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ITERATIVE APPROXIMATION OF SOLUTION OF GENERALIZED MIXED SET–VALUED VARIATIONAL INEQUALITY PROBLEM

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Abstract. In this paper, we consider a generalized mixed set-valued variational inequality problem which includes many important known variational inequality problems and equilibrium problem, and its related some auxiliary variational inequality problems. We prove the existence of solutions of the auxiliary variational inequality problems and suggest a two-step iterative algorithm and an inertial proximal iterative algorithm. Further, we discuss the convergence analysis of iterative algorithms. The theorems presented in this paper improve and generalize many known results for solving equilibrium problems, variational inequality and complementarity problems in the literature.

1. Introduction

Variational inequality theory has emerged as elegant and fascinating branch of applicable mathematics in recent years, because it describes a broad spectrum of interesting and important developments involving a link among various fields of mathematics, physics, economics, engineering, mechanics, etc. In last three decade, variational inequality theory has been extended and generalized in several directions, using new and powerful methods, to study a wide class of problems in a unified and general framework, see for example [7-9,12].

Equilibrium problems provide us with a unified, natural, innovative, and general framework to study a wide class of problems arising in finance, economics, transportation, and optimization. This theory has witnessed an explosive growth in theoretical advances and applications across all disciplines of pure and applied sciences. As a result of this interaction, we have a variety of techniques to study existence results for equilibrium problems, see for example [6-8,11].

One of the most important and interesting problems in the theories of variational inequality and equilibrium problems is to develop the methods which give efficient and implementable algorithms for solving variational inequalities and equilibrium problems. These methods include projection method and its variant forms, linear approximation, descent, and Newton's methods, and the method based on auxiliary principle technique.

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It is well known that the projection method and its variants can not be extended for mixed variational inequalities involving non-differentiable term. To overcome this drawback, one uses usually the auxiliary principle technique. This technique deals with finding a suitable auxiliary principle and prove that the solution of an auxiliary problem is the solution of the original problem by using the fixed-point approach. Glowinski, Lions and Tremolieres [9] used this technique to study the existence of a solution of mixed variational inequalities. Noor [14-20], Huang and Deng [10], Chidume *et al.* [4] extended this technique to suggest and analyze a number of iterative methods for solving various classes of variational inequalities and equilibrium problems.

Recently, Alvarez [1] and Alvarez and Attouch [2] have considered and studied inertial proximal methods for maximal monotone operators associated with the discretization of second order differential equations in time. Later, Noor [19,20] and Noor *et al.* [21] further extended this method for variational inequality and equilibrium problems using auxiliary principle technique.

Motivated by recent work going in this direction, we consider a generalized mixed set-valued variational inequality problem (in short, GMSVIP) which includes many important known variational inequality problems and equilibrium problem, and its related auxiliary variational inequality problems. By using the KKM-Fan lemma and the fixed point theorem, we prove the existence of solutions for these auxiliary problems. Further, these auxiliary problems enable us to suggest and analyze two-step iterative algorithm and inertial proximal iterative algorithm for finding the approximate solutions for GMSVIP. Furthermore, we discuss the convergence analysis of these iterative algorithms. The theorems presented in this paper, improve and generalize many known results for solving equilibrium problems, variational inequality and complementarity problems, see [14-21].

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let $CB(H)$ be a family of all non-empty closed and bounded subsets of H, $M(\cdot,\cdot)$ is the Hausdorff metric on $CB(H)$ defined by

$$M(C,D) = \max\{\sup_{x\in C}\inf_{y\in D} d(x,y), \sup_{y\in D}\inf_{x\in C} d(x,y)\},\ C,D \in CB(H),$$

and $K \subset H$ be nonempty, closed and convex set. For given nonlinear non-differentiable bifunction $b : H\times H \to R\cup\{+\infty\}$, nonlinear bifunction $F : H\times H \to R,$ nonlinear mapping $N : H\times H \to H$ and three set-valued mappings $T,A,B : H \to CB(H)$, we consider the following generalized mixed set-valued variational inequality problem (in short, GMSVIP):

Find $u \in K,\ x \in T(u),\ y \in A(u),\ z \in B(u)$ such that

$$F(x,v) + \langle N(y,z), v-u\rangle + b(v,u) - b(u,u) \geqslant 0,\ \ \forall\, v \in K. \tag{2.1}$$

Some special cases of GMSVIP (2.1)

(I) If $F(x,v) \equiv 0, \forall u,v \in K$, then GMSVIP (2.1) is reduced to a problem of finding $u \in K$, $y \in A(u)$, $z \in B(u)$ such that

$$\langle N(y,z), v-u\rangle + b(v,u) - b(u,u) \geqslant 0, \quad \forall v \in K,$$

similar to the problem studied by Chidume *et al.* [4] in the setting of Banach spaces.

(II) If $N(y,z) \equiv 0, \forall y,z \in H$, then GMSVIP (2.1) is reduced to a problem of finding $u \in K$, $x \in T(u)$ such that

$$F(x,v) + b(v,u) - b(u,u) \geqslant 0, \quad \forall v \in K,$$

which appears to be new.

(III) If $T \equiv I$, the identity mapping and $N(y,z) \equiv 0, \forall y,z \in H$, then GMSVIP (2.1) is reduced to a problem of finding $u \in K$ such that

$$F(u,v) + b(v,u) - b(u,u) \geqslant 0, \quad \forall v \in K,$$

which has been studied by Noor [20].

(IV) If $T \equiv I$, $N(y,z) \equiv 0, \forall y,z \in H$, and $b(v,u) \equiv \delta_K(u), \forall u,v \in H$ where

$$\delta_K(u) = \begin{cases} 0, & \text{if} \quad u \in K, \\ +\infty, & \text{otherwise} \end{cases}$$

is the indicator function of K, then GMSVIP (2.1) is reduced to a problem of finding $u \in K$ such that

$$F(u,v) \geqslant 0, \quad \forall v \in K,$$

which is the classical equilibrium problem studied by Blum and Oettli [3].

