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TWO EXTRAPOLATION THEOREMS FOR RELATED WEIGHTS AND APPLICATIONS

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Abstract. In this paper we prove two extrapolation theorems for related weights. The theorems proved by C. Segovia and J. L. Torrea in [C. Segovia and J. L. Torrea, *Weighted inequalities for commutators of fractional and singular integrals*, Publ. Mat. **35**, (1991), 209–235] are adapted for one-sided weights. We apply these extrapolation theorems to improve weighted inequalities for commutators (with symbol b depending on the related weights) of several one-sided operators such as the Weyl and the Riemann-Liouville fractional integrals, or one-sided maximal operators given by the convolution with a smooth function. We also characterize the symbols b for which the commutators of these one-sided operators are bounded.

1. Introduction

Extrapolation theorems have been a very useful tool in Harmonic Analysis. Rubio de Francia developed extrapolation technics for the A_p Muckenhoupt classes of weights in 1984 ([16]). Several authors had obtained generalizations of these results or had adapted his technics to solve a great kind of problems referring to weighted inequalities (see [5], [6], [11], [18], [9], [3]).

In this paper we prove two extrapolation theorems for related weights. Before stating the results we need some definitions. Throughout this paper the letter C will be a positive constant, not necessarily the same at each occurrence and M will be the Hardy-Littlewood maximal function, $Mf(x) = \sup_{h>0} \frac{1}{2h}\int_{x-h}^{x+h}|f|$. If $1 \leqslant p \leqslant \infty$, then its conjugate exponent will be denoted by p'. By a weight we understand a nonnegative locally integrable function, and A_p will be the classical Muckenhoupt class of weights (see [14]). Also, given an interval $I = (x, x+h)$, $h > 0$, we will denote by $I^+ = (x+h, x+2h)$ and $I^- = (x-h, x)$.

DEFINITION 1.1. The one-sided Hardy-Littlewood maximal operators M^+ and M^- are defined for locally integrable functions f by

$$M^+f(x) = \sup_{h>0} \frac{1}{h}\int_x^{x+h}|f|, \quad \text{and} \quad M^-f(x) = \sup_{h>0} \frac{1}{h}\int_{x-h}^{x}|f|.$$

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The good weights for these operators are the one-sided weights, A_p^+ and A_p^- :

$$\sup_{a<b<c} \frac{1}{(c-a)^p} \int_a^b \omega \left(\int_b^c \omega^{1-p'} \right)^{p-1} < \infty, \quad 1 < p < \infty, \qquad (A_p^+)$$

$$M^- \omega(x) \leqslant C\omega(x) \quad \text{a.e.}, \qquad (A_1^+)$$

and

$$A_\infty^+ = \cup_{p\geqslant 1} A_p^+. \qquad (A_\infty^+)$$

The classes A_p^- are defined in a similar way. It is interesting to note that $A_p = A_p^+ \cap A_p^-$, $A_p \subsetneq A_p^+$ and $A_p \subsetneq A_p^-$. Also $w \in A_p^+$ if and only if $w^{1-p'} \in A_{p'}^-$, $1 < p < \infty$. (See [17], [10], [11], [12] for more definitions and results.)

DEFINITION 1.2. The one-sided maximal fractional operator M_γ^+, $0 < \gamma < 1$, is defined, for locally integrable functions f, by

$$M_\gamma^+ f(x) = \sup_{h>0} \frac{1}{h^{1-\gamma}} \int_x^{x+h} |f|.$$

It is proved in [2] that $||M_\gamma^+ f||_{L^q(w^q)} \leqslant C||f||_{L^p(w^p)}$ if and only if $w \in A^+(p,q)$, for $1 < p < 1/\gamma$, $1/p - 1/q = \gamma$, where

$$\left(\frac{1}{h} \int_{x-h}^x \omega^q \right)^{1/q} \left(\frac{1}{h} \int_x^{x+h} \omega^{-p'} \right)^{1/p'} \leqslant C, \qquad (A^+(p,q))$$

$$||\omega \chi_{[x-h,x]}||_\infty \left(\frac{1}{h} \int_x^{x+h} \omega^{-p'} \right)^{1/p'} \leqslant C, \qquad (A^+(p,\infty))$$

for all $h > 0$ and $x \in \mathbb{R}$. The classes $A^-(p,q)$ are defined in a similar way and also $A(p,q) = A^+(p,q) \cap A^-(p,q)$, for all $1 \leqslant p < \infty$ and $1 < q \leqslant \infty$.

Now we are ready to state the extrapolation results.

THEOREM 1.1. *Let ν be a weight and T a sublinear operator defined in $\mathcal{C}_c^\infty(\mathbb{R})$ (the set of $\mathcal{C}^\infty$ functions with compact support) and satisfying*

$$||\beta T f||_\infty \leqslant C||f\alpha||_\infty,$$

for all β and α, such that $\alpha = \nu\beta$, $\beta^{-1} \in A_1^-$ and $\alpha^{-1} \in A_1$.

Then, for $1 < p < \infty$,

$$||Tf||_{L^p(w)} \leqslant C||f||_{L^p(v)},$$

holds whenever $w \in A_p^+$ and $v = \nu^p w \in A_p$.

THEOREM 1.2. *Let ν be a weight, $p_0 > 1$ and T be a sublinear operator defined in $\mathcal{C}_c^\infty(\mathbb{R})$ such that*

$$||\beta Tf||_\infty \leqslant C||f\alpha||_{p_0},$$

for all β, α, such that $\beta \in A^+(p_0,\infty)$ and $\alpha = \nu\beta \in A(p_0,\infty)$.

Then, if $1 < p < p_0$ and q is such that $\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0}$, the inequality

$$||Tf||_{L^q(w^q)} \leqslant C||f||_{L^p(v^p)},$$

holds whenever $w \in A^+(p,q)$ and $v = \nu w \in A(p,q)$.

In section 2 we state and prove several applications for these theorems, and in section 3 we give the proof of the extrapolation results.

2. Applications

First we give some definitions.

DEFINITION 2.1. For $b \in L^1(\mathbb{R})$ and $\nu \in A_\infty$, we say that $b \in BMO_\nu$ if

$$||b||_{BMO_\nu} = \sup_I \frac{1}{\nu(I)} \int_I |b - b_I| < \infty,$$

where I denotes any bounded interval and $b_I = \frac{1}{|I|}\int_I b$. (For $\nu = 1$ we get the classical BMO space.)

Observe that $b \in BMO_\nu$ if and only if $\sup_I \frac{1}{\nu(I)}\int_I |b - b_{I^+}| < \infty$, or equivalently, $\sup_I \frac{1}{\nu(I)}\int_{I\cup I^+} |b - b_{I^+}| < \infty$.

