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BESSEL POTENTIAL SPACES WITH VARIABLE EXPONENT

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Abstract. We show that a variable exponent Bessel potential space coincides with the variable exponent Sobolev space if the Hardy-Littlewood maximal operator is bounded on the underlying variable exponent Lebesgue space. Moreover, we study the Hölder type quasi-continuity of Bessel potentials of the first order.

1. Introduction

The (classical) Bessel potential space $\mathcal{L}^{\alpha,p}(\mathbb{R}^n)$, $1 < p < \infty$, consists of all functions u,

$$u = g_\alpha * f, \text{ where } f \in L^p(\mathbb{R}^n).$$

Here g_α is the Bessel kernel of the order $\alpha \geqslant 0$. It is well know that when α is a natural number the space $\mathcal{L}^{\alpha,p}(\mathbb{R}^n)$ (with the norm of u defined as $\|f\|_{L^p(\mathbb{R}^n)}$) coincides with the Sobolev space $W^{\alpha,p}(\mathbb{R}^n)$ and the corresponding norms are equivalent. The aim of this paper is to study this question in variable exponent case, that is, when the exponent p is a measurable function $p\colon \mathbb{R}^n \to [p_*, p^*]$, $1 < p_* \leqslant p^* < \infty$.

If the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is defined (see the next section), the variable exponent Sobolev space $W^{k,p(\cdot)}(\mathbb{R}^n)$ consists of all measurable functions $u \in L^{p(\cdot)}(\mathbb{R}^n)$ whose distributional derivatives up to the order k belong to $L^{p(\cdot)}(\mathbb{R}^n)$. These spaces have attracted steadily increasing interest over the past five years. The research was motivated by the differential equations with non-standard growth and coercivity conditions arising from modeling certain fluids called electrorheological (cf. [21]).

We define the variable exponent Bessel potential space $\mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^n)$ as in the classical situation. Assuming that the Hardy-Littlewood maximal operator M is bounded on the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ we show that the variable exponent Bessel potential space $\mathcal{L}^{k,p(\cdot)}(\mathbb{R}^n)$ and the Sobolev space $W^{k,p(\cdot)}(\mathbb{R}^n)$, $k \in \mathbb{N}$, coincide and their norms are equivalent.

As an application we study the Hölder type quasi-continuity of Bessel potentials of the first order. More precisely, we show that each function $u \in \mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n)$ coincides

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pointwise outside a small set (measured by the Bessel capacity) with a Hölder continuous function $w \in \mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n)$ and the norm of the difference $u - w$ in $\mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n)$ is small.

2. Variable exponent spaces

Let G be a measurable subset of $\mathbb{R}^n$ (with respect to n-dimensional Lebesgue measure), by $|G|$ we mean its n-volume, and χ_G will represent the characteristic function of G. For $r \in (0, \infty)$ and $x \in \mathbb{R}^n$ let $B(x, r)$ denote the open ball in $\mathbb{R}^n$ of radius r and center x.

By the symbol $\mathcal{P}(\mathbb{R}^n)$ we denote the family of all measurable functions $p(\cdot) : \mathbb{R}^n \to [1, \infty]$. For $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ put

$$p_* := \operatorname*{ess\,inf}_{x \in \mathbb{R}^n} p(x), \quad p^* := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} p(x).$$

Furthermore, we introduce a class $\mathcal{B}(\mathbb{R}^n)$ by

$$\mathcal{B}(\mathbb{R}^n) := \{p \in \mathcal{P}(\mathbb{R}^n); 1 < p_* \leqslant p^* < \infty\}.$$

Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Consider the functional

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, \mathrm{d}x$$

on all measurable functions f on $\mathbb{R}^n$. The *Lebesgue space with variable exponent* $L^{p(\cdot)}(\mathbb{R}^n)$ is defined as the set of all measurable functions f on $\mathbb{R}^n$ such that, for some $\lambda > 0$,

$$\varrho_{p(\cdot)}(f/\lambda) < \infty,$$

equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\{\, \lambda > 0; \varrho_{p(\cdot)}(f/\lambda) \leqslant 1 \,\}. \tag{2.1}$$

Recall that (cf. [14, (2.9) and (2.10)]),

$$\varrho_{p(\cdot)}(f/\|f\|_{p(\cdot)}) = 1 \quad \text{for every } f \text{ with } 0 < \|f\|_{p(\cdot)} < \infty, \tag{2.2}$$

$$\text{if} \quad \|f\|_{p(\cdot)} \leqslant 1 \quad \text{then} \quad \varrho_{p(\cdot)}(f) \leqslant \|f\|_{p(\cdot)}, \tag{2.3}$$

$$\varrho_{p(\cdot)}(f) \leqslant 1 \quad \text{if and only if} \quad \|f\|_{p(\cdot)} \leqslant 1, \tag{2.4}$$

$$\varrho_{p(\cdot)}(f) \to 0 \quad \text{if and only if} \quad \|f\|_{p(\cdot)} \to 0. \tag{2.5}$$

The *Hardy-Littlewood maximal operator* M is defined on locally integrable functions f on $\mathbb{R}^n$ by the formula

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, \mathrm{d}y. \tag{2.6}$$

DEFINITION 2.1. By $\mathcal{M}(\mathbb{R}^n)$ denote the class of all functions $p \in \mathcal{B}(\mathbb{R}^n)$ for which the operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, that is,

$$\|Mf\|_{p(\cdot)} \leqslant C\|f\|_{p(\cdot)} \tag{2.7}$$

with a positive constant C independent of f.

REMARK 2.2. For example, $p(\cdot) \in \mathcal{M}(\mathbb{R}^n)$ if the following two conditions are satisfied:

$$\begin{aligned} |p(x)-p(y)| &\leqslant \frac{c}{-\log(|x-y|)}, \quad |x-y| \leqslant 1/2, \\ |p(x)-p(y)| &\leqslant \frac{c}{\log(e+|x|)}, \quad |y| > |x|. \end{aligned} \tag{2.8}$$

For more details see [2], [3], [5], [15], [17] and [18] where various sufficient conditions for $p(\cdot) \in \mathcal{M}(\mathbb{R}^n)$ can be found.

Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $k \in \mathbb{N}$. We define the *Sobolev space with variable exponent* $W^{k,p(\cdot)}(\mathbb{R}^n)$ by

$$W^{k,p(\cdot)}(\mathbb{R}^n) := \{u;\ D^\beta u \in L^{p(\cdot)}(\mathbb{R}^n) \quad \text{if } |\beta| \leqslant k\},$$

equipped with the norm

$$\|u\|_{W^{k,p(\cdot)}} = \sum_{|\beta| \leqslant k} \|D^\beta u\|_{p(\cdot)},$$

where $\beta \in \mathbb{N}_0^n$ is a multi-index, $|\beta| = \beta_1 + \cdots + \beta_n$ and $D^\beta = \dfrac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \ldots \partial x_n^{\beta_n}}$.