Moreover, if

$$F(u,v) \equiv \langle Tu, v-u\rangle, \forall u,v \in K,$$

then classical equilibrium problem is reduced to a problem of finding $u \in K$ such that

$$\langle Tu, v-u\rangle \geqslant 0, \quad \forall\, v \in K,$$

which is the classical variational inequality studied by Stampacchia [22].

Now, we give the following concepts and known results.

DEFINITION 2.1. Let $b : H \times H \to R\bigcup\{+\infty\}$, $F : H \times H \to R$, $N : H \times H \to H$, $T, A, B : H \to CB(H)$. Then, for all $u,v,z \in H$, $x_1 \in T(u)$, $x_2 \in T(v)$, $y_1 \in A(u)$, $y_2 \in A(v)$ $z_1 \in B(u), z_2 \in B(v)$,

(i) N is said to be *mixed monotone with respect to* A *and* B, if

$$\langle N(y_1,z_1) - N(y_2,z_2), z-v\rangle \geqslant 0,$$

(ii) N is said to be a*-partially relaxed strongly mixed monotone with respect to* A *and* B, if there exists $a > 0$ such that

$$\langle N(y_1,z_1) - N(y_2,z_2), z-v\rangle \geqslant -a\|z-u\|^2,$$

(iii) F is said to be *monotone with respect to* A, if

$$F(x_1,v) + F(x_2,u) \leqslant 0,$$

(iv) F is said to be α*-partially relaxed strongly monotone with respect to* T, if there exists $\alpha > 0$ such that

$$F(x_1, v) + F(x_2, z) \leqslant \alpha\|z - u\|^2,$$

(v) F is said to be θ*-pseudomonotone with respect to* T where θ is a real-valued multivariable function, if

$$F(x_1, v) + \theta \geqslant 0 \text{ implies } -F(x_2, u) + \theta \geqslant 0,$$

(vi) T is said to be δ*-M-Lipschitz continuous*, if there exists $\delta > 0$ such that

$$M(T(u), T(v)) \leqslant \delta\|u - v\|,$$

where $M(\cdot,\cdot)$ is the Hausdorff metric on $CB(H)$,

(vii) b is said to be *skew-symmetric,* if

$$b(u,u) - b(u,v) - b(v,u) - b(v,v) \geqslant 0, \forall u, v \in H,$$

$(viii)$ F and N are said to be *simultaneously hemicontinuous*, if for $t > 0$, and $x_t \in T(u+tv)$, $y_t \in A(u+tv)$, $z_t \in B(u+tv)$, there exist $x_0 \in T(u)$, $y_0 \in A(u)$, $z_0 \in B(u)$, such that, for any $p \in H$,

$$F(x_t, p) + \langle N(y_t, z_t), p\rangle \rightarrow F(x_0, p) + \langle N(y_0, z_0), p\rangle$$

as $t \rightarrow 0_+$.

We note that if skew-symmetric bifunction b is bilinear then $b(u,u) \geqslant 0, \forall u \in H$.

REMARK 2.1. (i) If $z = u$, then partially relaxed strongly mixed monotonicity of N reduces to the mixed monotonicity of N.

(ii) Mixed monotonicity and partially relaxed strongly mixed monotonicity are the generalization of monotonicity and partially relaxed strongly monotonicity of N, respectively.

(iii) If $\theta \equiv 0$, θ-pseudomonotonicity of F reduces to simply pseudomonotonicity of F.

LEMMA 2.1. *For all* $\bar{u}, \bar{v} \in H$, *we have*

$$2\langle \bar{u}, \bar{v}\rangle = \|\bar{u} + \bar{v}\|^2 - \|\bar{u}\|^2 - \|\bar{v}\|^2. \tag{2.2}$$

LEMMA 2.2. (KKM-Fan Lemma [5]) *Let* E *be a subset of topological vector space* X. *Let* $F : E \rightarrow 2^E$ *be a set-valued mapping such that* $F(u)$ *is closed for each* $u \in E$, *and compact for atleast one* $u \in E$. *If the convex hull of every finite subset* $\{u_1, u_2, ..., u_n\}$ *of* E *is contained in the corresponding union* $\bigcup_{i=1}^n F(u_i)$, *then*

$$\bigcap_{u \in E} F(u) \neq \emptyset.$$

REMARK 2.2. $F : E \rightarrow 2^E$ is *KKM mapping* if and only if convex finite subset $\{u_1, u_2, ..., u_n\}$ of E is contained in the corresponding union $\bigcup_{i=1}^n F(u_i)$.

THEOREM 2.1. ([23]) *Let K be a nonempty and convex subset of Hausdörff topological vector space X and let $S : K \to 2^K$ be a set-valued mapping such that*

(i) *For each $u \in K$ $S(u)$ is a nonempty convex subset of K,*

(ii) *For each $v \in K$, $S^{-1}(v) := \{u \in K \ : \ v \in S(u)\}$ contains a relatively open subset O_v of K, where O_v may be empty for some $v \in K$,*

(iii) $\bigcup_{v \in K} O_v = K$,

(iv) *K contains a nonempty subset K_0 contained in a compact convex subset of K_1 of K such that the set $D = \bigcup_{v \in K_0} O_v^c$ is compact. (D may be nonempty and O_v^c denotes the complement of O_v in K).*

Then there exists $u_0 \in K$ such that $u_0 \in S(u_0)$.

3. Two-step iterative algorithm and results

In this section, we consider the following auxiliary variational inequality problem (in short, AVIP) related to GMSVIP (2.1):

(AVIP): For given $u \in K,\ x \in T(u),\ \ y \in A(u),\ \ z \in B(u)$, find $w \in K$ such that

$$\rho F(x,v) + \langle \rho N(y,z) + w - u, v - w\rangle + \rho b(v,w) - \rho b(w,w) \geqslant 0,\ \forall\, v \in K, \quad (3.1)$$

where $\rho > 0$ is a constant.

