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THE p–AFFINE SURFACE AREA

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Abstract. The main purpose of this article is to generalize Petty's affine projection inequality and monotonicity results related to affine surface area to p-affine surface area.

1. Introduction

During the past two decades the notion of affine surface area (from affine differential geometry) and the isoperimetric inequalities related to it, have attracted increased interest. There are a number of reasons for this. First, there are new applications (see, e.g. the survey of [6]). Another reason is the recently discovered extensions of affine surface to arbitrary convex bodies (see, e.g. [7-10], [14], [23-24], [26]). These extensions have led to recent verifications of the conjectured upper-semicontinuity and valuation property of classical as well as extended affine surface area (see [14], [23-24]). In addition, various isoperimetric inequalities involving affine surface area are very closely related to a variety of other important affine isoperimetric inequalities(see, e.g. [11-12], [18]).

In [14], Lutwak introduced extended affine surface area, and proved that all the well-known inequalities which involve affine surface (with theirs equality conditions) hold for arbitrary convex bodies. Some of his results can be stated as follows:

THEOREM A. *Let $K \in \mathcal{K}^n$. Then*

$$\omega_{n-1}^n \Omega(K)^{n+1} \leqslant n^{n+1} \omega_n^n V(\Pi K),$$

with equality if and only if K is an ellipsoid.

THEOREM B. *Let $K \in \mathcal{K}^n, L \in \mathcal{W}^n$. If $\Pi K \subset \Pi L$, then*

$$\Omega(K) \leqslant \Omega(L).$$

The aim of this article is to generalize above two theorems from affine surface area to p-affine surface area. Our main results can be stated as follows:

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THEOREM 1.1. *Let $K \in \mathcal{K}^n$. Then*

$$\Omega_p(K)^{n+p} \leqslant \omega_n^n n^{n+p} V(\Pi_p K)^p,$$

with equality if and only if K is an ellipsoid.

THEOREM 1.2. *Let $K \in \mathcal{K}^n, L \in \mathcal{W}_p^n$. If $\Pi_p K \subset \Pi_p L$, then*

$$\Omega_p(K) \leqslant \Omega_p(L).$$

Thus, this work may be seen as presenting addition evidence of the sentence said by Lutwak [17](there were natural extensions of affine surface areas in the Brunn-Minkowski-Firey theory).

The ideas and techniques of Lutwak [14] play a critical role throughout this paper.

Please see the next section for above interrelated notations, definitions and their background materials.

2. Notation and preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) and $\mathcal{K}_o^n$ denote the subset of $\mathcal{K}^n$ that contains the origin in their interiors in $\mathbb{R}^n$. Let $\mathcal{K}_c^n$ denote the set of convex bodies whose centroids lie at the origin. As usual, S^{n-1} denotes the unit sphere, B_n the unit ball, ω_n the volume of B_n.

If $K \in \mathcal{K}^n$, then the support function of K, $h_K = h(K,\cdot) : \mathbb{R}^n \longrightarrow (0,\infty)$, is defined by

$$h(K,u) = \max\{u \cdot x : x \in K\}, \quad u \in S^{n-1}$$

where $u \cdot x$ denotes the standard inner product of u and x.

For a compact subset L of $\mathbb{R}^n$, which is star-shaped with respect to the origin, we shall use $\rho(L,\cdot)$ to denote its radial function; i.e., for $u \in S^{n-1}$,

$$\rho(L,u) = \rho_L(u) = max\{\lambda > 0 : \lambda u \in L\}.$$

If $\rho(L,\cdot)$ is continuous and positive, L will be called a star body, and φ_o^n will be used to denote the class of star bodies in $\mathbb{R}^n$ containing the origin in their interiors.

For $K \in \mathcal{K}_o^n$, let K^* denote the polar of K; i.e.,

$$K^* = \{x \in \mathbb{R}^n : \ x \cdot y \leqslant 1, \text{ for all } y \in K\}.$$

It is easy to get that

$$\rho(K^*,\cdot) = 1/h(K,\cdot) \quad \text{and} \quad h(K^*,\cdot) = 1/\rho(K,\cdot). \tag{1.1}$$

2.1. Minkowski combination, Firey combination, and mixed volumes

For $K, L \in \mathcal{K}^n$ and $\lambda, \mu \geqslant 0$ (not both zero), the Minkowski linear combination $\lambda K + \mu L \in \mathcal{K}^n$ is defined by

$$h(\lambda K + \mu L,\cdot) = \lambda h(K,\cdot) + \mu h(L,\cdot).$$

The mixed volume, $V_1(K,L)$, of $K,L \in \mathcal{K}^n$ is defined by

$$nV_1(K,L) = \lim_{\varepsilon \to 0} \frac{V(K+\varepsilon L) - V(K)}{\varepsilon}. \tag{1.2}$$

Aleksandrov [1], Fenchel and Jessen [3] have shown that for each $K \in \mathcal{K}^n$, there is a positive Borel measure, $S(K,\cdot)$ on S^{n-1}, called the surface area measure of K, such that

$$V_1(K,Q) = \frac{1}{n}\int_{S^{n-1}} h(Q,u)dS(K,u),$$

for all $Q \in \mathcal{K}^n$.