The *Bessel kernel* g_α of order α, $\alpha > 0$, is defined by

$$g_\alpha(x) = \frac{\pi^{n/2}}{\Gamma(\alpha/2)} \int_0^\infty e^{-s-(\pi^2|x|^2)/s} s^{(\alpha-n)/2} \frac{\mathrm{d}s}{s}, \quad x \in \mathbb{R}^n. \tag{2.9}$$

Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $\alpha \geqslant 0$. The *Bessel potential space with variable exponent* $\mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^n)$ is, for $\alpha > 0$, defined by

$$\mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^n) := \{u = g_\alpha * f;\ f \in L^{p(\cdot)}(\mathbb{R}^n)\},$$

and is equipped with the norm

$$\|u\|_{\alpha;p(\cdot)} := \|f\|_{p(\cdot)}. \tag{2.10}$$

If $\alpha = 0$ we put $g_0 * f := f$ and $\mathcal{L}^{0,p(\cdot)}(\mathbb{R}^n) := L^{p(\cdot)}(\mathbb{R}^n)$ (normed by (2.1)).

We write $A \lesssim B$ (or $A \gtrsim B$) if $A \leqslant cB$ (or $cA \geqslant B$) for some positive constant c independent of appropriate quantities involved in the expressions A and B, and $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$.

3. Relationship between Sobolev and Bessel potential spaces

The main result of this section is the following theorem.

THEOREM 3.1. *Let $p \in \mathcal{M}(\mathbb{R}^n)$ and let $k \in \mathbb{N}$. Then*

$$\mathcal{L}^{k,p(\cdot)}(\mathbb{R}^n) = W^{k,p(\cdot)}(\mathbb{R}^n)$$

and the corresponding norms are equivalent.

Before we prove the main theorem we shall need some auxiliary results. First we introduce some notation.

If f belongs to the *Schwartz class* $\mathcal{S}$, the *Fourier transform* of f is the function $\mathcal{F}f$ or $\widehat{f}$ defined by

$$\mathcal{F}f(x) = \widehat{f}(x) = \int_{\mathbb{R}^n} f(y)\, e^{-2\pi i x\cdot y}\, \mathrm{d}y.$$

Let us summarize the basic properties of the Bessel kernel g_α, $\alpha > 0$:

$$g_\alpha \quad \text{is nonnegative, radially decreasing and} \quad \int_{\mathbb{R}^n} g_\alpha(y)\,\mathrm{d}y = 1, \tag{3.1}$$

$$\widehat{g}_\alpha(\xi) = (1+|\xi|^2)^{-\alpha/2}, \quad \xi \in \mathbb{R}^n, \tag{3.2}$$

$$g_\alpha * g_\beta = g_{\alpha+\beta}, \quad \alpha,\beta > 0.$$

Let δ_0 denotes the *Dirac delta measure* at zero. For $\alpha > 0$ we define the measure μ_α on measurable sets $E \subset \mathbb{R}^n$ by

$$\mu_\alpha(E) = \delta_0(E) + \sum_{k=1}^{\infty} b(\alpha,k) \int_E g_{2k}(y)\,\mathrm{d}y, \tag{3.3}$$

where $b(\alpha,k) = (-1)^k \binom{\alpha/2}{k} = \frac{(-1)^k}{k!} \prod_{j=0}^{k-1} ((\alpha/2)-j)$, $k = 1,2,\dots$. Since

$$\sum_{k=1}^{\infty} |b(\alpha,k)| < \infty, \tag{3.4}$$

the measure μ_α is a finite signed Borel measure on $\mathbb{R}^n$. For $\alpha = 0$ we set $\mu_0 = \delta_0$. This construction uses the Taylor expansion of the function $t \mapsto (1-t)^{\alpha/2}$, $\alpha > 0$, $t \in (0,1]$, to give

$$\frac{|x|^\alpha}{(1+|x|^2)^{\alpha/2}} = \Big(1 - \frac{1}{1+|x|^2}\Big)^{\alpha/2} = 1 + \sum_{k=1}^{\infty} b(\alpha,k)\,(1+|x|^2)^{-2k/2}, \quad x \in \mathbb{R}^n,$$

which implies that (for $\alpha > 0$)

$$\widehat{\mu_\alpha}(x) = \frac{|x|^\alpha}{(1+|x|^2)^{\alpha/2}}. \tag{3.5}$$

Obviously, (3.5) holds for $\alpha = 0$, too. (For more details see [19, p. 32] and [23, p. 134].)

We define the *Riesz transform* $\mathcal{R}_j f$, $j = 1,\dots,n$, of a function $f \in \mathcal{S}$ by the formula

$$\mathcal{R}_j f(x) = \big(\tfrac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}\big) \lim_{\varepsilon\to 0_+} \int_{|y|>\varepsilon} \frac{y_j}{|y|^{n+1}} f(x-y)\,\mathrm{d}y.$$

Recall that (cf. [23]),

$$\mathcal{F}(\mathcal{R}_j f)(x) = \frac{-ix_j}{|x|}\widehat{f}(x).$$

Let $\beta = (\beta_1,\dots,\beta_n) \in \mathbb{N}_0^n$ be a multi-index. Then the *multi-Riesz transform* $\mathcal{R}_\beta$ is defined as

$$\mathcal{R}_\beta f = \mathcal{R}_1^{\beta_1} \circ \cdots \circ \mathcal{R}_n^{\beta_n} f. \tag{3.6}$$

Let $f \in \mathcal{S}$. Then it is easy to verify (cf. [19]) that

$$\mathcal{F}(\mathcal{R}_\beta f)(x) = \Big(\frac{-ix_1}{|x|}\Big)^{\beta_1} \dots \Big(\frac{-ix_n}{|x|}\Big)^{\beta_n} \widehat{f}(x), \tag{3.7}$$

$$\mathcal{F}(\mathcal{R}_\beta\,(D^\beta f))(x) = \Big(\frac{-2\pi x_1^2}{|x|}\Big)^{\beta_1} \dots \Big(\frac{-2\pi x_n^2}{|x|}\Big)^{\beta_n} \widehat{f}(x), \tag{3.8}$$

$$\mathcal{F}(D^\beta f)(x) = (-2\pi i)^{|\beta|}\, x^\beta \widehat{f}(x) \tag{3.9}$$

$(x^\beta := x_1^{\beta_1} \dots x_n^{\beta_n})$.