DEFINITION 2.2. Let f be a locally integrable function. The one-sided sharp maximal function is defined by

$$f_{\#,+}(x) = \sup_{h>0} \frac{1}{h}\int_x^{x+h} \left(f(y) - \frac{1}{h}\int_{x+h}^{x+2h} f\right)^+ dy.$$

It is proved in [13] that

$$\begin{aligned} f_{\#,+}(x) &\leqslant \sup_{h>0}\inf_{a\in\mathbb{R}} \frac{1}{h}\int_x^{x+h} (f(y)-a)^+ dy + \frac{1}{h}\int_{x+h}^{x+2h} (a - f(y))^+ dy \\ &\leqslant C||f||_{BMO}. \end{aligned}$$

DEFINITION 2.3. Let $0 < \gamma < 1$. The Weyl fractional integral is defined by

$$I_\gamma^+ f(x) = \int_x^\infty \frac{f(y)}{(y-x)^{1-\gamma}} dy$$

and, for appropriate b, the commutator of the Weyl fractional integral is defined by

$$I_{\gamma,b}^+ f(x) = \int_x^\infty (b(x) - b(y)) \frac{f(y)}{(y-x)^{1-\gamma}} dy.$$

We shall also use for our purposes the following variant of the one-sided Hardy-Littlewood maximal operator:

DEFINITION 2.4. For $\varphi \in \mathcal{C}_c^\infty(-\infty,0]$, $\varphi \geqslant 0$ and nondecreasing in $(-\infty,0]$, let $\varphi_\varepsilon(x) = \varepsilon^{-1}\varphi(\varepsilon^{-1}x)$ for $\varepsilon > 0$. The maximal operator associated to φ is defined as

$$M_\varphi^+ f(x) = \sup_{\varepsilon>0} \varphi_\varepsilon * |f|(x).$$

It is not difficult to see that $M_\varphi^+ f$ is pointwise equivalent to $M^+ f$.

DEFINITION 2.5. Let φ be as in definition 2.4. For appropriate b we define the operators

$$M_{\varphi,b}^+ f(x) = \sup_{\varepsilon>0} \int_x^\infty |b(x)-b(y)|\varphi_\varepsilon(x-y)|f(y)|\,dy\,,$$

and

$$M_b^+ f(x) = \sup_{h>0} \frac{1}{h}\int_x^{x+h} |b(x)-b(y)||f(y)|\,dy\,.$$

Now we will give the definition of another maximal fractional operator.

DEFINITION 2.6. Let $0<\gamma<1$. Suppose $\varphi_\gamma \in \mathcal{C}^\infty((-\infty,0])$, $\varphi_\gamma \geqslant 0$, nondecreasing in $(-\infty,0]$ and such that $|\varphi_\gamma(x-y)-\varphi_\gamma(x)| \leqslant C|y||x|^{\gamma-2}$, for all x,y such that $|x|>2|y|$. The maximal operator associated to φ_γ is defined by

$$M_{\varphi_\gamma}^+ f(x) = \sup_{\varepsilon>0} \varphi_{\gamma,\varepsilon} * |f|(x).$$

DEFINITION 2.7. Let φ_γ as in definition 2.6. For appropriate b we define the operators

$$M_{\varphi_\gamma,b}^+ f(x) = \sup_{\varepsilon>0} \int_x^\infty |b(x)-b(y)|\varphi_{\gamma,\varepsilon}(x-y)|f(y)|\,dy$$

and

$$M_{\gamma,b}^+ f(x) = \sup_{h>0} \frac{1}{h^{1-\gamma}}\int_x^{x+h} |b(x)-b(y)||f(y)|\,dy\,.$$

Now we are ready to state the boundedness results for the operators just defined. The proofs are based on the extrapolation theorems of section 1.

In the next theorem we get a boundedness result for $M_{\varphi,b}^+$.

THEOREM 2.1. *Let φ be as in definition 2.4. Assume that $1<p<\infty$, $v\in A_p$, $w\in A_p^+$ are such that $\nu = \left(\frac{v}{w}\right)^{1/p} \in A_\infty$. Then, for $b \in BMO_\nu$, there exists $C>0$ such that*

$$\int_{\mathbb{R}} |M_{\varphi,b}^+ f|^p w \leqslant C \int_{\mathbb{R}} |f|^p v\,,$$

for all bounded f with compact support.

In the following theorem we get a boundedness result for the commutator of the one-sided fractional integral.

THEOREM 2.2. *Let* γ, p, q *be such that* $0<\gamma<1$, $1<p<\frac{1}{\gamma}$ *and* $\frac{1}{p}-\frac{1}{q}=\gamma$. *Assume that* $v\in A(p,q)$, $w\in A^{+}(p,q)$ *are such that* $\nu=\frac{v}{w}\in A_{\infty}$. *Then, for* $b\in BMO_{\nu}$, *there exists* $C>0$ *such that*

$$\int_{\mathbb{R}}|I^{+}_{\gamma,b}f|^{q}w^{q}\leqslant C\int_{\mathbb{R}}|f|^{p}v^{p},$$

for all bounded f *with compact support.*

Finally we state the result for $M^{+}_{\varphi_{\gamma},b}$.

THEOREM 2.3. *Let* γ, p, q *be such that* $0<\gamma<1$, $1<p<\frac{1}{\gamma}$ *and* $\frac{1}{p}-\frac{1}{q}=\gamma$. *Assume that* $v\in A(p,q)$, $w\in A^{+}(p,q)$ *and* $\nu=\frac{v}{w}\in A_{\infty}$. *Then, for* $b\in BMO_{\nu}$, *there exists* $C>0$ *such that*

$$\int_{\mathbb{R}}|M^{+}_{\varphi_{\gamma},b}f|^{q}w^{q}\leqslant C\int_{\mathbb{R}}|f|^{p}v^{p},$$

for all bounded f *with compact support.*

REMARK 1. Observe that the results in [8] are absolutely different. In [8] we dealt with only one weight (this allowed us to give results for commutators of higher order). On the other hand, we can not obtain the results in [8] (for order $k=1$) from the present Theorems since we can not take $w=v$.

REMARK 2. The results of Theorems 2.1, 2.2 and 2.3 for two-sided operators and related A_p weights are due to Segovia and Torrea (see [18] and [19]). The improvement in our theorems for the corresponding one-sided operators is that we take into consideration a wider class of weights. By taking $w\in A^{+}_{p}$ (or $w\in A^{+}(p,q)$), one improves not only on the left hand side of the inequality, but also on the right hand side. Notice the fact that $v=\nu^{p}w$ (or $v=\nu w$) gives

$$\int_{\mathbb{R}}|f|^{p}v=\int_{\mathbb{R}}|f\nu|^{p}w \quad (\text{ or } \quad \int_{\mathbb{R}}|f|^{p}v^{p}=\int_{\mathbb{R}}|f\nu|^{p}w^{p}).$$

An example showing that our class of weights is wider can be seen in [7].

REMARK 3. Theorems 2.1 and 2.2 in [7], i.e., the same result of Theorem 2.1, for one-sided singular integrals and for the one-sided discrete square function instead of M^{+}_{φ}, can be obtained applying the extrapolation Theorem 1.1 and following the same pattern as in the proof of our Theorem 2.1.