A convex body $K \in \mathcal{K}^n$ will be said to have a curvature function, $f(K,\cdot) : S^{n-1} \to \mathbb{R}$, if its surface area measure $S(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure, S, and

$$\frac{dS(K,\cdot)}{dS} = f(K,\cdot),$$

almost everywhere with respect to S. Let $\mathcal{F}^n, \mathcal{F}^n_o, \mathcal{F}^n_c$ denote set of all bodies in $\mathcal{K}^n, \mathcal{K}^n_o, \mathcal{K}^n_c$, respectively, that have positive continuous curvature functions.

For real $p \geqslant 1$, $K,L \in \mathcal{K}^n_o$, and $\lambda, \mu \geqslant 0$ (not both zero), the Firey linear combination $\lambda \cdot K +_p \mu \cdot L$, was defined (see [4]) by

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K,\cdot)^p + \mu h(L,\cdot)^P.$$

For $p \geqslant 1$, the p-mixed volume, $V_p(K,L)$, of $K,L \in \mathcal{K}^n_o$, was defined in [16] by

$$\frac{n}{p} V_p(K,L) = \lim_{\varepsilon \to 0} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}. \tag{1.3}$$

That this limit exists was demonstrated in [16]. Obviously,

$$V_p(K,K) = V(K). \tag{1.4}$$

It was shown in [15-16], that for each $K \in \mathcal{K}^n_o$, there is a positive Borel measure, $S_p(K,\cdot)$, on S^{n-1} such that

$$V_p(K,Q) = \frac{1}{n}\int_{S^{n-1}} h(Q,u)^p dS_p(K,u), \tag{1.5}$$

for all $Q \in \mathcal{K}^n_o$. It turns out that the measure $S_p(K,\cdot)$ is absolutely continuous with respect to the surface area measure $S(K,\cdot)$ of K, and has Radon-Nikodym derivative

$$\frac{dS_p(K,\cdot)}{dS(K,\cdot)} = h(K,\cdot)^{1-p}.$$

A convex body $K \in \mathcal{K}^n_o$ will be said to have a positive continuous p-curvature function, $f_p(K,\cdot) : S^{n-1} \to \mathbb{R}$, if $S_p(K)$ is absolutely continuous with respect to spherical Lebesgue measure, S, and

$$\frac{dS_p(K,\cdot)}{dS} = f_p(K,\cdot), \tag{1.6}$$

almost everywhere with respect to S. For $K \in \mathcal{K}^n_o$, $L \in \varphi^n_o$, and $p \geqslant 1$, $V_p(K,L^*)$ was defined by [14]:

$$V_p(K,L^*) = \frac{1}{n}\int_{S^{n-1}} \rho(L,u)^{-p} dS_p(K,u). \tag{1.7}$$

2.2. Affine surface area, extended affine surface area, and p-affine surface area

The affine surface area, $\Omega(K)$, of $K \in \mathcal{F}^n$ can be defined by:

$$\Omega(K) = \int_{S^{n-1}} f(K,u)^{n/(n+1)} dS(u).$$

In [14] (see also [9]), the extended affine surface area, $\Omega(K)$, of $K \in \mathcal{K}^n$, was defined by

$$n^{-\frac{1}{n}} \Omega(K)^{\frac{n+1}{n}} = \inf\{nV_1(K,Q^*)V(Q)^{\frac{1}{n}} : Q \in \varphi_o^n\}.$$

Following, Lutwak [17] defined the p-affine surface area, $\Omega_p(K)$, for $K \in \mathcal{F}_o^n$ by:

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K,u)^{n/(n+p)} dS(u). \tag{1.8}$$

and for $K \in \mathcal{K}_o^n$,

$$n^{-\frac{p}{n}} \Omega_p(K)^{\frac{n+p}{n}} = \inf\{nV_p(K,Q^*)V(Q)^{\frac{p}{n}} : Q \in \varphi_o^n\}. \tag{1.9}$$

