{(2m+\frac{1}{2})^{1/3}}. \end{aligned}$$

For $t \leq \frac{\pi}{2}$ and $m \geq 4$,

$$\left(1 - \frac{1}{6} \left(\frac{t+\pi}{2m+\frac{1}{2}} \right)^2 \right)^{1/3} \geq \left(1 - \frac{1}{6} \left(\frac{3\pi/2}{8.5} \right)^2 \right)^{1/3} = \left(1 - \frac{1}{6} \left(\frac{3\pi}{17} \right)^2 \right)^{1/3} > 0.98262$$

and

$$\frac{2}{9(2m+\frac{1}{2})^2} \leq \frac{2}{9 \times 8.5^2} < 0.00308.$$

Therefore,

$$\begin{aligned} 2 \sin \frac{x}{2} C_n(x) & \geq \left[0.98262 \frac{(t+\pi)^{1/3}}{2^{2/3}} - 0.93044 (\sin t + 0.00308) \right] \frac{1}{(2m+\frac{1}{2})^{1/3}} \\ & \geq \left[0.61901(t+\pi)^{1/3} - 0.93044 \sin t - 0.00287 \right] \frac{1}{(2m+\frac{1}{2})^{1/3}}. \end{aligned}$$

Let $u(t) = 0.61901(t+\pi)^{1/3} - 0.93044 \sin t - 0.00287$. Then, for $0 \leq t \leq \frac{\pi}{2}$,

$$\begin{aligned} u'(t) & = \frac{0.61901}{3}(t+\pi)^{-2/3} - 0.93044 \cos t \\ u''(t) & = -\frac{2 \times 0.61901}{9}(t+\pi)^{-5/3} + 0.93044 \sin t \end{aligned}$$

and

$$u'''(t) = \frac{6.1901}{27}(t+\pi)^{-8/3} + 0.93044 \cos t > 0.$$

Since $u''(0) < 0$, $u''(\pi/2) > 0$ and $u'(0) < 0$, $u'(\pi/2) > 0$, we see that $u'(x)$ has exactly one zero at $(0, \pi/2)$. Since

$$u' \left(\frac{\pi}{3} \right) = \frac{0.61901}{3} \left(\frac{\pi}{3} \right)^{-2/3} - 0.93044 \cos \frac{\pi}{3} = -0.2516 \dots$$

we can conclude that, for $t < \frac{\pi}{3}$,

$$u(t) \geq u\left(\frac{\pi}{3}\right) = 0.61901 \left(\frac{4\pi}{3}\right)^{1/3} - 0.93044 \sin \frac{\pi}{3} - 0.00287 = 0.18918 \dots$$

When $t > \frac{\pi}{3}$,

$$u(t) \geq 0.61901 \left(\frac{4\pi}{3}\right)^{1/3} - 0.93044 - 0.00287 = 0.06452 \dots$$

4.2. $\frac{\pi}{2} < t \leq \pi$

Now we consider $\frac{\pi}{2} < t \leq \pi$. In this interval, we use 1 to replace $-\sin(n + \frac{1}{2})x$. For $m \geq 4$, we have

$$\left(1 - \frac{1}{6} \left(\frac{t + \pi}{2m + \frac{1}{2}}\right)^2\right)^{1/3} \geq \left(1 - \frac{1}{6} \left(\frac{2\pi}{8.5}\right)^2\right)^{1/3} > 0.96867.$$

Also we have

$$\frac{t + \pi}{2m + \frac{1}{2}} < \frac{\pi}{4}.$$

We need to split this interval into two parts. For $\frac{\pi}{2} \leq t \leq \frac{2\pi}{3}$, by Lemma 2 and (12) we have