REMARK 4. Condition $b\in BMO_{\nu}$ is the natural one. Given $\nu\in A_{\infty}$, and assuming that there exists $w\in A^{+}_{p}$ with $v=\nu^{p}w\in A_{p}$, then, by factorization, it can be proved that $\nu\in A^{-}_{2}$ (see [11] and [20]). This fact, together with the doubling property for ν, easily gives that $\nu\in A_{2}$. It can be proved that $b\in BMO_{\nu}$ is necessary to obtain the boundedness of M^{+}_{b} and $M^{+}_{\gamma,b}$, $0<\gamma<1$. We shall state and prove this claim for M^{+}_{b}. In a similar way the same result can be obtained for $M^{+}_{\gamma,b}$.

THEOREM 2.4. *Let* $\nu \in A_2$ *and* $b \in L^1_{loc}(\mathbb{R})$. *The following conditions are equivalent:*

(*i*) M_b^+ *is bounded from* $L^p(\alpha)$ *to* $L^p(\beta)$, *for all* $1 < p < \infty$, $\alpha \in A_p$, $\beta \in A_p^+$ *such that* $\left(\frac{\alpha}{\beta}\right)^{1/p} = \nu$.

(*ii*) M_b^+ *is bounded from* $L^2(\nu)$ *to* $L^2(\nu^{-1})$.

(*iii*) $b \in BMO_\nu$.

Proof. $(iii) \Rightarrow (i)$ It is a consequence of Theorem 2.1.

$(i) \Rightarrow (ii)$ It is direct, by taking $p = 2$, $\alpha = \nu \in A_2$ and $\beta = \nu^{-1} \in A_2 \subset A_2^+$.

$(ii) \Rightarrow (iii)$ Recall that $b \in BMO_\nu$ is equivalent to prove that there exists C such that

$$\frac{1}{\nu(I)} \int_I |b - b_{I^+}| \leqslant C,$$

for any bounded interval I. Fixed I, let c be the right extreme of I^+. Then

$$\begin{aligned}\frac{1}{\nu(I)} \int_I |b(y) - b_{I^+}| dy &= \frac{1}{\nu(I)} \int_I \left| \frac{1}{|I^+|} \int_{I^+} (b(y) - b(x)) dx \right| dy \\ &\leqslant \frac{1}{\nu(I)} \int_I \frac{1}{|I^+|} \int_{I^+} |b(y) - b(x)|\, dx dy\ .\end{aligned}$$

Observe that, for $y \in I$,

$$\begin{aligned}\frac{1}{|I^+|} \int_{I^+} |b(x) - b(y)|\, dx &= \frac{1}{|I^+|} \int_y^c |b(x) - b(y)| \chi_{I^+}(x)\, dx \\ &\leqslant C M_b^+ \chi_{I^+}(y)\ .\end{aligned}$$

Therefore, by Hölder's inequality, (ii) and the fact that ν is doubling,

$$\begin{aligned}\frac{1}{\nu(I)} \int_I |b(y) - b_{I^+}|\, dy &\leqslant C \frac{1}{\nu(I)} \int_I M_b^+ \chi_{I^+}(y)\, dy \\ &\leqslant C \frac{1}{\nu(I)} \left(\int_I |M_b^+ \chi_{I^+}(y)|^2 \nu^{-1}(y) dy \right)^{1/2} \left(\int_I \nu \right)^{1/2} \\ &\leqslant C \frac{1}{\nu(I)} \left(\int_{\mathbb{R}} |\chi_{I^+}(y)|^2 \nu(y) dy \right)^{1/2} \left(\int_I \nu \right)^{1/2} \\ &= \frac{1}{\nu(I)} \left(\int_{I^+} \nu \right)^{1/2} \left(\int_I \nu \right)^{1/2} \leqslant C. \qquad \square\end{aligned}$$

To prove the above theorems we also need the following lemmas. The first one can be found in [10].

LEMMA 2.1. *Let* w *be a weight such that* $w^{-1} \in A_1^-$. *Then, there exists* $\varepsilon > 0$ *such that, for all* $1 \leqslant r \leqslant 1 + \varepsilon$, $w^{-r} \in A_1^- \subset A_r^-$ *and* $w^{r'} \in A_{r'}^+$.

LEMMA 2.2. *Assume* $b \in BMO_\nu$, $x \in \mathbb{R}$ *and* $h > 0$. *For each* $k \in \mathbb{Z}$, *set* $I_k = [x + 2^k h, x + 2^{k+1} h)$, *and* $J_k = [x, x + 2^{k+1} h)$. *Then for each* $l \in \mathbb{Z}$ *there exists* $\delta > 0$ *such that,*

$$|b_{J_{l-1}} - b_{I_k}| \leqslant C \frac{2^{k(1-\delta)}}{|J_{k-1}|} \int_{J_{k-1}} \nu \,,$$

for all $k > l$.

Proof. Fix $l \in \mathbb{Z}$ and set $I = J_{l-1}$ for simplicity. First of all observe that

$$|b_I - b_{I_k}| \leqslant |b_I - b_{I_l}| + \sum_{j=l}^{k-1} |b_{I_j} - b_{I_{j+1}}|.$$

Since $b \in BMO_\nu$ we get

$$\begin{aligned} |b_I - b_{I_l}| &= \left| \frac{1}{|I|} \int_I (b(x) - b_{I_l}) \, dx \right| \\ &\leqslant \frac{1}{|I|} \int_I |b(x) - b_{I_l}| \, dx \leqslant C \frac{\nu(I)}{|I|}. \end{aligned}$$

Then, using that $\nu \in A_\infty$, there exists $\delta > 0$ such that

$$\begin{aligned} |b_I - b_{I_l}| &\leqslant C \frac{\nu(I)}{\nu(J_{k-1})} \frac{|J_{k-1}|}{|I|} \frac{1}{|J_{k-1}|} \int_{J_{k-1}} \nu \\ &\leqslant C \left(\frac{|I|}{|J_{k-1}|} \right)^{\delta} \frac{|J_{k-1}|}{|I|} \frac{1}{|J_{k-1}|} \int_{J_{k-1}} \nu \\ &= C \left(\frac{|J_{k-1}|}{|I|} \right)^{1-\delta} \frac{1}{|J_{k-1}|} \int_{J_{k-1}} \nu \\ &= C \left(\frac{2^k}{2^l} \right)^{1-\delta} \frac{1}{|J_{k-1}|} \int_{J_{k-1}} \nu \\ &\leqslant C \frac{2^{k(1-\delta)}}{|J_{k-1}|} \int_{J_{k-1}} \nu \,. \end{aligned}$$

In the same way,

$$\begin{aligned} \sum_{j=l}^{k-1} |b_{I_j} - b_{I_{j+1}}| &\leqslant C \sum_{j=l}^{k-1} \frac{\nu(I_j)}{|I_j|} \\ &\leqslant C \sum_{j=l}^{k-1} \left(\frac{|J_{k-1}|}{|I_j|} \right)^{1-\delta} \frac{1}{|J_{k-1}|} \int_{J_{k-1}} \nu \\ &= C \sum_{j=l}^{k-1} \frac{(2^{1-\delta})^{k-j}}{|J_{k-1}|} \int_{J_{k-1}} \nu \\ &\leqslant C \frac{2^{k(1-\delta)}}{|J_{k-1}|} \int_{J_{k-1}} \nu \,. \qquad \square \end{aligned}$$