For $K \in \mathcal{K}^n$, and $L \in \mathcal{F}^n$, the mixed affine surface area of K and L, $\Omega_{-1}(K,L)$, was defined in [12-13] by

$$\Omega_{-1}(K,L) = \int_{S^{n-1}} f(L,u)^{-1/(n+1)} dS(K,u).$$

Similarly, the p-mixed affine surface, $\Omega_{p,-p}(K,L)$, of $K \in \mathcal{K}_o^n,\ L \in \mathcal{F}_o^n$, was defined in [19], by

$$\Omega_{p,-p}(K,L) = \int_{S^{n-1}} f_p(L,u)^{-p/(n+p)} dS_p(K,u). \tag{1.10}$$

From (1.6) and (1.8), it follows that for $K \in \mathcal{F}_o^n$,

$$\Omega_{p,-p}(K,K) = \Omega_p(K). \tag{1.11}$$

The following inequality of p-mixed affine surface will be used later. For $K \in \mathcal{K}_o^n,\ L \in \mathcal{F}_s^n$, if $n \neq p > 1$, then [19]

$$\Omega_{p,-p}(K,L)^n \geqslant \Omega_p(K)^{n+p}\, \Omega_p(L)^{-p}. \tag{1.12}$$

2.3. Projection body, L_p-projection body

For each $K \in \mathcal{K}^n$, the projection body, ΠK, of K is the unique origin symmetric convex body such that(see [5], [25])

$$h_{\Pi K}(u) = v(K|u^{\perp}),$$

for all $u \in S^{n-1}$, where $u^{\perp}$ denotes the hyperplane, through the origin, that is orthogonal to u and $v(K|u^{\perp})$ denotes $(n-1)$-dimensional volume of the orthogonal projection of K onto $u^{\perp}$.

Recently, Lutwak, Yang and Zhang based on the classical projection body, introduced the notion of L_p-projection body(see [15]). That is, for each $K \in \mathcal{K}^n$ and for each real number $p \geqslant 1$, define the L_p-projection body, $\Pi_p K$, of K to be the origin-symmetric convex body whose support function was given by

$$h^p_{\Pi_p K}(u) = \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} | u \cdot v |^p \, dS_p(K, v), \tag{1.13}$$

where $u, v \in S^{n-1}$, $c_{n,p} = \omega_{n+p}/\omega_2\omega_n\omega_{p-1}$. We write $\Pi^*_p K$, rather than $(\Pi_p K)^*$, for the polar of $\Pi_p K$. Write Π^n_p and Π^{*n}_p for the class of L_p projection bodies and polars of L_p projection bodies; i.e.,

$$\Pi^n_p = \{\Pi_p K : K \in \mathcal{K}^n\},$$

and

$$\Pi^{*n}_p = \{\Pi^*_p K : K \in \mathcal{K}^n\}.$$

In [17], Lutwak defined the class of bodies, $\mathcal{V}^n_p$

$$\mathcal{V}^n_p = \{K \in \mathcal{F}^n_o : \text{there exists a } Q \in \mathcal{K}^n_c \text{ with} f_p(K, \cdot) = h(Q, \cdot)^{-(n+p)}\}, \tag{1.14}$$

and proved that it was a centro-affine invariant class. Similarly, a class of bodies, $\mathcal{W}^n_p$, will be defined by

$$\mathcal{W}^n_p = \{K \in \mathcal{F}^n_o : \text{there exists a } Q \in \Pi^n_p \text{ with} f_p(K, \cdot) = h(Q, \cdot)^{-(n+p)}\}. \tag{1.15}$$

For $p = 1$, the classes $\mathcal{V}^n$ and $\mathcal{W}^n$ has been extensively investigated by Petty [21-22] and Lutwak [14].

2.4. L_p-polar curvature images

In [16], Lutwak gave a weak solution to the L_p-Minkowski problem with even data: If μ is an even positive Borel measure on S^{n-1}, which is not concentrated on a great sphere of S^{n-1}, and $p \in \mathbb{R}$ such that $p > 1$ and $p \neq n$, then there exists a unique centered L, such that $S_p(L, \cdot) = \mu$. Thus, given a continuous function $f : S^{n-1} \to (0, \infty)$, such that $f(\cdot)S(\cdot)$ is an even positive Borel measure on S^{n-1}, then there exists a body $L \in \mathcal{F}^n_s$, such that

$$f_p(L, \cdot) = f(\cdot). \tag{1.16}$$

Let φ^n_s denote the class of star bodies which are symmetric about the origin, and $\mathcal{F}^n_s$ the class of convex bodies in $\mathcal{F}^n$ which are symmetric about the origin.