REMARK 3.1. We note that if $w = u$, then clearly w is a solution of GMSVIP (2.1).

Next, we prove the following existence theorem for AVIP (3.1)

THEOREM 3.1. *Let $T, A, B : H \to CB(H)$, $N : H \times H \to H$, $F : H \times H \to R$ and $b : H \times H \to R \bigcup \{+\infty\}$ be nonlinear mappings. Assume that*

(i) *F is convex is second argument,*

(ii) *b is convex and continuous in first argument,*

(iii) *If there exist a non-empty compact subset D of H and $w_0 \in D \cap K$ such that for any $w \in K \setminus D$, we have*

$$\rho F(x,w_0) + \langle \rho N(y,z) + w - u, w_0 - w\rangle + \rho b(w_0,w) - \rho b(w,w) < 0$$

for given $u \in K,\ x \in T(u),\ y \in A(u),\ z \in B(u)$.

Then $AVIP$ (3.1) has a solution.

Proof. For given $u \in K,\ x \in T(u),\ y \in A(u),\ z \in B(u)$, define a set-valued mapping $G : K \to 2^K$ by

$$G(v) = \{w \in K : \rho F(x,v) + \langle \rho N(y,z) + w - u, v - w\rangle + \rho b(v,w) - \rho b(w,w) \geqslant 0\} \quad (3.2)$$

for $v \in K$.

We claim that G is a KKM-mapping. Indeed, let $\{w_1, w_2, \cdot, \cdot, \cdot, w_m\}$ be a finite subset of K and let $\alpha_i \geqslant 0,\ \ 1 \leqslant i \leqslant m$ with $\sum_{i=1}^m \alpha_i = 1$. Suppose that $w = \sum_{i=1}^m \alpha_i w_i \notin \bigcup_{i=1}^m G(w_i)$. Then

$$\rho F(x,w_i) + \langle \rho N(y,z) + w - u, w_i - w\rangle + \rho b(w_i,w) - \rho b(w,w) < 0, \ \forall i.$$

Let, for given $u \in K$, $x \in T(u)$, $y \in A(u)$, $z \in B(u)$,

$$V := \{p \in K : \rho F(x,p) + \langle \rho N(y,z) + w - u, p - w\rangle + \rho b(p,w) - \rho b(w,w) < 0\},$$

for fixed $w \in K$. Let $p_1,\ p_2 \in V$, we have

$$\rho F(x,p_i) + \langle \rho N(y,z) + w - u, p_i - w\rangle + \rho b(p_i,w) - \rho b(w,w) < 0, \text{ for } i = 1,2. \quad (3.3)$$

Since K is convex, $p_\lambda := \lambda p_1 + (1-\lambda)p_2 \in K,\ \forall \lambda \in [0,1]$. Also, since F and b are convex in second and first argument respectively, then we have

$$\begin{aligned}\rho F(x,p_\lambda) &+ \langle \rho N(y,z) + w - u, p_\lambda - w\rangle + \rho b(p_\lambda,w) - \rho b(w,w)\\ &\leqslant \lambda[\rho F(x,p_1) + \langle \rho N(y,z) + w - u, p_1 - w\rangle + \rho b(p_1,w) - \rho b(w,w)]\\ &\quad + (1-\lambda)[\rho F(x,p_2) + \langle \rho N(y,z) + w - u, p_2 - w\rangle + \rho b(p_2,w) - \rho b(w,w)]\\ &< 0,\end{aligned}$$

where we have used (3.3).

The preceding inequality implies that V is convex. Hence, we have, for given $u \in K$, $x \in T(u)$, $y \in A(u)$, $z \in B(u)$,

$$\rho F(x, \sum_{i=1}^{m} \alpha_i w_i) + \langle \rho N(y,z) + w - u, \sum_{i=1}^{m} \alpha_i w_i - w\rangle + \rho b(\sum_{i=1}^{m} \alpha_i w_i, w) - \rho b(w,w) < 0$$
$$\Longrightarrow \rho F(x,w) < 0,$$

which leads to a contradiction to the fact that, if we take $v = w$ in (3.1), we have $\rho F(x,w) \geqslant 0$. Thus $w \in \bigcup_{i=1}^{m} G(w_i)$. Since w was an arbitrary element of $Conv\{w_1, w_2, ..., w_m\}$, hence $Conv\{w_1, w_2, ..., w_m\} \subset \bigcup_{i=1}^{m} G(w_i)$, and our claim is then verified.

Next, the continuity of b implies the closedness of the set $G(v)$ for each $v \in K$.

Finally, we claim that, for $w_0 \in D \cap K$, $G(w_0)$ is compact. Indeed, suppose that there exists $\bar{w} \in G(w_0)$ such that $\bar{w} \notin D$. Since $w_0 \in D \cap K$ and $\bar{w} \in G(w_0)$, we have

$$\rho F(x,w_0) + \langle \rho N(y,z) + \bar{w} - u, w_0 - \bar{w}\rangle + \rho b(w_0,\bar{w}) - \rho b(\bar{w},\bar{w}) \geqslant 0. \quad (3.4)$$

Since $\bar{w} \notin D$, by hypothesis (iii), we have

$$\rho F(x,w_0) + \langle \rho N(y,z) + \bar{w} - u, w_0 - \bar{w}\rangle + \rho b(w_0,\bar{w}) - \rho b(\bar{w},\bar{w}) < 0,$$

which is a contradiction to (3.4). Hence $G(w_0) \subset D$. Since D is compact and $G(w_0)$ is closed, $G(w_0)$ is compact. By Lemma 2.2, it follows that

$$\bigcap_{v \in K} G(v) \neq \emptyset.$$

Thus, there exists $w \in K$ such that, for given $u \in K$, $x \in T(u)$, $y \in A(u)$, $z \in B(u)$, we have

$$\rho F(x,v) + \langle \rho N(y,z) + w - u, v - w \rangle + \rho b(v,w) - \rho b(w,w) \geqslant 0, \ \forall\, v \in K,$$

i.e., $w \in K$ is a solution of AVIP (3.1) and this complete the proof.