LEMMA 2.3. *Let $\nu \in A_\infty$. Assume that β and $\alpha = \nu\beta$ are such that $\beta^{-1} \in A_1^-$ and $\alpha^{-1} \in A_1$. Let $b \in BMO_\nu$. Then, there exists $\varepsilon > 0$ such that, for all $1 < r < 1+\varepsilon$,*

$$\left(\frac{1}{|I|}\int_I |b-b_I|^r \alpha^{-r}\right)^{1/r} \leqslant C\beta^{-1}(x)\ ,\ a.e.\ x \in I \cup I^+.$$

Proof. Since $\beta^{-1} \in A_1^-$ and $\alpha^{-1} \in A_1$, there exists $\varepsilon > 0$ such that $\beta^{-r} \in A_1^-$ and $\alpha^{-r} \in A_1$, for all $1 < r < 1+\varepsilon$. Let $s' > 1$ be such that $\alpha^{-r} \in RH_{s'}$ (see [12] and [15] for definition). Then, by Hölder's and John-Nirenberg's inequalities (see Prop. 6, Chap. III in [21]), we have that

$$\begin{aligned} \frac{1}{|I|}\int_I |b-b_I|^r \alpha^{-r} &\leqslant \left(\frac{1}{|I|}\int_I |b-b_I|^{rs}\right)^{\frac{1}{s}} \left(\frac{1}{|I|}\int_I \alpha^{-rs'}\right)^{\frac{1}{s'}} \\ &\leqslant C\left(\frac{\nu(I)}{|I|}\right)^r \frac{1}{|I|}\int_I \alpha^{-r}\ . \end{aligned} \tag{2.1}$$

Using now that $\nu \in A_\infty \subset A_\infty^+$, $\alpha^{-r} \in A_1 \subset A_r \subset A_r^+$ and $\beta^{-r} \in A_1^-$, Hölder's inequality gives,

$$\begin{aligned} \left(\frac{\nu(I)}{|I|}\right)^r \frac{1}{|I|}\int_I \alpha^{-r} &\leqslant \left(\frac{1}{|I|}\int_{I^+} \alpha\beta^{-1}\right)^r \frac{1}{|I|}\int_I \alpha^{-r} \\ &\leqslant \left(\frac{1}{|I|}\int_{I^+} \alpha^{r'}\right)^{\frac{r}{r'}} \frac{1}{|I|}\int_{I^+} \beta^{-r}\, \frac{1}{|I|}\int_I \alpha^{-r} \\ &\leqslant C\frac{1}{|I|}\int_{I^+} \beta^{-r} \leqslant C\,\beta^{-r}(x)\ , \end{aligned} \tag{2.2}$$

for almost every $x \in I \cup I^+$. Putting together inequalities (2.1) and (2.2) we get the desired result. □

We now pass to prove the theorems of this section.

Proof of Theorem 2.1. For $b \in L^\infty$ and f bounded of compact support we have that $M^+_{\varphi,b}f \in L^p(w)$. Using theorem 4 in [13],

$$\int_{\mathbb{R}} |M^+_{\varphi,b}f|^p w \leqslant C\int_{\mathbb{R}} |M^+(M^+_{\varphi,b}f)|^p w \leqslant C\int_{\mathbb{R}} |(M^+_{\varphi,b}f)_{\#,+}|^p w. \tag{2.3}$$

To prove the theorem for any $b \in BMO_\nu$ we proceed in the same way as in [7].
Let λ be an arbitrary constant. Then $b(x)-b(y) = (b(x)-\lambda)-(b(y)-\lambda)$ and

$$\begin{aligned} M^+_{\varphi,b}f(x) &= \sup_{\epsilon>0}\int_x^\infty |b(x)-b(y)|\varphi_\epsilon(x-y)|f(y)|\,dy \\ &\leqslant \sup_{\epsilon>0}|b(x)-\lambda|\int_x^\infty \varphi_\epsilon(x-y)|f(y)|\,dy \\ &\quad + \sup_{\epsilon>0}\int_x^\infty |\lambda-b(y)|\varphi_\epsilon(x-y)|f(y)|\,dy \\ &= |b(x)-\lambda|M^+_\varphi f(x) + M^+_\varphi((\lambda-b)f)(x). \end{aligned} \tag{2.4}$$

We will control $(M^+_{\varphi,b}f)_{\#,+}$ by sum of several one-sided maximal operators, which using Theorem 1.1, we shall prove that they are bounded from $L^p(v)$ to $L^p(w)$. Fix $x \in \mathbb{R}$ and $h > 0$. Set $J = [x, x+8h)$, $\lambda = b_J$ and write $f = f_1 + f_2$, where $f_1 = f\chi_J$. Then

$$\begin{aligned}
&\frac{1}{h}\int_x^{x+2h} |M^+_{\varphi,b}f(y) - M^+_\varphi((b-b_J)f_2)(x+2h)|\,dy \\
&\quad\leqslant \frac{1}{h}\int_x^{x+2h} |M^+_\varphi((b-b_J)f_1)(y)|\,dy \\
&\qquad + \frac{1}{h}\int_x^{x+2h} \left|M^+_\varphi((b-b_J)f_2)(y) - M^+_\varphi((b-b_J)f_2)(x+2h)\right|\,dy \\
&\qquad\quad + \frac{1}{h}\int_x^{x+2h} |b(y)-b_J||M^+_\varphi f(y)|\,dy \\
&\quad= I(x) + II(x) + III(x).
\end{aligned}$$

Observe that

$$II(x) \leqslant C\frac{1}{h}\int_x^{x+2h}\int_{x+8h}^{\infty} \frac{x+2h-y}{(z-(x+2h))^2}|b(z)-b_J||f(z)|\,dzdy, \tag{2.5}$$

by the conditions imposed on φ.

Consider the following sublinear operators:

$$M^+_1 f(x) = \sup_{h>0}\frac{1}{h}\int_x^{x+2h} |M^+_\varphi((b-b_J)f\chi_J)(y)|\,dy,$$
$$M^+_2 f(x) = \sup_{h>0}\frac{1}{h}\int_x^{x+2h}\int_{x+8h}^{\infty} \frac{x+2h-y}{(z-(x+2h))^2}|b(z)-b_J||f(z)|\,dzdy,$$

and

$$M^+_3 g(x) = \sup_{h>0}\frac{1}{h}\int_x^{x+2h} |b(y)-b_J||g(y)|\,dy$$

where, for each $h > 0$, J is the interval $[x, x+8h)$.