LEMMA 3.2. *Let $p(\cdot) \in \mathcal{M}(\mathbb{R}^n)$. Then there exists a positive constant C such that, for any $\alpha \geqslant 0$ and $f \in L^{p(\cdot)}(\mathbb{R}^n)$,*

$$\|g_\alpha * f\|_{p(\cdot)} \leqslant C\|f\|_{p(\cdot)}. \tag{3.10}$$

Proof. Using (3.1) and putting $\varepsilon = 1$ and $g_\alpha = \varphi = \psi$ in Theorem $2(a)$ on page 62 of [23] we obtain a point-wise estimate

$$(g_\alpha * f)(x) \leqslant Mf(x), \quad x \in \mathbb{R}^n \quad (\alpha \geqslant 0).$$

Hence, by (2.7) the inequality (3.10) follows. □

LEMMA 3.3. *Let $p(\cdot) \in \mathcal{M}(\mathbb{R}^n)$, $\alpha \geqslant 0$ and $\beta \in \mathbb{N}_0^n$. Then there exists a positive constant C such that, for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,*

$$\|\mu_\alpha * f\|_{p(\cdot)} \leqslant C\|f\|_{p(\cdot)}, \tag{3.11}$$

$$\|\mathcal{R}_\beta f\|_{p(\cdot)} \leqslant C\|f\|_{p(\cdot)}. \tag{3.12}$$

Proof. It is easy to calculate

$$(\mu_\alpha * f)(x) = f(x) + \sum_{k=1}^{\infty} b(\alpha,k)\,(g_{2k} * f)(x).$$

Then, by (3.10) and (3.4),

$$\|\mu_\alpha * f\|_{p(\cdot)} \leqslant \|f\|_{p(\cdot)} + \sum_{k=1}^{\infty} |b(\alpha,k)|\,\|g_{2k} * f\|_{p(\cdot)} \leqslant \|f\|_{p(\cdot)}\Big(1 + C\sum_{k=1}^{\infty} |b(\alpha,k)|\Big) \lesssim \|f\|_{p(\cdot)}$$

which proves (3.11).

To prove (3.12) we use the results of L. Diening and M. Růžička [8, Prop. 4.3] and L. Diening [6, Thm. 8.14] that under the assumption $p(\cdot) \in \mathcal{M}(\mathbb{R}^n)$ there exists a positive constant c such that, for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\|\mathcal{R}_j f\|_{p(\cdot)} \leqslant c\|f\|_{p(\cdot)}, \quad j = 1, \dots, n.$$

Applying (3.6) and iterating this inequality we obtain (3.12) with $C = c^{|\beta|}$. □

LEMMA 3.4. *Let $p(\cdot) \in \mathcal{M}(\mathbb{R}^n)$. Then*
(i) *$C_0^\infty(\mathbb{R}^n)$ is dense in $W^{k,p(\cdot)}(\mathbb{R}^n)$, $k \in \mathbb{N}$;*
(ii) *the Schwartz class $\mathcal{S}$ is dense in $\mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^n)$, $\alpha \geqslant 0$.*

Proof. The density in (i) follows from the assumption $p(\cdot) \in \mathcal{M}(\mathbb{R}^n)$ by [7, Cor. 2.5].

Let us prove (ii). If $\alpha = 0$, the result follows from density of $C_0^\infty(\mathbb{R}^n)$ in $L^{p(\cdot)}(\mathbb{R}^n)$ (cf. [14, Thm. 2.11]). Let $\alpha > 0$ and $u \in \mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^n)$. Then there is a function $f \in L^{p(\cdot)}(\mathbb{R}^n)$ such that $u = g_\alpha * f$. By density of $C_0^\infty(\mathbb{R}^n)$ in $L^{p(\cdot)}(\mathbb{R}^n)$ we can find a sequence $(f_j)_{j=1}^\infty \subset C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}$ converging to f in $L^{p(\cdot)}(\mathbb{R}^n)$. Since the mapping $f \mapsto g_\alpha * f$ maps $\mathcal{S}$ onto $\mathcal{S}$ (cf. [23]), the functions $u_j := g_\alpha * f_j$, $j \in \mathbb{N}$, belong to $\mathcal{S}$. Moreover,

$$\|u - u_j\|_{\alpha;p(\cdot)} = \|f - f_j\|_{p(\cdot)} \to 0 \quad \text{as } j \to \infty$$

and the assertion follows. □

LEMMA 3.5. *Let $f \in \mathcal{S}$ and $k \in \mathbb{N}$. Then*

$$f = g_k * \sum_{m=0}^{k} \binom{k}{m} g_{k-m} * \mu_m * (-2\pi)^{-m} \sum_{|\beta|=m} \binom{m}{\beta} \mathcal{R}_\beta\,(D^\beta f),$$

where $\binom{m}{\beta} = \frac{m!}{\beta_1!\beta_2!\ldots\beta_n!}$.

Proof. (Cf. [19, Lemma 5.15]) Let $f \in \mathcal{S}$. Using the Binomial Theorem we have

$$1 = \frac{\sum_{m=0}^{k} \binom{k}{m} \sum_{|\beta|=m} \binom{m}{\beta} x_1^{2\beta_1} \ldots x_n^{2\beta_n}}{\left(1+|x|^2\right)^{2k/2}}$$

$$= \frac{1}{\left(1+|x|^2\right)^{\frac{k}{2}}} \sum_{m=0}^{k} \binom{k}{m} \frac{1}{\left(1+|x|^2\right)^{\frac{k-m}{2}}} \frac{|x|^m}{\left(1+|x|^2\right)^{\frac{m}{2}}} (-2\pi)^{-m} \sum_{|\beta|=m} \binom{m}{\beta} \left(\tfrac{-2\pi x_1^2}{|x|}\right)^{\beta_1} \ldots \left(\tfrac{-2\pi x_n^2}{|x|}\right)^{\beta_n}.$$

Consequently, by (3.8), (3.5) and (3.2), we obtain

$$\widehat{f}(x) = \widehat{g}_k(x) \sum_{m=0}^{k} \binom{k}{m} \widehat{g}_{k-m}(x)\, \widehat{\mu}_m(x)\, (-2\pi)^{-m} \sum_{|\beta|=m} \binom{m}{\beta} \mathcal{F}(\mathcal{R}_\beta(D^\beta f))(x).$$

The result then follows by applying the inverse Fourier transform. □

LEMMA 3.6. *Let $f \in \mathcal{S}$, $k \in \mathbb{N}$ and $\beta \in \mathbb{N}_0^n$, $|\beta| \leqslant k$. Then*

$$D^\beta(g_k * f) = (2\pi)^{|\beta|}\, g_{k-|\beta|} * \mu_{|\beta|} * \mathcal{R}_\beta f.$$