Based on Remark 3.1, Theorem 3.1 and Nadler's technique [13], we suggest and analyze the following two-step iterative algorithm for finding the approximate solution of GMSVIP (2.1).

Iterative algorithm 3.1. Let $T, A, B : H \to CB(H)$, $N : H \times H \to H$, $F : H \times H \to R$ and $b : H \times H \to R \bigcup \{+\infty\}$ be given. For a given $u_0 \in H$, $x_0 \in T(u_0)$, $y_0 \in A(u_0)$, $z_0 \in B(u_0)$, compute the approximate solution (u_n, x_n, y_n, z_n) by the iterative schemes:

$$\begin{aligned} \rho F(x_n, v) + \langle \rho N(y_n, z_n) + u_{n+1} - w_n, v - u_{n+1} \rangle \\ + \rho b(v, u_{n+1}) - \rho b(u_{n+1}, u_{n+1}) \geqslant 0, \ \ \forall v \in K, \end{aligned} \tag{3.5}$$

$$x_n \in T(w_n) : \|x_{n+1} - x_n\| \leqslant (1 + (1+n)^{-1}) M(T(w_{n+1}), T(w_n)),$$
$$y_n \in A(w_n) : \|y_{n+1} - y_n\| \leqslant (1 + (1+n)^{-1}) M(A(w_{n+1}), A(w_n)),$$
$$z_n \in B(w_n) : \|z_{n+1} - z_n\| \leqslant (1 + (1+n)^{-1}) M(B(w_{n+1}), B(w_n)),$$

$$\begin{aligned} \beta F(\xi_n, v) + \langle \beta N(\eta_n, \gamma_n) + w_n - u_n, v - w_n \rangle \\ + \beta b(v, w_n) - \beta b(w_n, w_n) \geqslant 0, \ \ \forall v \in K, \end{aligned} \tag{3.6}$$

$$\xi_n \in T(w_n) : \|\xi_{n+1} - \xi_n\| \leqslant (1 + (1+n)^{-1}) M(T(u_{n+1}), T(u_n)),$$
$$\eta_n \in A(u_n) : \|\eta_{n+1} - \eta_n\| \leqslant (1 + (1+n)^{-1}) M(A(u_{n+1}), A(u_n)),$$
$$\gamma_n \in B(u_n) : \|\gamma_{n+1} - \gamma_n\| \leqslant (1 + (1+n)^{-1}) M(B(u_{n+1}), B(u_n)),$$

where $n = 1, 2, 3, ...$ and $\rho > 0, \beta > 0$ are constants.

Before discussing the convergence analysis of Iterative algorithm 3.1, we prove the following theorem:

THEOREM 3.2. *Let (u, x, y, z), where $u \in K$, $x \in T(u)$, $y \in A(u)$, $z \in B(u)$ be a solution of GMSVIP (2.1) and let (u_n, ξ_n, η_n, γ_n) be an approximate solution obtained by Iterative algorithm 3.1. Let N is a-partially relaxed strongly mixed monotone with respect to A and B, let F is α-partially relaxed strongly monotone with respect to T, and let b be skew-symmetric. If conditions $(i)-(iii)$ of Theorem 3.1 hold, then*

$$\|u_{n+1} - u\|^2 \leqslant \|w_n - u\|^2 - (1 - 2\rho(\alpha + a))\|u_{n+1} - w_n\|^2, \tag{3.7}$$

$$\|w_n - u\|^2 \leqslant \|u_n - u\|^2 - (1 - 2\beta(\alpha + a))\|w_n - u_n\|^2, \tag{3.8}$$

where $\rho > 0$ and $\beta > 0$ are constants.

Proof. By assumption, (u, x, y, z) satisfies

$$\rho F(x,v) + \langle \rho N(y,z), v - u \rangle + \rho b(v,u) - \rho b(u,u) \geqslant 0, \ \forall\, v \in K, \tag{3.9}$$

$$\beta F(x,v) + \langle \beta N(y,z), v - u \rangle + \beta b(v,u) - \beta b(u,u) \geqslant 0, \ \forall\, v \in K, \tag{3.10}$$

where $\rho > 0, \beta > 0$ are constants.

Now, taking $v = u_{n+1}$ in (3.9) and $v = u$ in (3.5), and then adding the resultant inequalities, we have

$$\langle u_{n+1}-w_n, u-u_{n+1}\rangle \geqslant -\rho[F(x_n,u)+F(x,u_{n+1})]+\rho\langle N(y_n,z_n)-N(y,z), u_{n+1}-u\rangle \\ + \rho[b(u,u) - b(u_{n+1},u) - b(u,u_{n+1}) + b(u_{n+1},u_{n+1})]. \tag{3.11}$$

Since F is α-partially relaxed strongly monotone with respect to T, N is a-partially relaxed strongly mixed monotone with respect to A ans B and b is skew-symmetric, then (3.11) reduces to

$$\langle u_{n+1} - w_n, u - u_{n+1}\rangle \geqslant -\rho(\alpha + a)\|u_{n+1} - w_n\|^2. \tag{3.12}$$

Now, Setting $\bar{u} = u - u_{n+1}$, $\bar{v} = u_{n+1} - w_n$ in (2.2), we obtain

$$\langle u_{n+1} - w_n, u - u_{n+1}\rangle = \frac{1}{2}\{\|u - w_n\|^2 - \|u_{n+1} - w_n\|^2 - \|u - u_{n+1}\|^2\}. \tag{3.13}$$

Combining (3.12) and (3.13), we have

$$\|u_{n+1} - u\|^2 \leqslant \|w_n - u\|^2 - (1 - 2\rho(\alpha + a))\|u_{n+1} - w_n\|^2,$$

the requried (3.7).