The above inequalities and definitions give that

$$(M^+_{\varphi,b}f)_{\#,+}(x) \leqslant C\left(M^+_1 f(x) + M^+_2 f(x) + M^+_3(M^+_\varphi f)(x)\right). \tag{2.6}$$

Boundedness of M^+_1:

Let $f \in C^\infty_c(\mathbb{R})$. Let β be a weight such that $\beta^{-1} \in A^-_1$ and, defining $\alpha = v\beta$, then $\alpha^{-1} \in A_1$. Using Hölder's inequality, the fact that M^+_φ is bounded from $L^r(\mathbb{R})$

to $L^r(\mathbb{R})$ (for all $1 < r < \infty$) and Lemma 2.3 we obtain,

$$\begin{aligned}\frac{1}{h}\int_x^{x+2h} |M_\varphi^+((b-b_J)f\chi_J)(y)|\,dy &\leqslant \left(\frac{1}{h}\int_x^{x+2h} |M_\varphi^+((b-b_J)f\chi_J)(y)|^r dy\right)^{1/r} \\ &\leqslant C\left(\frac{1}{h}\int_x^{x+8h} |b-b_J|^r|f|^r\,dy\right)^{1/r} \\ &\leqslant C||f\alpha||_\infty\left(\frac{1}{h}\int_J |b-b_J|^r\alpha^{-r}dy\right)^{1/r} \\ &\leqslant C||f\alpha||_\infty\beta^{-1}(x)\ .\end{aligned}$$

That is,

$$||\beta M_1^+ f||_\infty \leqslant C||f\alpha||_\infty\ .$$

So, by Theorem 1.1,

$$||M_1^+ f||_{L^p(w)} \leqslant C||f||_{L^p(v)} \tag{2.7}$$

holds, whenever $w \in A_p^+$ and $v = v^p w \in A_p$.

Boundedness of M_3^+:

Let $g \in C_c^\infty(\mathbb{R})$. Let β be a weight such that $\beta^{-1} \in A_1^-$, and such that defining $\alpha = v\beta$, then $\alpha^{-1} \in A_1$. By Hölder's inequality and Lemma 2.3,

$$\begin{aligned}\frac{1}{h}\int_x^{x+2h} |b(y)-b_J||g(y)|\,dy &\leqslant C||g\alpha||_\infty\frac{1}{8h}\int_x^{x+8h} |b(y)-b_J|\alpha^{-1}(y)\,dy \\ &\leqslant C||g\alpha||_\infty\left(\frac{1}{8h}\int_x^{x+8h} |b(y)-b_J|^r\alpha^{-r}(y)\,dy\right)^{\frac{1}{r}} \\ &\leqslant C||g\alpha||_\infty\beta^{-1}(x)\ .\end{aligned}$$

That is,

$$||\beta M_3^+ g||_\infty \leqslant C||g\alpha||_\infty\ .$$

By Theorem 1.1,

$$||M_3^+ g||_{L^p(w)} \leqslant C||g||_{L^p(v)}\ , \tag{2.8}$$

provided that $w \in A_p^+$ and $v = v^p\beta \in A_p$.

Boundedness of M_2^+:

Let $f \in C_c^\infty(\mathbb{R})$. Let β be a weight such that $\beta^{-1} \in A_1^-$, and such that defining $\alpha = v\beta$, then $\alpha^{-1} \in A_1$. For each $j \in \mathbb{N}$, write $I_j = [x+2^j h, x+2^{j+1}h)$ and $J_j = [x, x+2^{j+1}h)$. Then

$$\begin{aligned}&\frac{1}{h}\int_x^{x+2h}\int_{x+8h}^\infty \frac{x+2h-y}{(z-(x+2h))^2}|b(z)-b_J||f(z)|\,dzdy \\ &\qquad\leqslant C\frac{1}{h}\int_x^{x+2h} h\sum_{j=3}^\infty \int_{I_j}\frac{|b(z)-b_J|}{(z-(x+2h))^2}|f(z)|\,dzdy \\ &\qquad\leqslant Ch||f\alpha||_\infty\sum_{j=3}^\infty \frac{2^{j+1}}{(2^j-2)^2h^2}\frac{1}{2^{j+1}}\int_{I_j}|b(z)-b_J|\alpha^{-1}(z)\,dz\end{aligned}$$

$$\begin{aligned}&\leqslant C||f\alpha||_\infty \sum_{j=3}^{\infty} \frac{2^{j+1}}{(2^j-2)^2}\left(\frac{1}{2^{j+1}h}\int_{I_j}|b(z)-b_{I_j}|\alpha^{-1}(z)\,dz\right.\\&\qquad\left.+\frac{1}{2^{j+1}h}\int_{I_j}|b_{I_j}-b_J|\alpha^{-1}(z)\,dz\right) \qquad (2.9)\\&= C||f\alpha||_\infty \sum_{j=3}^{\infty} \frac{2^{j+1}}{(2^j-2)^2}\left(IV(x)+V(x)\right)\ .\end{aligned}$$

By Hölder's inequality and Lemma 2.3,

$$IV(x) \leqslant C\left(\frac{1}{2^jh}\int_{I_j}|b-b_{I_j}|^r\alpha^{-r}\right)^{1/r} \leqslant C\beta^{-1}(x)\ . \qquad (2.10)$$

On the other hand, using Lemma 2.2,

$$V(x) \leqslant |b_{I_j}-b_J|\frac{1}{2^{j+1}h}\int_{I_j}\alpha^{-1} \leqslant C\frac{2^{j(1-\delta)}}{|J_{j-1}|}\int_{J_{j-1}} v\ \frac{1}{2^{j+1}h}\int_{I_j}\alpha^{-1}\ .$$

Since $\alpha^{-1} \in A_1 \subset A_1^-$, observe that,

$$\frac{1}{2^{j+1}h}\int_{I_j}\alpha^{-1} \leqslant C\alpha^{-1}(y)\ ,$$

for almost all $y \in J_{j-1}$. Therefore, using again that $\beta^{-1} \in A_1^-$,

$$V(x) \leqslant C\frac{2^{j(1-\delta)}}{|J_{j-1}|}\int_{J_{j-1}} v\alpha^{-1} \leqslant C2^{j(1-\delta)}\beta^{-1}(x)\ . \qquad (2.11)$$

Put together inequalities (2.9), (2.10) and (2.11) to get,

$$\begin{aligned}M_2^+f(x) &\leqslant C||f\alpha||_\infty \sum_{j=3}^{\infty}\frac{2^{j+1}}{(2^j-2)^2}(\beta^{-1}(x)+2^{j(1-\delta)}\beta^{-1}(x))\\&\leqslant C\beta^{-1}(x)||f\alpha||_\infty\sum_{j=3}^{\infty}\left(\frac{1}{2^j}+\frac{1}{2^{j\delta}}\right)\\&\leqslant C\beta^{-1}(x)||f\alpha||_\infty\ .\end{aligned}$$

As before,

$$||\beta M_2^+f||_\infty \leqslant C||f\alpha||_\infty\ .$$

So, by Theorem 1.1,

$$||M_2^+f||_{L^p(w)} \leqslant C||f||_{L^p(v)} \qquad (2.12)$$

holds, whenever $w \in A_p^+$ and $v = v^pw \in A_p$. Going back to (2.3) and collecting (2.6), (2.7), (2.8) and (2.12), get

$$\begin{aligned}\int_{\mathbb{R}} |M^+_{\varphi,b} f|^p w &\leqslant C \int_{\mathbb{R}} |(M^+_{\varphi,b} f)_{\#,+}|^p w \\ &\leqslant C \int_{\mathbb{R}} (M^+_1 f + M^+_2 f + M^+_3 (M^+_{\varphi} f))^p w \\ &\leqslant C \left(||f||_{L^p(v)} + ||f||_{L^p(v)} + ||M^+_{\varphi} f||_{L^p(v)} \right) \\ &\leqslant C ||f||_{L^p(v)} . \qquad \square\end{aligned}$$