$$\begin{aligned} C_n(x) - C_\infty(x) &\geq -\frac{b_{2m}}{2 \sin \frac{x}{2}} \left[1 + \frac{1}{3(t + \pi)} \left(\frac{4}{3(t + \pi)} \left(1 - \sin\left(t + \frac{t + \pi}{2m + \frac{1}{2}}\right) \right) + \cos \frac{\pi}{3} \right) \right] \\ &\geq -\frac{b_{2m}}{2 \sin \frac{x}{2}} \left[1 + \frac{1}{3(t + \pi)} \left(\frac{4}{3(t + \pi)} + \frac{1}{2} \right) \right] \\ &\geq -\frac{b_{2m}}{2 \sin \frac{x}{2}} \left[1 + \frac{4}{9(t + \pi)^2} + \frac{1}{6(t + \pi)} \right] \\ &\geq -\frac{b_{2m}}{2 \sin \frac{x}{2}} \left[1 + \frac{4}{9\left(\frac{3\pi}{2}\right)^2} + \frac{1}{9\pi} \right] \geq -\frac{1.05539b_{2m}}{2 \sin \frac{x}{2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} 2 \sin \frac{x}{2} C_n(x) &\geq \frac{0.96867}{2^{2/3}} \cdot \frac{(t + \pi)^{1/3}}{(2m + \frac{1}{2})^{1/3}} - 1.05539b_{2m} \\ &\geq \left[0.61022(t + \pi)^{1/3} - 0.93044 \times 1.05539 \right] \frac{1}{(2m + \frac{1}{2})^{1/3}} \\ &\geq \left[0.61022 \left(\frac{\pi}{2} + \pi\right)^{1/3} - 0.98198 \right] \frac{1}{(2m + \frac{1}{2})^{1/3}} > \frac{0.04107}{(2m + \frac{1}{2})^{1/3}} > 0. \end{aligned}$$

When $\frac{2\pi}{3} \leq t \leq \pi$, we have

$$\begin{aligned}
 C_n(x) - C_\infty(x) &\geq -\frac{b_{2m}}{2 \sin \frac{x}{2}} \left[1 + \frac{1}{6(m+1) \sin x} \left(\frac{4}{3(t+\pi)} \left(1 - \sin\left(t + \frac{t+\pi}{2m+\frac{1}{2}}\right) \right) + 1 \right) \right] \\
 &\geq -\frac{b_{2m}}{2 \sin \frac{x}{2}} \left[1 + \frac{1}{6(m+1) \sin x} \left(\frac{4}{3(t+\pi)} \left(1 + \sin \frac{\pi}{4} \right) + 1 \right) \right] \\
 &\geq -\frac{b_{2m}}{2 \sin \frac{x}{2}} \left[1 + \frac{4 + 2\sqrt{2}}{9(t+\pi)^2} + \frac{1}{3(t+\pi)} \right] \\
 &\geq -\frac{b_{2m}}{2 \sin \frac{x}{2}} \left[1 + \frac{4 + 2\sqrt{2}}{(5\pi)^2} + \frac{1}{5\pi} \right] \geq -\frac{1.09134b_{2m}}{2 \sin \frac{x}{2}}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 2 \sin \frac{x}{2} C_n(x) &\geq 0.96867 \frac{(t+\pi)^{1/3}}{2^{2/3}} \frac{1}{(2m+\frac{1}{2})^{1/3}} - 1.09134b_{2m} \\
 &\geq \left[0.61022(t+\pi)^{1/3} - 0.93044 \times 1.09134 \right] \frac{1}{(2m+\frac{1}{2})^{1/3}} \\
 &\geq \left[0.61716 \left(\frac{5\pi}{3} \right)^{1/3} - 1.01546 \right] \frac{1}{(2m+\frac{1}{2})^{1/3}} > \frac{0.04416}{(2m+\frac{1}{2})^{1/3}} > 0.
 \end{aligned}$$

5. Proof of Theorem 1: $\frac{2\pi}{2m+\frac{1}{2}} \leq x \leq \frac{\pi}{2}$.

Now we consider $\frac{2\pi}{2m+\frac{1}{2}} \leq x \leq \frac{\pi}{2}$, i.e., $t \geq \pi$ but $x \leq \frac{\pi}{2}$. We use 1 to dominate $-\sin(n + \frac{1}{2})x$, $-\cos t$ and $\sin(t + \frac{t+\pi}{2m+\frac{1}{2}})$. Then

$$\begin{aligned}
 C_n(x) - C_\infty(x) &\geq -\frac{b_{2m}}{2 \sin \frac{x}{2}} \left[1 + \frac{1}{6(m+1) \sin x} \left(\frac{8}{3(t+\pi)} + 1 \right) \right] \\
 &\geq -\frac{b_{2m}}{2 \sin \frac{x}{2}} \left[1 + \frac{8}{9(t+\pi)^2} + \frac{1}{3(t+\pi)} \right] \\
 &\geq -\frac{b_{2m}}{2 \sin \frac{x}{2}} \left[1 + \frac{4 + \sqrt{2}}{9(2\pi)^2} + \frac{1}{6\pi} \right] \geq -\frac{1.07557b_{2m}}{2 \sin \frac{x}{2}}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 2 \sin \frac{x}{2} C_n(x) &\geq \left(\frac{\sin x}{4} \right)^{1/3} - 1.07557b_{2m} \geq 2^{-2/3} \left(\sin \frac{2\pi}{2m+\frac{1}{2}} \right)^{1/3} - 1.07557b_{2m} \\
 &\geq \left[\left(\frac{\pi}{2} \right)^{1/3} \left(1 - \frac{1}{6} \left(\frac{2\pi}{2m+\frac{1}{2}} \right)^2 \right)^{1/3} - 1.07557 \times 0.93044 \right] \frac{1}{(2m+\frac{1}{2})^{1/3}}
 \end{aligned}$$