Similarly, taking $v = w_n$ in (3.10) and $v = u$ in (3.6) we have

$$\beta F(x,w_n) + \langle \beta N(y,z), w_n - u\rangle + \beta b(w_n,u) - \beta b(u,u) \geqslant 0, \tag{3.14}$$

$$\beta F(\xi_n,u) + \langle \beta N(\eta_n,\gamma_n) + w_n - u_n, u - w_n\rangle + \beta b(u,w_n) - \beta b(w_n,w_n) \geqslant 0. \tag{3.15}$$

From (3.14), (3.15), and the arguments used for obtaining (3.12), we have

$$\langle w_n - u_n, u - w_n\rangle \geqslant -\beta(\alpha + a)\|u_n - w_n\|^2. \tag{3.16}$$

Finally, taking $\bar{v} = w_n - u_n$ and $\bar{u} = u - w_n$ in (2.2), then from resultant and (3.16), we have

$$\|w_n - u\|^2 \leqslant \|u_n - u\|^2 - (1 - 2\beta(\alpha + a))\|w_n - u_n\|^2,$$

the requried (3.8). This complete the proof.

THEOREM 3.3. *Let H be finite dimensional space and let $K \subseteq H$ be a non-empty, closed and convex set. Let the mappings F, N, b satisfy the conditions of Theorem* 3.2*, F, N be continuous and T, A, B be M-Lipschitz continuous with constants $l_1, l_2, l_3 > 0$, respectively, and (u,x,y,z), where $u \in K$, $x \in T(u)$, $y \in A(u)$, $z \in B(u)$, be a solution of GMSVIP* (2.1)*. If* $0 < \rho < \frac{1}{2(\alpha+a)}$, $0 < \beta < \frac{1}{2(\alpha+a)}$ *and the conditions* $(i)-(iii)$ *of Theorem* 3.1 *hold, then the sequences $\{u_n\}$, $\{\xi_n\}$, $\{\eta_n\}$, $\{\gamma_n\}$ generated by Iterative algorithm* 3.1 *converge strongly to u, x, y, z, respectively.*

Proof. Since $0 < \rho, \beta < \frac{1}{2(\alpha+a)}$, it follows from (3.7) and (3.8) that the sequences $\{\|w_n - u\|\}$ and $\{\|u - u_n\|\}$ are nonincreasing, and consequently, $\{u_n\}$ and $\{w_n\}$ are bounded. Furthermore, we have

$$\sum_{n=0}^{\infty}(1-2\rho(\alpha+a))\|u_{n+1}-w_n\|^2 \leqslant \|w_0-u\|^2,$$

$$\sum_{n=0}^{\infty}(1-2\beta(\alpha+a))\|w_n-u_n\|^2 \leqslant \|u_0-u\|^2,$$

which implies that

$$\lim_{n\to\infty}\|u_{n+1}-w_n\|=0$$

and

$$\lim_{n\to\infty}\|w_n-u_n\|=0.$$

Thus

$$\lim_{n\to\infty}\|u_{n+1}-u_n\| \leqslant \lim_{n\to\infty}\|u_{n+1}-w_n\|+\lim_{n\to\infty}\|w_n-u_n\|=0. \tag{3.17}$$

Let $\tilde{u}$ be a limit point of bounded sequence $\{u_n\}$, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $u_{n_j}\to\tilde{u}$ as $n_j\to\infty$, consequently, $\{u_{n_j}\}$ is a Cauchy sequence in H. Next, from Iterative algorithm 3.1 and Lipschitz continuity of T,A,B, we have that $\{\xi_{n_j}\}$, $\{\eta_{n_j}\}$ and $\{\gamma_{n_j}\}$ are Cauchy sequences in H. Let $\xi_{n_j}\to\tilde{x}$, $\eta_{n_j}\to\tilde{y}$ and $\gamma_{n_j}\to\tilde{z}$. Since

$$\begin{aligned} d(\tilde{x},T(\tilde{u})) &\leqslant \|\tilde{x}-\xi_{n_j}\|+M(T(u_{n_j}),T(\tilde{u}))\\ &\leqslant \|\tilde{x}-\xi_{n_j}\|+l_1\|u_{n_j}-\tilde{x}\| \\ &\to\ 0 \text{ as } \ n_j\to\infty, \end{aligned} \tag{3.18}$$

where $d(\tilde{x},T(\tilde{u}))=\inf\{\|\tilde{x}-z\| : z\in T(\tilde{u})\}$. Hence, (3.18) implies that $\tilde{x}\in T(\tilde{u})$.

Similary, we can obtain $\tilde{y}\in A(\tilde{u})$ and $\tilde{z}\in B(\tilde{u})$. Now, replacing w_n by u_{n_j} in (3.5) and (3.6), and taking limit $n_j\to\infty$, then using (3.17), and the continuity of F,N,T,A,B,b, we have

$$F(\tilde{x},v)+\langle N(\tilde{y},\tilde{z}),v-\tilde{u}\rangle+b(v,\tilde{u})-b(\tilde{u},\tilde{u})\geqslant 0,\ \ \forall\, v\in K,$$

i.e. $(\tilde{u},\tilde{x},\tilde{y},\tilde{z})$ is a solution of GMSVIP (2.1), and

$$\|u_{n+1}-\tilde{u}\|^2\leqslant\|u_n-\tilde{u}\|^2.$$

Thus, it follows from the preceding inequality that the sequence $\{u_n\}$ has exactly one limit point $\tilde{u}$ and

$$\lim_{n\to\infty}u_n=\tilde{u}.$$

This completes the proof.

4. Inertial proximal iterative algorithm and results

First, we prove the following Minty-type lemma for GMSVIP (2.1).