Proof of Theorem 2.2. Fix $b \in BMO_v$ and $\lambda \in \mathbb{R}$. Then, as in (2.4),

$$I^+_{\gamma,b} f(x) = I^+_{\gamma}((b-\lambda) f)(x) + (b(x)-\lambda) I^+_{\gamma} f(x) .$$

If $b \in L^\infty$ and f is bounded with compact support, then $I^+_{\gamma,b} f \in L^q(\beta^q)$, and by theorem 4 in [13],

$$\int_{\mathbb{R}} |I^+_{\gamma,b} f|^q \beta^q \leqslant C \int_{\mathbb{R}} |M^+(I^+_{\gamma,b} f)|^q \beta^q \leqslant C \int_{\mathbb{R}} |(I^+_{\gamma,b} f)_{\#,+}|^q \beta^q. \tag{2.13}$$

To prove the theorem for any $b \in BMO_v$ proceed as in [7].

Let us bound $(I^+_{\gamma,b} f)_{\#,+}$ pointwise. Fix $x \in \mathbb{R}$ and $h > 0$. Set $J = [x, x+8h)$ and write $f = f_1 + f_2$, where $f_1 = f \chi_J$. Then, with $\lambda = b_J$,

$$\begin{aligned}&\frac{1}{h} \int_x^{x+2h} \left| I^+_{\gamma,b} f(y) - I^+_{\gamma}((b-b_J) f_2)(x+2h) \right| dy \\ &\leqslant \frac{1}{h} \int_x^{x+2h} |I^+_{\gamma}((b-b_J) f_1)(y)| dy \\ &\qquad + \frac{1}{h} \int_x^{x+2h} \left| I^+_{\gamma}((b-b_J) f_2)(y) - I^+_{\gamma}((b-b_J) f_2)(x+2h) \right| dy \\ &\qquad + \frac{1}{h} \int_x^{x+2h} |b(y) - b_J| |I^+_{\gamma} f(y)| dy \\ &= I(x) + II(x) + III(x).\end{aligned} \tag{2.14}$$

It is clear that

$$III(x) \leqslant M^+_3 (I^+_{\gamma} f)(x),$$

where M^+_3 is as in the proof of Theorem 2.1. We already know that M^+_3 is bounded from $L^p(\alpha)$ to $L^p(\beta)$, whenever $\beta \in A^+_p$ and $\alpha = v^p \beta \in A_p$, $1 < p < \infty$. Since $w \in A^+(p,q)$ and $v \in A(p,q)$, then $w^q \in A^+_q$ and $v^q = v^q w^q \in A_q$, by [2]. So

$$||M^+_3 (I^+_{\gamma} f)||_{L^q(w^q)} \leqslant C ||I^+_{\gamma} f||_{L^q(v^q)} \leqslant C ||f||_{L^p(v^p)} ,$$

for all $f \in \mathcal{C}^\infty_c(\mathbb{R})$.

To control $I(x)$ let us define

$$M^+_4 f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+2h} |I^+_{\gamma}((b-b_J) f \chi_J)(y)| dy,$$

where, for each $x \in \mathbb{R}$ and $h > 0$, $J = [x, x+8h)$. Let us prove that M_4^+ is bounded from $L^p(v^p)$ to $L^q(w^q)$ using Theorem 1.2.

Let $\beta \in A^+(\frac{1}{\gamma}, \infty)$ and such that $\alpha = \nu\beta \in A(1/\gamma, \infty)$. Then $\beta^{\frac{-1}{1-\gamma}} \in A_1^-$ and $\alpha^{\frac{-1}{1-\gamma}} = (\nu\beta)^{\frac{-1}{1-\gamma}} \in A_1$. Therefore, there exists $t_1 > 1$ with the properties that $\alpha^{\frac{-t_1}{1-\gamma}} \in A_1$, $s = \frac{t_1}{1-\gamma} > 1$ and Lemma 2.3 holds for such s. Let r be such that $1/r - 1/s = \gamma$. Then, using Hölder's inequality, the fact that I_γ^+ is bounded from $L^r(\mathbb{R})$ to $L^s(\mathbb{R})$ and Lemma 2.3,

$$\begin{aligned}
&\frac{1}{h}\int_x^{x+2h} |I_\gamma^+((b-b_J)f\chi_J)(y)|dy \\
&\quad \leqslant \left(\frac{1}{h}\int_x^{x+2h} |I_\gamma^+((b-b_J)f\chi_J)(y)|^s dy\right)^{1/s} \\
&\quad \leqslant Ch^\gamma \left(\frac{1}{h}\int_x^{x+8h} |(b(y)-b_J)f(y)|^r \alpha^r \alpha^{-r} dy\right)^{1/r} \\
&\quad \leqslant Ch^\gamma \left(\frac{1}{h}\int_x^{x+8h} |b-b_J|^{r\frac{s}{r}} \alpha^{-r\frac{s}{r}}\right)^{\frac{1}{s}} \left(\frac{1}{h}\int_x^{x+8h} |f|^{\frac{1}{\gamma}} \alpha^{\frac{1}{\gamma}}\right)^{\gamma} \\
&\quad \leqslant C||f\alpha||_{\frac{1}{\gamma}} \beta^{-1}(x) \,.
\end{aligned} \tag{2.15}$$

As a consequence,

$$||\beta M_4^+ f||_\infty \leqslant C||f\alpha||_{\frac{1}{\gamma}} \,.$$

Then, by Theorem 1.2,

$$||M_4^+ f||_{L^q(w^q)} \leqslant C||f||_{L^p(v^p)},$$

whenever $w \in A^+(p,q)$, $v = \nu w \in A(p,q)$, $\frac{1}{p} - \frac{1}{q} = \gamma$ and $f \in \mathcal{C}_c^\infty(\mathbb{R})$.

Finally, let estimate

$$II(x) = \frac{1}{h}\int_x^{x+2h} \left|\int_{x+8h}^\infty \sigma(t,y)dt\right| dy,$$

where

$$\sigma(t,y) = (b(t)-b_J)f(t)\left(\frac{1}{(t-y)^{1-\gamma}} - \frac{1}{(t-(x+2h))^{1-\gamma}}\right).$$

Consider the following sublinear operator in $\mathcal{C}_c^\infty(\mathbb{R})$:

$$M_5^+ f(x) = \sup_{h>0} \frac{1}{h}\int_x^{x+2h} \left|\int_{x+8h}^\infty \sigma(t,y)dt\right| dy.$$