$$\begin{aligned} &\geq \left[\left(\frac{\pi}{2}\right)^{1/3} \left(1 - \frac{1}{6} \left(\frac{2\pi}{10.5}\right)^2\right)^{1/3} - 1.07557 \times 0.93044 \right] \frac{1}{(2m + \frac{1}{2})^{1/3}} \\ &\geq \frac{1.12603 - 1.07557 \times 0.93044}{(2m + \frac{1}{2})^{1/3}} > 0. \end{aligned}$$

Now we consider $m = 2, 3$. For $m = 2$, and $t \leq \frac{\pi}{2}$, we have

$$x = \frac{t + \pi}{4.5} \geq \frac{2\pi}{9}.$$

Thus ($n = 3$ or 4)

$$\begin{aligned} 2 \sin \frac{x}{2} C_n(x) n &\geq \sin \frac{\pi + 2\pi/9}{6} (2 \sin x)^{1/3} - b_4 \left[\sin \frac{\pi}{3} + \frac{8}{9\pi^2} \right] \\ &= \sin \frac{11\pi}{54} \left(2 \sin \frac{2\pi}{9}\right)^{1/3} - \frac{5}{9} \left[1 + \frac{8}{9\pi^2}\right] \\ &\geq 0.6493 - 0.6056 > 0. \end{aligned}$$

For $m = 2$ and $t \geq \frac{\pi}{2}$, we have

$$x = \frac{t + \pi}{4.5} \geq \frac{\pi}{3}.$$

Therefore,

$$2 \sin \frac{x}{2} C_n(x) \geq \sin \frac{\pi}{4} \left(2 \sin \frac{\pi}{3}\right)^{1/3} - \frac{5}{9} \left[1 + \frac{2}{9\pi} + \pi \frac{8}{9(3\pi/2)^2}\right] \geq 0.8491 - 0.6171 > 0.$$

For $m = 3$ we shall follow the same routine as $m = 2$. First assume that $t \leq \frac{\pi}{2}$. Then ($n = 5$, or 6)

$$x = \frac{t + \pi}{6.5} \geq \frac{2\pi}{13},$$

which follows

$$\frac{\pi + x}{6} \geq \frac{5\pi}{26}.$$

Therefore

$$\begin{aligned} 2 \sin \frac{x}{2} C_n(x) &\geq \sin \frac{5\pi}{26} \left(2 \sin \frac{2\pi}{13}\right)^{1/3} - b_6 \left[1 + \frac{8}{9\pi^2}\right] \\ &\geq 0.5543 - \frac{40}{81} \left[1 + \frac{8}{9\pi^2}\right] = 0.5543 - 0.5384 > 0. \end{aligned}$$

If $t \geq \frac{\pi}{2}$, then

$$x = \frac{t + \pi}{6.5} \geq \frac{3\pi}{13}.$$

Hence

$$2 \sin \frac{x}{2} C_n(x) \geq \sin \frac{8\pi}{39} \left(2 \sin \frac{3\pi}{13}\right)^{1/3} - \frac{40}{81} \left[1 + \frac{2\pi}{9} + \frac{32}{81\pi^2}\right] \geq 0.66 - 0.5486 > 0.$$

6. Proof of Theorem 1: $\frac{\pi}{2} \leq x \leq \pi$

In this section we consider $\frac{\pi}{2} \leq x < \pi$. First we assume that $n \geq 4$ and then we check for $n = 2$ and 3. It is trivial for $n = 0, 1$.

6.1. $n \geq 4$.