Proof. (Cf. [19, Lemma 5.17]) Let $f \in \mathcal{S}$. By (3.9), (3.2) and (3.5),

$$\begin{aligned}
\mathcal{F}(D^\beta(g_k * f))(x) &= (-2\pi i)^{|\beta|}\, x^\beta \widehat{g}_k(x) \widehat{f}(x) \\
&= (2\pi)^{|\beta|} \frac{1}{\left(1+|x|^2\right)^{(k-|\beta|)/2}} \frac{|x|^{|\beta|}}{\left(1+|x|^2\right)^{|\beta|/2}} \left(\tfrac{-ix_1}{|x|}\right)^{\beta_1} \ldots \left(\tfrac{-ix_n}{|x|}\right)^{\beta_n} \widehat{f}(x) \\
&= (2\pi)^{|\beta|}\, \widehat{g}_{k-|\beta|}(x)\, \widehat{\mu}_{|\beta|}(x)\, \mathcal{F}(\mathcal{R}_\beta(f))(x).
\end{aligned}$$

The result now follows by applying the inverse Fourier transform. □

Proof of Theorem 3.1. (i) Let $f \in \mathcal{L}^{k,p(\cdot)}(\mathbb{R}^n)$. In view of Lemma 3.4 we can assume that $f \in \mathcal{S}$. Then there is a function $h \in \mathcal{S}$ such that $f = g_\alpha * h$. By (2.10), Lemma 3.5, Lemma 3.2 and Lemma 3.3

$$\begin{aligned}\|f\|_{W^{k,p(\cdot)}} &= \sum_{|\beta|\leqslant k} \|D^\beta f\|_{p(\cdot)} = \sum_{|\beta|\leqslant k} \left\|D^\beta (g_k * h)\right\|_{p(\cdot)} \\ &= \sum_{|\beta|\leqslant k} \left\|(2\pi)^{|\beta|}\, g_{k-|\beta|} * \mu_{|\beta|} * \mathcal{R}_\beta\, h\right\|_{p(\cdot)} \leqslant c\,\|h\|_{p(\cdot)} = c\,\|f\|_{k;p(\cdot)},\end{aligned}$$

where $c > 0$ is a suitable constant independent of f.

(ii) We prove the reverse inequality. Let $f \in W^{k,p(\cdot)}(\mathbb{R}^n)$. Again, by Lemma 3.4, we can assume that $f \in \mathcal{S}$. Then, by Lemma 3.5, Lemma 3.2 and Lemma 3.3

$$\begin{aligned}\|f\|_{k;p(\cdot)} &= \left\| \sum_{m=0}^{k} g_{k-m} * \mu_m * (-2\pi)^{-m} \sum_{|\beta|=m} \tbinom{m}{\beta} \mathcal{R}_\beta\,(D^\beta f)\right\|_{p(\cdot)} \\ &\leqslant c \sum_{|\beta|\leqslant k} \|D^\beta f\|_{p(\cdot)} = c\,\|f\|_{W^{k,p(\cdot)}}\end{aligned}$$

with a suitable constant $c > 0$ independent of f. □

4. Capacity

Let $E \subset \mathbb{R}^n$, $\alpha > 0$ and $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Define a *capacity* in $\mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^n)$ by

$$Cap_{\alpha,p(\cdot)}(E) = \inf \varrho_{p(\cdot)}(f),$$

where the infimum is taken over all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ with $g_\alpha * f \geqslant 1$ on E. Since g_α is non-negative (cf. (3.1)) we can assume that $f \geqslant 0$.

LEMMA 4.1. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. The capacity $Cap_{\alpha,p(\cdot)}$ is an outer measure. That is,*

(i) $Cap_{\alpha,p(\cdot)}(\emptyset) = 0$;

(ii) *if* $E_1 \subset E_2$, *then* $Cap_{\alpha,p(\cdot)}(E_1) \leqslant Cap_{\alpha,p(\cdot)}(E_2)$;

(iii) *if* $E_i \subset \mathbb{R}^n$, $i = 1, 2, \ldots,$ *then*

$$Cap_{\alpha,p(\cdot)}\Big(\bigcup_{i=1}^{\infty} E_i\Big) \leqslant \sum_{i=1}^{\infty} Cap_{\alpha,p(\cdot)}(E_i).$$

Proof. The property (i) immediately follows on putting $f \equiv 0$. The property (ii) follows from the fact that every test function of E_2 is also a test function of E_1.

Next we prove (iii), following [13]. We may assume that $\sum_{i=1}^{\infty} Cap_{\alpha,p(\cdot)}(E_i) < \infty$. Let $\varepsilon > 0$ be fixed. For every $i \in \mathbb{N}$ choose $f_i \in L^{p(\cdot)}(\mathbb{R}^n)$ such that $g_\alpha * f_i \geqslant 1$ on E_i and

$$\int_{\mathbb{R}^n} |f_i(x)|^{p(x)}\,\mathrm{d}x \leqslant Cap_{\alpha,p(\cdot)}(E_i) + \frac{\varepsilon}{2^i}.$$

Put $f := \sup_i f_i$ and $E := \cup_{i=1}^{\infty} E_i$. If $x \in E$ then $x \in E_i$ for some $i \in \mathbb{N}$ and $(g_\alpha * f)(x) \geqslant (g_\alpha * f_i)(x) \geqslant 1$. Thus, f is a test function for E. Set $h_k = \max_{1 \leqslant i \leqslant k} f_i$ and define $X_i = \{x \in \mathbb{R}^n; h_k(x) = f_i(x)\}$. Consequently, $\mathbb{R}^n = \cup_{i=1}^{k} X_i$ and

$$\int_{\mathbb{R}^n} |h_k(x)|^{p(x)}\,\mathrm{d}x \leqslant \sum_{i=1}^{k} \int_{X_i} |f_i(x)|^{p(x)}\,\mathrm{d}x \leqslant \sum_{i=1}^{k} \int_{\mathbb{R}^n} |f_i(x)|^{p(x)}\,\mathrm{d}x$$
$$\leqslant \sum_{i=1}^{k} Cap_{\alpha,p(\cdot)}(E_i) + \varepsilon \leqslant \sum_{i=1}^{\infty} Cap_{\alpha,p(\cdot)}(E_i) + \varepsilon.$$

Since $h_k \nearrow f$, by the Monotone Convergence Theorem we have

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)}\,\mathrm{d}x \leqslant \sum_{i=1}^{\infty} Cap_{\alpha,p(\cdot)}(E_i) + \varepsilon.$$

Letting $\varepsilon \to 0$ we obtain the assertion. $\square$

The ordinary *Sobolev capacity* is defined by

$$C_{p(\cdot)}(E) = \inf \varrho_{W^{1,p(\cdot)}}(u)$$

where

$$\varrho_{W^{1,p(\cdot)}}(u) := \int_{\mathbb{R}^n} \left(|u(x)|^{p(x)} + |\nabla u(x)|^{p(x)} \right) \mathrm{d}x$$

and the infimum is taken over all $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ for which there is an open set $G \supset E$ such that $u \geqslant 1$ a. e. on G.