For each $j \in \mathbb{N}$, set $I_j = [x+2^j h, x+2^{j+1}h)$ and $J_j = [x, x+2^{j+1}h)$. Then, by the mean value theorem,

$$
\begin{aligned}
\frac{1}{h}\int_x^{x+2h}\left|\int_{x+8h}^{\infty}\sigma(t,y)dt\right|dy &\leqslant \frac{1}{h}\int_x^{x+2h}\sum_{j=3}^{\infty}\int_{I_j}|\sigma(t,y)|\,dtdy \\
&\leqslant C\sum_{j=3}^{\infty}\int_{I_j}|b(t)-b_J||f(t)|\frac{2h}{(h(2^j-2))^{2-\gamma}}dt \\
&\leqslant C\sum_{j=3}^{\infty}\frac{h^{\gamma}}{(2^{j-1})^{1-\gamma}}\left(\frac{2}{2^{j-1}h}\int_{I_j}|b(t)-b_J||f(t)|dt\right) \\
&\leqslant C\sum_{j=3}^{\infty}\frac{h^{\gamma}}{(2^{j-1})^{1-\gamma}}\left(\frac{2}{2^{j-1}h}\int_{I_j}|b(t)-b_{I_j}||f(t)|dt\right) \\
&\quad + C\sum_{j=3}^{\infty}\frac{h^{\gamma}}{(2^{j-1})^{1-\gamma}}\left(\frac{2}{2^{j-1}h}\int_{I_j}|b_{I_j}-b_J||f(t)|dt\right) \\
&\leqslant C\sum_{j=3}^{\infty}\frac{h^{\gamma}}{(2^{j-1})^{1-\gamma}}(IV(x)+V(x))\ .
\end{aligned} \tag{2.16}
$$

Let $\beta\in A^+(\frac{1}{\gamma},\infty)$ and such that $\alpha=\nu\beta\in A(\frac{1}{\gamma},\infty)$. Then $\alpha^{\frac{-1}{1-\gamma}}\in A_1$ which implies that $\alpha^{-1}\in A_1$. Choose $r>1-\gamma$ such that Lemma 2.3 holds for $\frac{r}{1-\gamma}$. Then, by Hölder's inequality,

$$
\begin{aligned}
IV(x) &\leqslant \left(\frac{2}{2^{j-1}h}\int_{I_j}|f|^{\frac{1}{\gamma}}\alpha^{\frac{1}{\gamma}}\right)^{\gamma}\left(\frac{2}{2^{j-1}h}\int_{I_j}|b(t)-b_{I_j}|^{\frac{1}{1-\gamma}}\alpha^{\frac{-1}{1-\gamma}}\right)^{1-\gamma} \\
&\leqslant C(2^jh)^{-\gamma}||f\alpha||_{\frac{1}{\gamma}}\left(\frac{1}{2^jh}\int_{I_j}|b(t)-b_{I_j}|^{\frac{r}{1-\gamma}}\alpha^{\frac{-r}{1-\gamma}}\right)^{\frac{1-\gamma}{r}} \\
&\leqslant C(2^jh)^{-\gamma}||f\alpha||_{\frac{1}{\gamma}}\beta^{-1}(x)\ .
\end{aligned} \tag{2.17}
$$

Using again Lemma 2.2 and Hölder's inequality,

$$
\begin{aligned}
V(x) &\leqslant \frac{1}{2^jh}|b_{I_j}-b_J|\int_{I_j}|f(t)|dt \\
&\leqslant C\frac{2^{j(1-\delta)}}{|J_{j-1}|}\int_{J_{j-1}}\nu\left(\frac{1}{2^jh}\int_{I_j}|f|^{\frac{1}{\gamma}}\alpha^{\frac{1}{\gamma}}\right)^{\gamma}\left(\frac{1}{2^jh}\int_{I_j}\alpha^{\frac{-1}{1-\gamma}}\right)^{1-\gamma} \\
&\leqslant C2^{j(1-\delta)}(2^jh)^{-\gamma}||f\alpha||_{\frac{1}{\gamma}}\frac{1}{|J_{j-1}|}\int_{J_{j-1}}\nu\left(\frac{1}{2^jh}\int_{I_j}\alpha^{\frac{-1}{1-\gamma}}\right)^{1-\gamma}\ .
\end{aligned}
$$

Since $\alpha^{\frac{-1}{1-\gamma}}\in A_1\subset A_1^-$,

$$
\frac{1}{2^jh}\int_{I_j}\alpha^{\frac{-1}{1-\gamma}}\leqslant C\alpha^{\frac{-1}{1-\gamma}}(y)\ ,
$$

for all $y \in J_{j-1}$. Then, using now that $\beta^{-1} \in A_1^-$,

$$\begin{aligned} V(x) &\leqslant C2^{j(1-\delta)}(2^j h)^{-\gamma}||f\alpha||_{\frac{1}{\gamma}} \frac{1}{|J_{j-1}|}\int_{J_{j-1}} v\alpha^{-1} \\ &\leqslant C2^{j(1-\delta)}(2^j h)^{-\gamma}||f\alpha||_{\frac{1}{\gamma}}\beta^{-1}(x) . \end{aligned} \tag{2.18}$$

Put together inequalities (2.16), (2.17) and (2.18), to get

$$\begin{aligned} &\frac{1}{h}\int_x^{x+2h}\left|\int_{x+8h}^{\infty}\sigma(t,y)dt\right|dy \\ &\leqslant C||f\alpha||_{\frac{1}{\gamma}}\beta^{-1}(x)\sum_{j=3}^{\infty}\frac{h^\gamma}{(2^{j-1})^{1-\gamma}}\left((2^j h)^{-\gamma}+2^{j(1-\delta)}(2^j h)^{-\gamma}\right) \\ &\leqslant C||f\alpha||_{\frac{1}{\gamma}}\beta^{-1}(x)\sum_{j=3}^{\infty}\left(\frac{1}{2^j}+\frac{1}{2^{j\delta}}\right) \\ &\leqslant C||f\alpha||_{\frac{1}{\gamma}}\beta^{-1}(x) . \end{aligned}$$

Taking supremum first on $h>0$ and then on $x \in \mathbb{R}$, get,

$$||\beta M_5^+ f||_\infty \leqslant C||f\alpha||_{\frac{1}{\gamma}} ,$$

So, by Theorem 1.2,

$$||M_5^+ f||_{L^q(w^q)} \leqslant C||f||_{L^p(v^p)} ,$$

whenever $w \in A^+(p,q)$ and $v = vw \in A(p,q)$, $\frac{1}{p}-\frac{1}{q}=\gamma$ and $f \in \mathcal{C}_c^\infty(\mathbb{R})$. □

The proof of Theorem 2.3 follows the same pattern as Theorem 2.2, so we omit it.

3. Proof of the extrapolation theorems

To prove our results on extrapolation, we need the following lemma (see [16] and [18]).

LEMMA 3.1. *Let μ be a weight, and let $1<r<\infty$. Let $W \in A_r^-$, such that $\mu^r W \in A_r$. Then, for all $u \in L^{r'}(W)$, there exists $U \in L^{r'}(W)$ such that*
- (i) $u(x) \leqslant U(x)$, *a.e.*
- (ii) $||U||_{L^{r'}(W)} \leqslant C||u||_{L^{r'}(W)}$,
- (iii) $UW \in A_1^-$ *and* $U\mu W \in A_1$.