Assume that $n \geq 4$. Denote

$$D_k(x) = \sum_{j=0}^k \cos jx = \frac{1}{2} + \frac{\sin(k + \frac{1}{2})x}{2 \sin \frac{x}{2}}.$$

Then by means of summation by parts (we set $m = [\frac{n}{2}]$ and recall that $b_{2k} = b_{2k+1}$),

$$\begin{aligned} C_n(x) &= \sum_{k=0}^n b_k \cos kx \\ &= \sum_{k=0}^{n-1} (b_k - b_{k+1})D_k(x) + b_n D_n(x) \\ &= \sum_{k=0}^m (b_{2k} - b_{2k+2})D_{2k+1}(x) + b_n D_n(x) \\ &= \sum_{k=0}^{m-1} (b_{2k} - 2b_{2k+2} + b_{2k+4}) \times \\ &\quad \times \sum_{j=0}^k D_{2j+1}(x) + (b_{2m} - b_{2m+2}) \sum_{j=0}^m D_{2j+1}(x) + b_n D_n(x). \end{aligned} \tag{16}$$

It is easy to check that $b_{2k} - 2b_{2k+2} + b_{2k+4} > 0$. We show that $\sum_{j=0}^k D_{2j+1}(x) > 0$. We have

$$\begin{aligned} \sum_{j=0}^k D_{2j+1}(x) &= \frac{k+1}{2} + \frac{1}{2 \sin \frac{x}{2}} \sum_{j=0}^k k \sin(2j + \frac{1}{2})x \\ &= \frac{k+1}{2} + \frac{\cos \frac{x}{2} - \cos(2k + \frac{5}{2})x}{4 \sin \frac{x}{2} \sin x} \\ &= \frac{1}{4 \sin \frac{x}{2} \sin x} \left[2(k+1) \sin \frac{x}{2} \sin x + \cos \frac{x}{2} - \cos(2k + \frac{5}{2})x \right] \\ &= \frac{1}{4 \sin \frac{x}{2} \sin x} \left[(k+2) \cos \frac{x}{2} - (k+1) \cos \frac{3x}{2} - \cos(2k + \frac{5}{2})x \right]. \end{aligned}$$

Set $t = \pi - x$ and define

$$F_k(t) = (k+2) \sin \frac{t}{2} + (k+1) \sin \frac{3t}{2} - \sin(2k + \frac{5}{2})t.$$

Then $\sum_{j=0}^k D_{2j+1}(x)$ is a positive multiple of $F_k(t)$. We have $F'_k(0) = 0$, $F''_k(0) = 0$ and

$$F'''_k(t) = -\frac{k+2}{8} \cos \frac{t}{2} - \frac{27(k+1)}{8} \cos \frac{3t}{2} + \left(2k + \frac{5}{2}\right)^3 \cos\left(2k + \frac{5}{2}\right)t.$$

When $(2k + \frac{5}{2})t < \frac{2\pi}{5}$, i.e. $t < \frac{8\pi}{20k+25}$,

$$\begin{aligned} F'''_k(t) &> \left(2k + \frac{5}{2}\right)^3 \cos \frac{2\pi}{5} - \frac{28k+29}{8} \\ &> \left(8k^3 + 30k^2 + \frac{15}{2}k + \frac{125}{8}\right) \times 0.3 - (3.5k + 3.625) > 0 \end{aligned}$$

for all $k \geq 0$. Thus, for $0 < t < \frac{8\pi}{20k+25}$, $F_k(t) > 0$. When $\frac{8\pi}{20k+25} \leq t \leq \frac{\pi}{2}$,

$$\begin{aligned} F_k(t) &> (k+2) \left[\frac{4\pi}{20k+25} - \frac{1}{6} \left(\frac{4\pi}{20k+25} \right)^3 \right] \\ &\quad + (k+1) \left[\frac{12\pi}{20k+25} - \frac{1}{6} \left(\frac{12\pi}{20k+25} \right)^3 \right] - 1 > 0 \end{aligned}$$

for all $k \geq 0$. Therefore, for $\frac{\pi}{2} \leq x < \pi$, $\sum_{j=0}^k D_{2j+1}(x) > 0$. By this (16) can be simplified as following:

$$\begin{aligned} C_n(x) &> (b_0 - 2b_2 + b_4)D_1(x) + b_n D_n(x) \\ &= \frac{2}{9}(1 + \cos x) + b_n \left[\frac{1}{2} + \frac{\sin(k + \frac{1}{2})x}{2 \sin \frac{x}{2}} \right] \\ &\geq \frac{2}{9}(1 + \cos x) + b_n \left[\frac{1}{2} - \frac{1}{2 \sin \frac{x}{2}} \right] \\ &\geq \frac{2}{9}(1 + \cos x) + b_4 \left[\frac{1}{2} - \frac{1}{2 \sin \frac{x}{2}} \right] \\ &= \frac{2}{9}(1 + \cos x) + \frac{5}{9} \left[\frac{1}{2} - \frac{1}{2 \sin \frac{x}{2}} \right] \\ &= \frac{1}{18 \sin \frac{x}{2}} \left[4 \sin x \cos \frac{x}{2} + 5 \sin \frac{x}{2} - 5 \right]. \end{aligned}$$