It is possible to show that if $p \in \mathcal{B}(\mathbb{R}^n)$ then $C_{p(\cdot)}$ is an outer measure and an Choquet capacity [12, Corollaries 3.3 and 3.4]. Relationship between the capacities $Cap_{1,p(\cdot)}$ and $C_{p(\cdot)}$ is formulated in the next lemma.

LEMMA 4.2. *Assume that $p(\cdot) \in \mathcal{M}(\mathbb{R}^n)$ and $E \subset \mathbb{R}^n$. Then*

$$Cap_{1,p(\cdot)}(E) \leqslant c \max\{C_{p(\cdot)}(E)^{\frac{p_*}{p^*}}, C_{p(\cdot)}(E)^{\frac{p^*}{p_*}}\}$$

and

$$C_{p(\cdot)}(E) \leqslant C \max\{Cap_{p(\cdot)}(E)^{\frac{p_*}{p^*}}, Cap_{p(\cdot)}(E)^{\frac{p^*}{p_*}}\}.$$

Here c and C are positive constants independent of E.

Proof. Let $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$, $u \geqslant 1$ on an open neighborhood of E, be a test function for $C_{p(\cdot)}(E)$. By Theorem 3.1 there exists $f \in L^{p(\cdot)}(\mathbb{R}^n)$ so that $u = g_1 * f$, and

$$\|u\|_{W^{1,p(\cdot)}} \approx \|f\|_{p(\cdot)}. \tag{4.1}$$

Obviously, f is a test function for $Cap_{p(\cdot)}(E)$ and

$$Cap_{p(\cdot)}(E) \leqslant \int_{\mathbb{R}^n} |f(x)|^{p(x)}\,\mathrm{d}x. \tag{4.2}$$

By (2.2) it is easy to see that for a function $g \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |g(x)|^{p(x)}\,\mathrm{d}x \leqslant \max\left\{\|g\|_{p(\cdot)}^{p^*}, \|g\|_{p(\cdot)}^{p_*}\right\} \tag{4.3}$$

and

$$\|g\|_{p(\cdot)} \leqslant \max\left\{\Big(\int_{\mathbb{R}^n} |g(x)|^{p(x)}\,\mathrm{d}x\Big)^{1/p^*}, \Big(\int_{\mathbb{R}^n} |g(x)|^{p(x)}\,\mathrm{d}x\Big)^{1/p_*}\right\}. \tag{4.4}$$

Applying (4.2), (4.3), (4.1) and (4.4), we arrive at

$$\begin{aligned} Cap_{p(\cdot)}(E) &\leqslant \int_{\mathbb{R}^n} |f(x)|^{p(x)}dx \leqslant \max\left\{\|f\|_{p(\cdot)}^{p_*}, \|f\|_{p(\cdot)}^{p^*}\right\} \\ &\lesssim \max\left\{\|u\|_{W^{1,p(\cdot)}}^{p_*}, \|u\|_{W^{1,p(\cdot)}}^{p^*}\right\} \lesssim \max\left\{\varrho_{W^{1,p(\cdot)}}(u)^{p_*/p^*}, \varrho_{W^{1,p(\cdot)}}(u)^{p^*/p_*}\right\}, \end{aligned}$$

and the first inequality follows.

Let $\varepsilon > 0$. Take $f \geqslant 0, f \in L^{p(\cdot)}(\mathbb{R}^n)$, such that $g_1 * f \geqslant 1$ on E and

$$\varrho_{p(\cdot)}(f) \leqslant Cap_{1,p(\cdot)}(E) + \varepsilon.$$

Since $f \geqslant 0$, the function $g_1 * f$ is lower semi-continuous and so, the set $E_\varepsilon = \{x \in \mathbb{R}^n; \frac{g_1 * f}{1-\varepsilon} > 1\}$ is open and contains E. Thus,

$$C_{p(\cdot)}(E) \leqslant \varrho_{W^{1,p(\cdot)}}\left(\tfrac{g_1 * f}{1-\varepsilon}\right) \leqslant (1-\varepsilon)^{-p^*}\varrho_{W^{1,p(\cdot)}}(g_1 * f).$$

Letting $\varepsilon \to 0_+$, we obtain

$$C_{p(\cdot)}(E) \leqslant \varrho_{W^{1,p(\cdot)}}(g_1 * f).$$

Now, by Theorem 3.1, we have

$$\begin{aligned} C_{p(\cdot)}(E) &\leqslant \varrho_{W^{1,p(\cdot)}}(g_1 * f) \leqslant \max\left\{\|g_1 * f\|_{W^{1,p(\cdot)}}^{p_*}, \|g_1 * f\|_{W^{1,p(\cdot)}}^{p^*}\right\} \\ &\lesssim \max\left\{\|g_1 * f\|_{1;p(\cdot)}^{p_*}, \|g_1 * f\|_{1;p(\cdot)}^{p^*}\right\} = \max\left\{\|f\|_{p(\cdot)}^{p_*}, \|f\|_{p(\cdot)}^{p^*}\right\} \\ &\leqslant \max\{\varrho_{p(\cdot)}(f)^{p_*/p^*}, \varrho_{p(\cdot)}(f)^{p^*/p_*}\} \end{aligned}$$

which completes the proof. $\square$

5. Hölder type quasi-continuity

In this section we study point-wise behavior of functions from $\mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n)$. First we investigate a quasi-continuity of such functions.

PROPOSITION 5.1. *Let $p(\cdot) \in \mathcal{M}(\mathbb{R}^n)$. Every $u \in \mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n)$ is quasi-continuous. That is, for every $\varepsilon > 0$, there exists a set $F \subset \mathbb{R}^n$, $Cap_{1,p(\cdot)}(F) \leqslant \varepsilon$, so that u restricted to $\mathbb{R}^n \setminus F$ is continuous.*

Proof. Let $u = g_1 * f \in \mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n)$. Then, by Lemma 3.4, there is a sequence $u_i = g_1 * f_i \in \mathcal{S}$ converging to u in $\mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n)$. We may assume without loss of generality, by considering a subsequence if necessary, that

$$\|u_i - u_{i+1}\|_{1;p(\cdot)} \leqslant 4^{-i}, \quad i = 1,2,\ldots. \tag{5.1}$$

Put

$$E_i = \{x \in \mathbb{R}^n; |u_i(x) - u_{i+1}(x)| > 2^{-i}\}, \quad i = 1,2,\ldots, \quad \text{and} \quad F_j = \bigcup_{i=j}^{\infty} E_i.$$