Proof. Define first the following operator

$$S(h) = W^{-1}M^+(Wh) + (\mu W)^{-1}M(\mu Wh) .$$

We claim that this operator is bounded from $L^{r'}(W)$ to $L^{r'}(W)$. Indeed, observe that $W \in A_r^-$ if, and only if, $W^{1-r'} \in A_{r'}^+$. Then M^+ is bounded from $L^{r'}(W^{1-r'})$ to $L^{r'}(W^{1-r'})$. Also, $\mu^r W \in A_r$ implies that $(\mu^r W)^{1-r'} = \mu^{-r'}W^{1-r'} \in A_{r'}$. Therefore M is bounded from $L^{r'}(\mu^{-r'}W^{1-r'})$ to $L^{r'}(\mu^{-r'}W^{1-r'})$. As a consequence,

$$\begin{aligned}\int (S(h))^{r'} W &\leqslant C \int W^{-r'} (M^+(Wh))^{r'} W + C \int (\mu W)^{-r'} (M(\mu W h))^{r'} W \\ &= C \int (M^+(Wh))^{r'} W^{1-r'} + C \int (M(\mu W h))^{r'} \mu^{-r'} W^{1-r'} \\ &\leqslant C \int (Wh)^{r'} W^{1-r'} + C \int (\mu W h)^{r'} \mu^{-r'} W^{1-r'} \\ &= C \int |h|^{r'} W .\end{aligned}$$

Then, by lemma 5.1 in [4], given $u \in L^{r'}(W)$, there exists $U \in L^{r'}(W)$ such that $||U||_{L^{r'}(W)} \leqslant 2||u||_{L^{r'}(W)}$, $U(x) \geqslant u(x)$ and $S(U)(x) \leqslant CU(x)$ a.e. $x \in \mathbb{R}$. Then

$$W^{-1}M^+(WU)(x) \leqslant CU(x) \text{ and } (\mu W)^{-1}M^+(\nu W U)(x) \leqslant CU(x),$$

a.e. $x \in \mathbb{R}$. In other words, $WU \in A_1^-$ and $\mu W U \in A_1$. $\square$

Proof of Theorem 1.1. Fix $w \in A_p^+$, such that $v = \nu^p w \in A_p$. Let $f \in L^p(v)$ and consider

$$g(x) = \begin{cases} \dfrac{w^{-\frac{1-p'}{p}}(x)|f(x)|\nu(x)w^{1/p}(x)}{||f||_{L^p(v)}}, & \text{if } f(x) \neq 0, \\ w^{-\frac{1-p'}{p}}(x)e^{-\frac{\pi x^2}{p}}, & \text{if } f(x) = 0. \end{cases}$$

Observe that $g \in L^p(w^{1-p'})$ and $||g||_{L^p(w^{1-p'})} \leqslant 2$. On the other hand, $w^{1-p'} \in A_{p'}^-$ and $\nu^{-p'} w^{1-p'} \in A_{p'}$. Then, by Lemma 3.1, there exists $G \in L^p(w^{1-p'})$, such that

(i) $g(x) \leqslant G(x)$, a.e.

(ii) $||G||_{L^p(w^{1-p'})} \leqslant C||g||_{L^p(w^{1-p'})}$,

(iii) $Gw^{1-p'} \in A_1^-$ and $G\nu^{-1}w^{1-p'} \in A_1$.

Let $\beta = (Gw^{1-p'})^{-1}$. Then $\beta^{-1} \in A_1^-$ and, defining $\alpha = \nu\beta$, we also have $\alpha^{-1} = (\nu\beta)^{-1} \in A_1$ and

$$||\beta T f||_\infty \leqslant C||f\alpha||_\infty .$$

So,

$$\begin{aligned}||f||_{L^p(v)} &= ||g^{-1}w^{-\frac{1-p'}{p}}|f|v^{\frac{1}{p}}||_\infty \\ &\geqslant ||G^{-1}w^{-\frac{1-p'}{p}}|f|\nu w^{\frac{1}{p}}||_\infty \\ &= ||(Gw^{1-p'})^{-1}\nu|f|\,||_\infty = ||f\alpha||_\infty \\ &\geqslant C||\beta T f||_\infty = C||G^{-1}w^{p'-1}Tf||_\infty \\ &\geqslant C\left(\int G^p w^{1-p'}\right)^{1/p} ||G^{-1}w^{p'-1}Tf||_\infty \\ &\geqslant C\left(\int G^{-p}w^{p(p'-1)}|Tf|^p G^p w^{1-p'}\right)^{1/p} \\ &= C||Tf||_{L^p(w)} . \qquad \square\end{aligned}$$

Proof of Theorem 1.2. Fix $1<p<p_0$, q such that $\frac{1}{p}-\frac{1}{q}=\frac{1}{p_0}$ and $w\in A^+(p,q)$ such that $v=\nu w\in A(p,q)$. Observe that, for $r=1+q/p'$ and $s=1+p'/q$ ($s'=q/p_0'$), we have that $w^q\in A_r^+$, $v^q\in A_r$, $w^{-p'}\in A_s^-$ and $v^{-p'}=\nu^{-p'}w^{-p'}\in A_s$. Let $f\in L^p(v^p)$ and consider

$$h(x)=\left(\frac{|f(x)|\nu(x)w^{p'}(x)}{||f||_{L^p(v^p)}}\right)^{\frac{pp_0'}{q}}.$$

Observe that $h\in L^{s'}(w^{-p'})$. In fact $||h||_{L^{s'}(w^{-p'})}=1$. Therefore, by Lemma 3.1, there exists $H\in L^{s'}(w^{-p'})$ such that

(i) $h(x)\leqslant H(x)$, a.e.

(ii) $||H||_{L^{s'}(w^{-p'})}\leqslant C||h||_{L^{s'}(w^{-p'})}=1$,

(iii) $Hw^{-p'}\in A_1^-$ and $H\nu^{\frac{-p'}{s}}w^{-p'}\in A_1$.

Let $\beta=(Hw^{-p'})^{-\frac{1}{p_0'}}$ and consider $\alpha=\nu\beta$. Then $\beta\in A^+(p_0,\infty)$ and $\alpha\in A(p_0,\infty)$. As a consequence,

$$\begin{aligned}
||f||_{L^p(v^p)} &= \left(\int|f|^{p_0}\left(h^{-1/p_0'}\nu w^{p'/p_0'}\right)^{p_0}\right)^{1/p_0}\\
&\geqslant \left(\int|f|^{p_0}\left(H^{-1/p_0'}\nu w^{p'/p_0'}\right)^{p_0}\right)^{1/p_0}\\
&= ||f||_{L^{p_0}(\alpha^{p_0})}\geqslant C||\beta Tf||_\infty\\
&= C||TfH^{-1/p_0'}w^{p'/p_0'}||_\infty\\
&\geqslant C||TfH^{-1/p_0'}w^{p'/p_0'}||_\infty\left(\int H^{q/p_0'}w^{-p'}\right)^{1/q}\\
&\geqslant C\left(\int|Tf|^qH^{-q/p_0'}w^{qp'/p_0'}H^{q/p_0'}w^{-p'}\right)^{1/q}\\
&= C||Tf||_{L^q(w^q)}. \qquad \square
\end{aligned}$$

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