L_p-polar curvature image: $\Lambda^*_p : \varphi^n_s \to \mathcal{F}^n_s$ was defined in [19] as follows: Suppose $n \neq p > 1$, and $K \in \varphi^n_s$. Let $f(\cdot) = \omega_n \rho(K, u)^{n+p}/V(K)$, then $f(\cdot)S(\cdot)$ is an even positive Borel measure on S^{n-1}, by (1.16), there exists a unique convex body $\Lambda^*_p K \in \mathcal{F}^n_s$, such that

$$f_p(\Lambda^*_p K, \cdot) = \frac{\omega_n}{V(K)} \rho(K, \cdot)^{n+p}. \tag{1.17}$$

It was shown in [19] that p-affine surface area can be defined from p-mixed affine surface:Let $K \in \mathcal{K}_o^n$ and $n \neq p > 1$. Then

$$\Omega_p(K)^{(n+p)/n} = \inf\{\ \Omega_{p,-p}(K,L)\Omega_p(L)^{p/n} : L \in \mathcal{F}_s^n\}. \tag{1.18}$$

From (1.10) and (1.17), it follows immediately that for $n \neq p > 1$, $K \in \mathcal{K}_o^n$, $L \in \varphi_s^n$,

$$\omega_n^p \Omega_{p,-p}(K,\Lambda_p^* L)^{n+p} = n^{n+p} V(L)^p V_p(K,L^*)^{n+p}. \tag{1.19}$$

3. Extension of Petty's affine projection inequality

Petty's affine projection inequality [20] states that for $K \in \mathcal{F}^n$,

$$\omega_{n-1}^n \Omega(K)^{n+1} \leqslant n^{n+1} \omega_n^n V(\Pi K), \tag{2.1}$$

with equality if and only if K is an ellipsoid.

In [14], Lutwak shown that this inequality holds for all convex bodies. In this section, we generalize it to p-affine surface area.

THEOREM 3.1. *For $K \in \mathcal{K}^n$, then*

$$\Omega_p(K)^{n+p} \leqslant n^{n+p} \omega_n^n V(\Pi_p K)^p, \tag{2.2}$$

with equality if and only if K is an ellipsoid.

To prove Theorem 3.1, the following lemmas will be needed:

LEMMA 3.2. ([15], L_p-Petty projection inequality) *Let $K \in \mathcal{K}^n$. Then for $p > 1$,*

$$V(K)^{(n-p)/p} V(\Pi_p^* K) \leqslant \omega_n^{n/p}, \tag{2.3}$$

with equality if and only if K is an ellipsoid.

From the integral representation (1.5), definition (1.13), Fubini's theorem, we have

LEMMA 3.3. *Suppose $K, L \in \mathcal{K}^n$, then for $p > 1$*

$$V_p(K,\Pi_p L) = V_p(L,\Pi_p K). \tag{2.4}$$

For $p = 1$, the identity of (2.4) was presented in [12,14].

Proof of Theorem 3.1. From definition (1.9), it follows that for $Q \in \varphi_o^n$,

$$n^{-p} \Omega_p(K)^{n+p} \leqslant n^n V_p(K,Q^*)^n V(Q)^p.$$

Suppose $L \in \mathcal{K}^n$. Take $\Pi_p^* L$ for Q, and notice that $Q^{**} = Q$, we get

$$\Omega_p(K)^{n+p} \leqslant n^{n+p} V_p(K,\Pi_p L)^n V(\Pi_p^* L)^p. \tag{2.5}$$

By (2.3) and (2.4),

$$\Omega_p(K)^{n+p} \leqslant n^{n+p} w_n^n V_p(L,\Pi_p K)^n V(L)^{p-n},$$

with equality implies that L is an ellipsoid. Now take $\Pi_p K$ for L, use (1.4), and the result is the desired inequality (2.2). Note that equality in (2.2) would imply that $\Pi_p K$ is an ellipsoid. Suppose equality holds in (2.2):

$$\Omega_p(K)^{n+p} = n^{n+p}\omega_n^n V(\Pi_p K)^p.$$

Hence $\Pi_p K$ is a centered ellipsoid, and $V(\Pi_p K)V(\Pi_p^* K) = w_n^2$. From definition (1.9), it follows that for all $Q \in \varphi_o^n$,

$$n^n\omega_n^n V^p(\Pi_p K) = n^{-p}\Omega_p(K)^{n+p} \leqslant n^n V_p^n(K, Q^*)V(Q)^p.$$

Take K^* for Q, and get

$$\omega_n^n V^p(\Pi_p K) \leqslant V(K)^n V(K^*)^p.$$

By the Blaschke-Santaló inequality (see [5],[25]), it shows that

$$\omega_n^n V^p(\Pi_p K) \leqslant \omega_n^{2p} V(K)^{n-p}.$$

But, as noted previously, $V(\Pi_p K) = \omega^2 V(\Pi_p^* K)^{-1}$. Hence the last inequality is

$$\omega_n^{n/p} \leqslant V(K)^{(n-p)/p} V(\Pi_p^* K).$$

The equality condition of (2.3) shows that K must be an ellipsoid.