Continuity of the functions u_i implies that the sets E_i, F_j, $i,j = 1,2\ldots$, are open. By Theorem 2.2 of [12], the functions $|u_i - u_{i+1}|$ belong to $W^{1,p(\cdot)}(\mathbb{R}^n)$ if $u_i \in W^{1,p(\cdot)}(\mathbb{R}^n)$, $i = 1,2,\ldots$, and so, using Theorem 3.1, we have that $2^i|u_i - u_{i+1}| \in \mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n)$. Hence, for every $i = 1,2,\ldots$, there is a function $h_i \in L^{p(\cdot)}(\mathbb{R}^n)$ such that $2^i|u_i - u_{i+1}| = g_1 * h_i$. Using (5.1), (2.3) and definition of the norm (2.10), we obtain

$$Cap_{1,p(\cdot)}(E_i) \leqslant \int_{\mathbb{R}^n} h_i(x)^{p(x)}\,\mathrm{d}x \leqslant \|h_i\|_{p(\cdot)} = 2^i\|u_i - u_{i+1}\|_{1;p(\cdot)} \leqslant 2^{-i}, \quad i = 1,2,\ldots.$$

Given $\varepsilon > 0$, choose $j \in \mathbb{N}$ so that $2^{1-j} < \varepsilon$. Now, Lemma 4.1 (iii) implies that

$$Cap_{1,p(\cdot)}(F_j) \leqslant \sum_{i=j}^{\infty} Cap_{1,p(\cdot)}(E_i) \leqslant \sum_{i=j}^{\infty} 2^{-i} \leqslant 2^{1-j} < \varepsilon.$$

Moreover, for every $x \in \mathbb{R}^n \setminus F_j$ and every $k,l \in \mathbb{N}$, $k > l \geqslant j$,

$$|u_l(x) - u_k(x)| \leqslant \sum_{i=l}^{k-1} |u_i(x) - u_{i+1}(x)| \leqslant \sum_{i=l}^{k-1} 2^{-i} < 2^{1-l}.$$

Hence, the sequence $\{u_i\}_i$ is uniformly convergent in $\mathbb{R}^n \setminus F_j$ which implies that the function $v := \lim_{i\to\infty} u_i$ restricted to $\mathbb{R}^n \setminus F_j$ is continuous. Put

$$G := \{x \in \mathbb{R}^n; |u(x) - v(x)| > 0\}, \quad F := F_j \cup G.$$

Obviously, the function u restricted to $\mathbb{R}^n \setminus F$ is continuous, and so, it remains to prove that

$$Cap_{1,p(\cdot)}(G \setminus F_j) = 0. \tag{5.2}$$

Given $k \in \mathbb{N}$, by the uniform convergence of $\{u_i\}_i$ to v on $\mathbb{R}^n \setminus F_j$ we find $i_0 \in \mathbb{N}$ such that, for any $i \geqslant i_0$,

$$\{x \in \mathbb{R}^n \setminus F_j;\ |u(x) - v(x)| > 2/k\} \subset \{x \in \mathbb{R}^n \setminus F_j;\ |u(x) - u_i(x)| > 1/k\}.$$

Moreover, for any $i \geqslant i_0$,

$$\{x \in \mathbb{R}^n \setminus F_j;\ |u(x) - u_i(x)| > 1/k\} \subset \{x \in \mathbb{R}^n \setminus F_j;\ g_1 * |k(f - f_i)|(x) > 1\}.$$

Consequently, using (ii) of Lemma 4.1 and the definition of capacity $Cap_{1,p(\cdot)}$, we obtain, for any $i \geqslant i_0$,

$$\begin{aligned} &Cap_{1,p(\cdot)}\big(\{x \in \mathbb{R}^n \setminus F_j\,;\ |u(x) - v(x)| > 2/k\}\big) \\ &\quad \leqslant Cap_{1,p(\cdot)}\big(\{x \in \mathbb{R}^n \setminus F_j\,;\ g_1 * |k(f - f_i)|(x) > 1\}\big) \\ &\quad \leqslant \varrho_{p(\cdot)}(k|f - f_i|). \end{aligned}$$

Letting $i \to \infty$, we obtain, in view of the fact that $\|f - f_i\|_{p(\cdot)} = \|u - u_i\|_{1;p(\cdot)} \to 0$ and (2.5), that

$$Cap_{1,p(\cdot)}\big(\{x \in \mathbb{R}^n \setminus F_j\,;\ |u(x) - v(x)| > 2/k\}\big) = 0.$$

Since $k \in \mathbb{N}$ was arbitrary, the assertion (5.2) now follows by (iii) of Lemma 4.1. □

Our second aim is to generalize the result of J. Malý [16] on Hölder type quasi-continuity. The idea of the proof of Malý can be applied in spaces $\mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n)$, too.

DEFINITION 5.2. Let Ω be a bounded domain in $\mathbb{R}^n$. Say that $u : \Omega \to \mathbb{R}$ is an α*-Hölder-continuous function* on Ω if

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} < \infty.$$

First, let us formulate the result.

THEOREM 5.3. *Let $p \in \mathcal{M}(\mathbb{R}^n)$ and let $p^* \leqslant n$. Suppose that*

$$0 < \alpha < \beta \frac{p_*(p_* - \varepsilon_0)}{p^*(p^* - \varepsilon_0)}, \tag{5.3}$$

where $0 < \varepsilon_0 < p_* - 1$ and $0 < \beta < \varepsilon_0/p_*$. If $u \in \mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n)$ then, for any $\varepsilon > 0$, there exists an α-Hölder continuous function w on $\mathbb{R}^n$ such that

$$\|u - w\|_{1;p(\cdot)} \leqslant \varepsilon, \quad Cap_{1,p(\cdot)-\varepsilon_0}\big(\{x \in \mathbb{R}^n; w(x) \neq u(x)\}\big) \leqslant \varepsilon.$$

Before proving Theorem 5.3 we need some preliminary results. The following embedding theorem can be found in [24, Section 2.7.1].

LEMMA 5.4. *Let $r > n$ and let $u \in \mathcal{L}^{1,r}(\mathbb{R}^n)$. Then there is a positive constant $c = c(n,r)$ such that*

$$|u(x) - u(y)| \leqslant c\,\|u\|_{1;r}\,|x - y|^{1-n/r} \quad \text{and} \quad |u(x)| \leqslant c\,\|u\|_{1;r} \tag{5.4}$$

for all $x, y \in \mathbb{R}^n$.