LEMMA 4.1. *Let the bifunction F be θ-pseudomonotone with respect to θ where θ is defined as*

$$\theta(u,y,z,v)=\langle N(y,z),v-u\rangle+b(v,u)-b(u,u),\ \forall u,v,y,z\in H,$$

$F(x,v) \geqslant 0$, $\forall v \in K, x \in T(v)$, and convex in the second argument. Let F and N be simultaneously hemicontinuous, let N be mixed monotone with respect to A and B and let b be convex in the first argument. Then GMSVIP (2.1) is equivalent to finding $u \in K$ such that, for each $v \in K$, there exist $x_1 \in T(v)$, $y_1 \in A(v)$, $z_1 \in B(v)$ such that

$$-F(x_1,u) + \langle N(y_1,z_1), v-u\rangle + b(v,u) - b(u,u) \geqslant 0. \tag{4.1}$$

Proof. Let (u,y,z,v) be a solution of GMSVIP (2.1), then

$$F(x,v) + \langle N(y,z), v-u\rangle + b(v,u) - b(u,u) \geqslant 0, \ \forall\, v \in K.$$

Since F is θ-pseudomonotone and N is mixed monotone, preceding inequality implies that for each $v \in K$, there exist $x_1 \in T(v)$, $y_1 \in A(v)$, $z_1 \in B(v)$ such that

$$\begin{aligned} -F(x_1,u) + b(v,u) - b(u,u) &\geqslant -\langle N(y,z), v-u\rangle \\ &\geqslant -\langle N(y_1,z_1), v-u\rangle . \end{aligned}$$

Conversely, let problem (4.1) has a solution. Since K is convex, for any $t \in (0,1]$, $v_t := u + t(v-u) \in K$. Hence there exist $x_t \in T(v_t)$, $y_t \in A(v_t)$, $z_t \in B(v_t)$ such that

$$-F(x_t,u) + \langle N(y_t,z_t), v_t-u\rangle + b(v_t,u) - b(u,u) \geqslant 0.$$

$$\begin{aligned} F(x_t,u) - \langle N(y_t,z_t), v_t-u\rangle &\leqslant tb(v,u) + (1-t)b(u,u) - b(u,u) \\ &= t[b(v,u) - b(u,u)]. \end{aligned} \tag{4.2}$$

Now, using (4.2), we have

$$\begin{aligned} 0 &\leqslant F(x_t,v_t) - \langle N(y_t,z_t), v_t-u_t\rangle \\ &\leqslant tF(x_t,v) + (1-t)F(x_t,u) - t\langle N(y_t,z_t), v_t-v\rangle - (1-t)\langle N(y_t,z_t), v_t-u\rangle \\ &\leqslant tF(x_t,v) - t\langle N(y_t,z_t), v_t-v\rangle + t(1-t)[b(v,u) - b(u,u)]. \end{aligned}$$

Dividing preceding inequality by $t > 0$ and taking the limit as $t \to 0_+$, then there exist $x_0 \in T(u)$, $y_0 \in A(u)$, $z_0 \in B(u)$ such that

$$F(x_0,v) + \langle N(y_0,z_0), v-u\rangle + b(v,u) - b(u,u) \geqslant 0, \ \forall v \in K,$$

where we have used simultaneously hemicontinuity of F and N. This complete the proof.

REMARK 4.1. From Lemma 4.1, we observe that GMSVIP (2.1) and problem (4.1) both have the same solution set. Problem (4.1) is called dual generalized mixed set-valued variational inequality problem (in short, DGMSVIP). We can easily observe that the solution set of DGMSVIP (4.1) is closed and convex.

Next, we consider the following auxiliary variational inequality problem (AVIP) related to GMSVIP (2.1):

(AVIP): For given $u \in K$, find $w \in K$, $x_1 \in T(w)$, $y_1 \in A(w)$, $z_1 \in B(w)$ such that

$$\rho F(x_1,v) + \langle \rho N(y_1,z_1) + w - u, v-w\rangle + \rho b(v,w) - \rho b(w,w) \geqslant 0, \ \forall\, v \in K. \tag{4.3}$$

REMARK 4.2. If $w = v$, clearly w is a solution of GMSVIP (2.1).

The following theorem ensures the existence of solution of AVIP (4.3).

THEOREM 4.1. *Let K be a nonempty, closed and convex subset of Hilbert space H and let $F : H \times H \to R$, $b : H \times H \to R \cup \{+\infty\}$, $N : H \times H \to H$, $T, A, B : H \to 2^H$. Assume that*

(i) *F is continuous and convex in the second argument and θ-pseudomonotone with respect to θ, where θ is defined as $\theta(v,y,z,w,u) = \langle N(y,z) + \rho^{-1}(v-u), w - v\rangle + b(w,v) - b(v,v)$, $\forall u, v, w, y, z \in H$, $F(x,v) \geqslant 0$, $\forall v \in K, x \in T(v)$,*

(ii) *N is mixed monotone with respect to A and B,*

(iii) *b is convex in first argument and continuous,*

(iv) *For given $u \in K$, there exists $w \in K$ such that $x_1 \in T(w), y_1 \in A(w)$, $z_1 \in B(w)$ and*

$$-\rho F(x_1, v) + \langle \rho N(y_1, z_1) + v - u, w - v\rangle + \rho b(w,v) - \rho b(v,v) < 0,$$

(v) *There exists a nonempty set K_0 contained in a compact and convex subset K_1 of K such that*

$$D = \bigcap_{w \in K_0} \bigcap_{x_1 \in T(w)} \bigcap_{y_1 \in A(w)} \bigcap_{z_1 \in B(w)} \{v \in K : -\rho F(x_1, v) + \langle \rho N(y_1, z_1) + v - u, w - v\rangle + \rho b(w,v) - \rho b(v,v) \geqslant 0\}$$

is either empty or compact.

Then AVIP (4.3) has a solution.