Define

$$G(x) = 4 \sin x \cos \frac{x}{2} + 5 \sin \frac{x}{2} - 5.$$

Then $G(\pi/2) > 0$ and $G(\pi) = 0$. In addition,

$$G'(x) = 4 \cos x \cos \frac{x}{2} - 2 \sin x \sin \frac{x}{2} + \frac{5}{2} \cos \frac{x}{2} = \left(12 \cos^2 \frac{x}{2} - 10.5\right) \cos \frac{x}{2} < 0$$

for $\frac{\pi}{2} < x < \pi$. Hence $G(x) \geq 0$ for $\frac{\pi}{2} < x < \pi$. This proves $C_n(x) > 0$ on $[\frac{\pi}{2}, \pi)$ for $n \geq 4$.

6.2. $n = 2$ and 3 .

It remains to prove Theorem 1 for $n = 2$ and 3 on $[\frac{\pi}{2}, \pi)$. For $n = 2$,

$$\begin{aligned} C_2(x) &= 1 + \cos x + \frac{2}{3} \cos 2x \\ &= \frac{4}{3} \cos^2 x + \cos x + \frac{1}{3} \\ &= \frac{1}{3} \left(2 \cos x + \frac{3}{4} \right)^2 + \frac{1}{3} - \frac{3}{16} > 0. \end{aligned}$$

For $n = 3$, noting that $-1 < \cos x \leq 0$ when $\frac{\pi}{2} \leq x < \pi$,

$$\begin{aligned} C_3(x) &= C_2(x) + \frac{2}{3} \cos 3x \\ &= \frac{4}{3} \cos^2 x + \cos x + \frac{1}{3} + \frac{2}{3} (4 \cos^3 x - 3 \cos x) \\ &= \frac{8}{3} \cos^3 x + \frac{4}{3} \cos^2 x - \cos x + \frac{1}{3} \\ &\geq -\frac{8}{3} \cos^2 + \frac{4}{3} \cos^2 x + \cos^2 x + \frac{1}{3} \\ &= \frac{1 - \cos^2 x}{3} > 0. \end{aligned}$$

Now we have completed the proof of Theorem 1.

REFERENCES

- [1] G. BROWN, E. HEWITT, *A class of positive trigonometric sums*. Math. Ann. **268**, (1984), 91–22.
- [2] G. BROWN, K-Y. WANG AND D. WILSON, *Positivity of some basic cosine sums*. Math. Proc. Camb. Phil. Soc., **114**, (1993), 383–391.
- [3] G. BROWN, Q. YIN, *Positivity of a class of cosine sums*. Acta Sci. Math. (Szged), **67**, (2001), 221–247.
- [4] L. FEJÉR, *Einige Sätze, die sich auf das Vorzeichen einer ganzen rationalen Funktion beziehen; nebst Anwendungen dieser Sätze auf die Abschnitte und Abschnittsmittelwerte von ebenen und räumlichen harmonischen Entwicklungen und von beschränkten Potenzreihen*. Monatsh. Math. Physik, **35**, (1928), 305–344.
- [5] G. PÓLYA, G. SZEGÖ, *Problems and theorems in analysis*, Berlin, New York, Springer, 1972–76.
- [6] L. VIETORIS, *Über das Vorzeichen gewisser trigonometrischer Summen*. Sitzungsber. Öst. Akad. Wiss., **167**, (1958), 125–135.
- [7] W. H. YOUNG, *On certain series of Fourier*. Proc. London Math. Soc. 2 **11**, (1913), 357–366.
- [8] A. ZYGMUND, *Trigonometric Series, 2nd ed.*, Cambridge University Press, 1959.

(Received May 27, 2003)

Qinghe Yin
Mathematical Science Institute
Australian National University
ACT 0200
Australia

e-mail: Qinghe.Yin@maths.anu.edu.au