LEMMA 5.5. *Assume that $p(\cdot) \in \mathcal{M}(\mathbb{R}^n)$, $p^* \leqslant n$, $0 < \varepsilon_0 < p_* - 1$ and $0 < \beta < \varepsilon_0/p_*$. Let $u \in \mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n)$ with $\|u\|_{1;p(\cdot)} \leqslant 1$ and $u = g_1 * f$. Then there exist bounded functions $u_R \in \mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n)$, $0 < R < 1$, $u_R = g_1 * f_R$, so that*

$$\lim_{R\to 0_+} \|u - u_R\|_{1;p(\cdot)} = 0, \tag{5.5}$$

$$\varrho_{p(\cdot)-\varepsilon_0}(f-f_R) \leqslant cR^{\beta p_*(p_*-\varepsilon_0)/p^*}, \tag{5.6}$$

and

$$|u_R(x)-u_R(y)| \leqslant C|x-y|^\beta \quad \textit{if} \quad |x-y| \leqslant R. \tag{5.7}$$

Proof. Given $u = g_1 * f \in \mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n)$, where $f \in L^{p(\cdot)}(\mathbb{R}^n)$, $\|u\|_{1;p(\cdot)} = \|f\|_{p(\cdot)} \leqslant 1$. Let $R \in (0,1)$ and $\lambda_R = R^{-\beta(p_*-\varepsilon_0)/\varepsilon_0}$. Put

$$M_R = \{x \in \mathbb{R}^n; |f(x)| \geqslant \lambda_R\}.$$

Clearly, (2.3) yields

$$|M_R| \leqslant \varrho_{p(\cdot)}(f/\lambda_R) \leqslant \lambda_R^{-p_*}. \tag{5.8}$$

We set

$$f_R(x) := \begin{cases} 0 & \text{if} \quad x \in M_R \\ f(x) & \text{if} \quad x \in \mathbb{R}^n \setminus M_R \end{cases} \qquad \text{and} \quad u_R := g_1 * f_R.$$

Obviously, the functions u_R are bounded and belong to $\mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n)$. Since $|M_R| \to 0$ as $R \to 0_+$ (cf. (5.8)), we have by (2.5)

$$\|u-u_R\|_{1;p(\cdot)} = \|f-f_R\|_{p(\cdot)} = \|f\chi_{M_R}\|_{p(\cdot)} \to 0 \quad \text{as } R \to 0_+ .$$

By the Hölder inequality (cf. [14, Thm. 2.1]), (2.4) and $0 < \varepsilon_0 < p_* - 1$ we arrive at

$$\varrho_{p(\cdot)-\varepsilon_0}(f-f_R) = \int_{M_R} |f(x)|^{p(x)-\varepsilon_0}\,dx \lesssim \| |f(x)|^{p(x)-\varepsilon_0} \|_{p(\cdot)/(p(\cdot)-\varepsilon_0)} \, \|\chi_{M_R}\|_{p(\cdot)/\varepsilon_0} \leqslant |M_R|^{\varepsilon_0/p^*}.$$

Hence and from (5.8) we obtain (5.6).

It remains to prove (5.7). Put

$$r = \frac{n\varepsilon_0 - \beta p_*(p_*-\varepsilon_0)}{\varepsilon_0 - \beta p_*}.$$

Since $0 < \beta < \varepsilon_0/p_*$ and $p_* \leqslant p^* \leqslant n$ we find that $r > n$ and $r - p(x) \geqslant 0$, $x \in \mathbb{R}^n$. As the functions $|f_R|$ are bounded by λ_R and $\varrho_{p(\cdot)}(f_R) \leqslant \varrho_{p(\cdot)}(f) \leqslant \|f\|_{p(\cdot)} \leqslant 1$, it implies

$$\varrho_r(f_R) = \int_{\mathbb{R}^n} |f_R(x)|^{p(x)} |f_R(x)|^{r-p(x)}\,dx \leqslant \lambda_R^{r-p_*} \varrho_{p(\cdot)}(f_R) \leqslant \lambda_R^{r-p_*},$$

that is, $u_R \in \mathcal{L}^{1,r}(\mathbb{R}^n)$ and $\|u_R\|_{1,r} \leqslant \lambda_R^{(r-p_*)/r}$ for all $R \in (0,1)$. Consequently, using (5.4), we obtain, for $|x-y| \leqslant R$,

$$\begin{aligned} |u_R(x)-u_R(y)| &\leqslant c|x-y|^{1-n/r}\|u_R\|_{1,r} \leqslant c|x-y|^{1-n/r}\lambda_R^{(r-p_*)/r} \\ &\leqslant c|x-y|^{1-n/r}R^{-\beta(p_*-\varepsilon_0)(r-p_*)/(r\varepsilon_0)} \leqslant c|x-y|^{1-n/r-\beta(p_*-\varepsilon_0)(r-p_*)/(r\varepsilon_0)} \\ &= c|x-y|^\beta \end{aligned}$$

and (5.7) is verified. □

Similarly as in the classical case, the space $W^{1,p(\cdot)}(\mathbb{R}^n)$ is also closed under truncation.

LEMMA 5.6. *Let $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ and let $\tau \geqslant 0$. Then the function $v := \max\big\{-\tau, \min\{\tau, u\}\big\}$ belongs to $W^{1,p(\cdot)}(\mathbb{R}^n)$ and satisfies $\|u-v\|_{W^{1,p(\cdot)}} \leqslant \|u\|_{W^{1,p(\cdot)}}$.*

Proof. The assertions immediately follow from the fact that the space $W^{1,p(\cdot)}(\mathbb{R}^n)$ is a lattice (see [12, Thm. 2.2]). $\square$

Proof of Theorem 5.3. Fix $u \in \mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n)$ and $\varepsilon > 0$. We may assume that $\|u\|_{1;p(\cdot)} \leqslant 1$. Let $u_R = g_1 * f_R$ have the same meaning as in Lemma 5.5 and let α be a number from (5.3). Denote $u_j := u_{R_j}$ and $f_j := f_{R_j}$, where the sequence $\{R_j\} \subset (0,1]$ is chosen so that

$$R_0 = 1, \quad R_{j+1}^{\alpha} \leqslant \tfrac{1}{2} R_j^{\alpha}, \quad j = 0, 1, \dots, \tag{5.9}$$

and (cf. (5.5))

$$\sum_{j=1}^{\infty} \|u_{j+1} - u_j\|_{1;p(\cdot)} < \infty. \tag{5.10}$$

For $j \in \mathbb{N}$ put

$$E_j = \{x \in \mathbb{R}^n; |u_{j+1}(x) - u_j(x)| > R_j^{\alpha}\}.$$

Then, using the inequality

$$g_1 * (R_j^{-\alpha}|f_{j+1} - f_j|)(x) \geqslant R_j^{-\alpha}\big|g_1 * (f_{j+1} - f_j)(x)\big| = R_j^{-\alpha}|u_{j+1}(x) - u_j(x)| > 1, \quad x \in E_j,$$

and (5.6), we get

$$\begin{aligned} Cap_{1,p(\cdot)-\varepsilon_0}(E_j) &\leqslant \varrho_{p(\cdot)-\varepsilon_0}(R_j^{-\alpha}|f_{j+1} - f_j|) \leqslant R_j^{-\alpha(p^*-\varepsilon_0)} \varrho_{p(\cdot)-\varepsilon_0}(|f - f_j| + |f_{j+1} - f|) \\ &\leqslant 2^{p^*-1}\, 2\, c\, R_j^{-\alpha(p^*-\varepsilon_0)} R_j^{\beta p_*(p_*-\varepsilon_0)/p^*} = c 2^{p^*} R_j^{\beta p_*(p_*-\varepsilon_0)/p^* - \alpha(p^*-\varepsilon_0)}. \end{aligned}$$