Proof. We establish the proof by an indirect method of showing a contradiction. Assume that AVIP (4.3) has no solution, then for each $w \in K$ there exist $x_1 \in T(w)$, $y_1 \in A(w)$, $z_1 \in B(w)$, $v \in K$ such that

$$\rho F(x_1, v) + \langle \rho N(y_1, z_1) + w - u, v - w\rangle + \rho b(v,w) - \rho b(w,w) < 0.$$

Clearly, for given $u \in K$, the set

$$G(w) := \{v \in K : \rho F(x_1, v) + \langle \rho N(y_1, z_1) + w - u, v - w\rangle + \rho b(v,w) - \rho b(w,w) < 0\}, \; x_1 \in T(w), y_1 \in A(w), z_1 \in B(w),$$

is nonempty. Since F is convex in second argument and b is convex in first argument, the set $G(w)$ is convex for each $w \in K$. Thus, $G : K \to 2^K$ is a set-valued mapping such that for each $w \in K$, $G(w)$ is nonempty and convex.

Now, for each $w \in K$,

$$\begin{aligned} G^{-1}(w) &:= \{v \in K, w \in G(v)\} \\ &= \{v \in K : \text{there exist } x_2 \in T(v), y_2 \in A(v), z_2 \in B(v), \\ &\quad \rho F(x_2, w) + \langle \rho N(y_2, z_2) + v - u, w - v\rangle + \rho b(w,v) - \rho b(v,v) < 0\}. \end{aligned}$$

Using θ-pseudomonotonicity of F and mixed monotonicity of N, for each $w \in K$, the complement of $G^{-1}(w)$ is in K, that is,

$$\begin{aligned}
[G^{-1}(w)]^c &= \{v \in K, w \notin G(v)\} \\
&= \{v \in K : \text{there exist } x_2 \in T(v), y_2 \in A(v), z_2 \in B(v), \\
&\qquad \rho F(x_2, w) + \langle \rho N(y_2, z_2) + v - u, w - v\rangle + \rho b(w, v) - \rho b(v, v) \geqslant 0\} \\
&\subseteq \{v \in K : \text{there exist } x_1 \in T(w), y_1 \in A(w), z_1 \in B(w), \\
&\qquad \rho F(x_2, w) + \langle \rho N(y_1, z_1) + v - u, w - v\rangle + \rho b(w, v) - \rho b(v, v) \geqslant 0\} \\
&=: S(w) \subset K.
\end{aligned}$$

Evidently, the continuity of F and b yield the relativity closedness of $S(w)$. Hence, for each $w \in K$, $O_w := [S(w)]^c$ is a relatively open subset of K. Now, by assumption (iv), it follows that $\bigcup_{w\in K} O_w = K$. Indeed, if $v \in K$, by assumption (iv), there exist $w \in K, x_1 \in T(w)$, $y_1 \in A(w)$, $z_1 \in B(w)$ such that $v \in [S(w)]^c = O_w$. Thus, $v \in \bigcup_{w\in K} O_w$. Finally,

$$D := \bigcap_{w\in K} \bigcap_{x_1\in T(w)} \bigcap_{y_1\in A(w)} \bigcap_{z_1\in B(w)} S(w) = \bigcap_{w\in K} \bigcap_{x_1\in T(w)} \bigcap_{y_1\in A(w)} \bigcap_{z_1\in B(w)} O_w^c$$

is compact or empty by assumption (v).

Hence, from Theorem 2.1, there exists $w \in K$ such that $w \in G(w)$, which implies $F(x_1, w) < 0$, a contradiction to our assumption. Hence, AVIP (4.3) admits a solution. This completes the proof.

Based on Remark 4.2, Theorem 4.1 and Nadler's technique [13], we suggest and analyze the following inertial proximal iterative algorithm for GMSVIP (2.1).

Iterative algorithm 4.1. Let $F : H \times H \to R$, $b : H \times H \to R \cup \{+\infty\}$, $N : H \times H \to H$, $T, A, B, : H \to CB(H)$. For $u_0 \in K$, $x_0 \in T(u_0)$, $y_0 \in A(u_0)$, $z_0 \in B(u_0)$, compute the approximate solution (u_n, x_n, y_n, z_n) by the iterative scheme:

$$\begin{aligned}
&\rho F(x_{n+1}, v) + \langle \rho N(y_{n+1}, z_{n+1}) + u_{n+1} - u_n - \alpha_n(u_n - u_{n-1}), v - u_{n+1}\rangle \\
&\qquad + \rho b(v, u_{n+1}) - \rho b(u_{n+1}, u_{n+1}) \geqslant 0, \quad \forall v \in K,
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
x_n \in T(u_n) &: \|x_{n+1} - x_n\| \leqslant (1 + (1+n)^{-1}) M(T(u_{n+1}), T(u_n)), \\
y_n \in A(u_n) &: \|y_{n+1} - y_n\| \leqslant (1 + (1+n)^{-1}) M(A(u_{n+1}), A(u_n)), \\
z_n \in B(u_n) &: \|z_{n+1} - z_n\| \leqslant (1 + (1+n)^{-1}) M(B(u_{n+1}), B(u_n)),
\end{aligned}$$

where $n = 0, 1, 2, ...$, $u_{-1} \equiv u_0$, $\alpha_0 \in [0, 1)$, $0 \leqslant \alpha_n < \alpha_0\ \forall n \in N$, and $\rho > 0$ is a constant.

If $\alpha_n \to 0$ as $n \to \infty$, Iterative algorithm 4.1 reduces to the following iterative algorithm:

Iterative algorithm 4.2. Let $F : H \times H \to R$, $b : H \times H \to R \cup \{+\infty\}$, $N : H \times H \to H$, $T, A, B, : H \to CB(H)$. For $u_0 \in K$, $x_0 \in T(u_0)$, $y_0 \in A(u_0)$, $z_0 \in B(u_0)$, compute the approximate solution (u_n, x_n, y_n, z_n) by the iterative scheme:

$$\begin{aligned}
&\rho F(x_{n+1}, v) + \langle \rho N(y_{n+1}, z_{n+1}) + u_{n+1} - u_n, v - u_{n+1}\rangle \\
&\qquad + \rho b(v, u_{n+1}) - \rho b(u_{n+1}, u_{n+1}) \geqslant 0, \forall v \in K,
\end{aligned} \tag{4.5}$$

$$x_n \in T(u_n) : \|x_{n+1} - x_n\| \leqslant (1 + (1+n)^{-1})M(T(u_{n+1}), T(u_n)),$$
$$y_n \in A(u_n) : \|y_{n+1} - y_n\| \leqslant (1 + (1+n)^{-1})M(A(u_{n+1}), A(u_n)),$$
$$z_n \in B(u_n) : \|z_{n+1} - z_n\| \leqslant (1 + (1+n)^{-1})M(B(u_{n+1}), B(u_n)),$$

where $n = 1, 2, 3, ...$ and $\rho > 0$.

Iterative algorithm 4.2 is known as the proximal method for solving GMSVIP (2.1). This includes as special cases, a number of new and known proximal methods for solving various classes of variational inequality and equilibrium problems, see for example [10,18,19] and the relevant references cited therein.

Now, we prove the following theorem, which is useful for discussing convergence analysis for Iterative algorithms 4.1-4.2.

THEOREM 4.2. *Let* $F : H \times H \to R$, $b : H \times H \to R \cup \{+\infty\}$, $N : H \times H \to H$, $T, A, B, : H \to CB(H)$. *Let the conditions of Lemma* 4.1 *and Theorem* 4.1 *hold and let* b *be skew-symmetric. If* (u, x, y, z) *is a solution of GMSVIP* (2.1) *and* (u_n, x_n, y_n, z_n) *is an approximate solution obtained from Iterative algorithm* 4.1, *then*

$$\begin{aligned} \|u_{n+1} - u\|^2 \leqslant \|u_n - u\|^2 - \|u_{n+1} - u_n - \alpha_n(u_n - u_{n-1})\|^2 \\ + \alpha_n\{\|u_n - u\|^2 - \|u_{n-1} - u\|^2 + 2\|u_n - u_{n-1}\|^2\}. \end{aligned} \tag{4.6}$$

Proof. By assumption, (u, x, y, z) satisfies

$$F(x, v) + \langle N(y, z), v - u\rangle + b(v, u) - b(u, u) \geqslant 0, \ \ \forall\, v \in K.$$

Since F is θ-pseudomonotone and N is mixed monotone with respect to A and B, then preceding inequality implies that for $v \in K$, there exist $x_1 \in T(v)$, $y_1 \in A(v)$, $z_1 \in B(v)$ such that

$$-F(x_1, u) + \langle N(y_1, z_1), v - u\rangle + b(v, u) - b(u, u) \geqslant 0. \tag{4.7}$$

Taking $v = u_{n+1}$ in (4.7) and $v = u$ in (4.4), and then adding the resultant inequalities, we have

$$\begin{aligned} &\langle u_{n+1} - u_n - \alpha_n(u_n - u_{n-1}), u - u_{n+1}\rangle \\ &\qquad \geqslant -\rho F(x_{n+1}, u) - \langle N(y_{n+1}, z_{n+1}), u - u_{n+1}\rangle \\ &\qquad \geqslant \rho[b(u, u) - b(u_{n+1}, u) - b(u, u_{n+1}) + b(u_{n+1}, u_{n+1})] \\ &\qquad \geqslant 0, \end{aligned} \tag{4.8}$$

since b is skew-symmetric. The desired inequality (4.6) follows by making use of Lemma 2.1 and rearranging the terms in (4.8). This complete the proof.

Finally, we discuss the convergence analysis for Iterative algorithm 4.1.

THEOREM 4.3. *Let* H *be a finite dimensional Hilbert space and let the conditions of Theorem* 4.2 *hold. Let* F, N *be continuous and* $T, A, B,$ *be* M*-Lipschitz continuous with constants* l_1, l_2, $l_3 > 0$, *respectively. Let* (u, x, y, z) *be a solution of GMSVIP* (2.1) *and let* (u_n, x_n, y_n, z_n) *be an approximate solution obtained by Iterative algorithm* 4.1. *If* $\sum_{n=1}^{\infty} \|u_{n+1} - u_n\|^2 \leqslant \infty$, *then* (u_n, x_n, y_n, z_n) *converge strongly to* (u, x, y, z).

Proof. First, we consider the case $\alpha_n = 0\ \forall n$. In this case, it follows from (4.6) that the sequence $\{\|u_n - u\|\}$ is nonincreasing and hence $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leqslant \|u_0 - u\|^2,$$

which implies that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0.$$

Using the same arguments used in the proof of Theorem 3.3, we can easily shown that (u_n, x_n, y_n, z_n) converge strongly to $(\tilde{u}, \tilde{x}, \tilde{y}, \tilde{z})$ a solution of GMSVIP (2.1).

Now, we consider the case $0 \leqslant \alpha_n \leqslant \alpha_0 \in [0,1)$. From (4.6) and hypothesis, it follows that

$$\sum_{n=1}^{\infty} \|u_{n+1}-u_n-\alpha_n(u_n-u_{n-1})\|^2 \leqslant \|u_0-u\|^2+\sum_{n=1}^{\infty}\{\alpha_0\|u_n-u\|^2+2\|u_n-u_{n-1}\|^2\}$$
$$\leqslant \infty,$$

which implies that

$$\lim_{n\to\infty} \|u_{n+1} - u_n - \alpha_n(u_n - u_{n-1})\|^2 = 0.$$

Replacing the above argument as in the case $\alpha_n = 0$, we get the desired result. This complete the proof.

REMARK 4.3. (i) Theorems 3.2-3.3 and Theorems 4.2-4.3 generalize and improve the corresponding results of [16, 20].

(ii) In [15-21], authors have not established the existence of solution for auxiliary problems. They developed iterative algorithms without proving the existence of solutions of auxiliary problems. In this paper, we have also proved the existence of solutions for auxiliary problems which are essential for developing the iterative algorithms.

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