Since the exponent on R_j is positive, we can find by (5.9) a number $j_0 \in \mathbb{N}$ such that

$$\sum_{j=j_0}^{\infty} Cap_{1,p(\cdot)-\varepsilon_0}(E_j) < \varepsilon \tag{5.11}$$

and (cf. (5.5) and (5.10))

$$\|u - u_{j_0}\|_{1;p(\cdot)} + \sum_{j=j_0}^{\infty} \|u_{j+1} - u_j\|_{1;p(\cdot)} \leqslant \varepsilon. \tag{5.12}$$

For $j \in \mathbb{N}$ define the functions

$$v_j = \max\big\{-R_j^{\alpha}, \min\{R_j^{\alpha}, u_{j+1} - u_j\}\big\}.$$

The functions v_j belong (by Lemma 5.6 and Theorem 3.1) to $\mathcal{L}^{1,p(\cdot)}(\mathbb{R}^n)$ and it is easy to see that

$$E_j = \{x \in \mathbb{R}^n; v_j(x) \neq u_{j+1}(x) - u_j(x)\}, \quad j \in \mathbb{N}. \tag{5.13}$$

Set

$$w := u_{j_0} + \sum_{j=j_0}^{\infty} v_j \quad \text{and} \quad E := \bigcup_{j=j_0}^{\infty} E_j.$$

Then, by (5.13), $w = u$ outside E, that is,

$$E \supset \{x \in \mathbb{R}^n; u(x) \neq v(x)\}.$$

Moreover, by (5.11) and Lemma 4.1 (iii),

$$Cap_{1,p(\cdot)-\varepsilon_0}(E) \leqslant \varepsilon.$$

Concerning the α-Hölder continuity of w, observe that it is enough to estimate $|w(x) - w(y)|$ for $|x-y| \leqslant R_{j_0}$ as the function w is bounded. Choose $x, y \in \mathbb{R}^n$ such that $0 < |x-y| \leqslant R_{j_0}$ and find $k \in \mathbb{N}$, $k \geqslant j_0$, so that

$$R_{k+1} < |x-y| \leqslant R_k. \tag{5.14}$$

By (5.7),

$$|u_{j_0}(x) - u_{j_0}(y)| \leqslant C|x-y|^{\beta}.$$

If $j_0 \leqslant j \leqslant k$ we deduce from (5.7) that

$$|v_j(x) - v_j(y)| \leqslant C\,|x-y|^{\beta}.$$

If $j > k$ then, by (5.9) and (5.14),

$$|v_j(x) - v_j(y)| \leqslant |v_j(x)| + |v_j(y)| \leqslant 2R_j^{\alpha} \leqslant 2^{k-j+2} R_{k+1}^{\alpha} \leqslant 2^{k-j+2}|x-y|^{\alpha}.$$

Using (5.9), we obtain

$$k \leqslant \frac{\alpha}{\log 2} \log \frac{1}{R_k} \leqslant \frac{\alpha}{(\beta-\alpha)\log 2} R_k^{\alpha-\beta} \leqslant \frac{\alpha}{(\beta-\alpha)\log 2} |x-y|^{\alpha-\beta}.$$

Consequently,

$$\begin{aligned}
|w(x) - w(y)| &\leqslant |u_{j_0}(x) - u_{j_0}(y)| + \sum_{j=j_0}^{k} |v_j(x) - v_j(y)| + \sum_{j=k+1}^{\infty} |v_j(x) - v_j(y)| \\
&\leqslant C(k+1)\,|x-y|^{\beta} + \sum_{j=k+1}^{\infty} 2^{k-j+2}|x-y|^{\alpha} \\
&\lesssim |x-y|^{\alpha}\Big(1 + \sum_{j=k+1}^{\infty} 2^{k-j+2}\Big) \lesssim |x-y|^{\alpha}.
\end{aligned}$$

Finally, by (5.12),

$$\begin{aligned}
\|u - w\|_{1;p(\cdot)} &\leqslant \|u - u_{j_0}\|_{1;p(\cdot)} + \sum_{j=j_0}^{\infty} \|v_j\|_{1;p(\cdot)} \\
&\leqslant \|u - u_{j_0}\|_{1;p(\cdot)} + \sum_{j=j_0}^{\infty} \|u_{j+1} - u_j\|_{1;p(\cdot)} \leqslant \varepsilon
\end{aligned}$$

and the assertions are verified. □

REMARK 5.7. The assumption $p^* \leqslant n$ in Theorem 5.3 is quite natural. If $p_* > n$ then every Sobolev class has a continuous representative and the classical Morey's inequality [10, Theorem 3, on p. 143] together with [14, Theorem 2.8] implies that

$$|u(y) - u(z)| \leqslant Cr^{1-\frac{n}{p_*}} \left(\int_{B(x,r)} |\nabla u|^{p_*} dx \right)^{\frac{1}{p_*}} \leqslant C(1 + |B(x,r)|) r^{1-\frac{n}{p_*}} \|\nabla u\|_{p(\cdot)}$$

for every $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$, $r > 0$ and $y, z \in B(x,r)$. Thus every Sobolev class has a locally $(1 - \frac{n}{p_*})$-Hölder continuous representative which, by [11, Theorem 4.7], is

$$v(x) = \lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\, \mathrm{d}y.$$

The last claim follows by an observation that if two continuous functions coincide almost everywhere then they actually coincide everywhere.

Related results concerning Sobolev spaces $W^{1,p(\cdot)}(\Omega)$ on an open bounded subset Ω of $\mathbb{R}^n$ with $p(x) > n$ for all $x \in \Omega$ are derived in [9].

REMARK 5.8. One of the referees pointed out to us a recent preprint [1] where the spaces of Bessel potentials over the spaces $L^{p(\cdot)}(\mathbb{R}^n)$ were investigated. Among other results the authors independently proved Theorem 3.1 and showed that $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^n)$ if $\alpha \geqslant 0$, $p(\cdot)$ satisfies conditions (2.8) and $1 < p_* \leqslant p^* < n/\alpha$ (cf. Lemma 3.4 (*ii*)).

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