As an application, we establish a connection between p-affine surface area and L_p-mixed volumes of two L_p-projection bodies.

THEOREM 3.4. *For $K, L \in \mathcal{K}^n$, then*

$$\left[\Omega_p(L)^{n-p}\Omega_p(K)^p\right]^{n+p} \leqslant n^{n(n+p)}\omega_n^{n^2} V_p(\Pi_p K, \Pi_p L)^{np}, \tag{2.6}$$

with equality if and only if K and L are homothetic ellipsoids.

Proof. As in the proof of Theorem 3.1, by (2.5) and (2.3), we have

$$V(L)^{n-p}\Omega_p(K)^{n+p} \leqslant n^{n+p}\omega_n^n V_p(K, \Pi_p L)^n.$$

Take $\Pi_p L$ for L and use Lemma 3.3, we have

$$V(\Pi_p L)^{n-p}\Omega_p(K)^{n+p} \leqslant n^{n+p}\omega_n^n V_p(\Pi_p K, \Pi_p L)^n.$$

Applying Theorem 3.1, we get

$$\left[\Omega_p(L)^{n-p}\Omega_p(K)^p\right]^{n+p} \leqslant n^{n(n+p)}\omega_n^{n^2} V_p(\Pi_p K, \Pi_p L)^{np},$$

with equality if and only if K and L are homothetic ellipsoids.

4. Monotonicity results

Winternitz (see [2]) proved that if $K \in \mathcal{F}^n$ (actually a more restrictive condition) and E is an ellipsoid such that

$$K \subset E,$$

then it follows that

$$\Omega(K) \leqslant \Omega(E).$$

Petty [21] proved the following extension of Winternitz's monotonicity result. If $K \in \mathcal{F}^n, L \in \mathcal{V}^n$, and

$$K \subset L,$$

then

$$\Omega(K) \leqslant \Omega(L). \tag{3.1}$$

In [14], Lutwak showed that this is also correct when K is an arbitrary convex body and proved that: If $K \in \mathcal{K}^n, L \in \mathcal{W}^n$, and

$$\Pi K \subset \Pi L,$$

then

$$\Omega(K) \leqslant \Omega(L). \tag{3.2}$$

In this section, we generalize (3.1) and (3.2) to p-affine area.

THEOREM 4.1. *If $K \in \mathcal{K}^n, L \in \mathcal{V}_p^n$, and*

$$K \subset L, \tag{3.3}$$

then

$$\Omega_p(K) \leqslant \Omega_p(L). \tag{3.4}$$

THEOREM 4.2. *If $K \in \mathcal{K}^n, L \in \mathcal{W}_p^n$, and*

$$\Pi_p K \subset \Pi_p L, \tag{3.5}$$

then

$$\Omega_p(K) \leqslant \Omega_p(L). \tag{3.6}$$

To prove the inequality (3.4), the following lemma will be needed.

LEMMA 4.3. $\Lambda_p^*(\mathcal{K}_c^n) = \mathcal{V}_p^n$.

Proof. If $K \in \mathcal{V}_p^n$, then by the definition (1.14), there exists a $Q \in \mathcal{K}_c^n$ with $f_p(K,\cdot) = h(Q,\cdot)^{-(n+p)}$. From (1.1), it follows that

$$f_p(K) = \rho_{Q^*}^{n+p}.$$

It is now obvious that K is the L_p-polar curvature image of a dilate of Q^*. This shows that $\mathcal{V}_p^n \subset \Lambda_p^*(\mathcal{K}_c^n)$. That $\Lambda_p^*(\mathcal{K}_c^n) \subset \mathcal{V}_p^n$ is an easy consequence of definition (1.17) and (1.1).

In the same way, one easily shows that

$$\Lambda_p^*(\Pi_p^{*n}) = \mathcal{W}_p^n. \tag{3.7}$$

Proof of Theorem 4.1. Since $L \in \mathcal{V}_p^n$. By Lemma 4.3, there exists a $Q \in \mathcal{K}_c^n$, such that $L = \Lambda_p^* Q$. Since $Q^* \in \mathcal{K}^n$, from the monotonicity of L_p-mixed volumes and (3.3), we have

$$V_p(K,Q^*) \leqslant V_p(L,Q^*).$$

But from (1.19) and (1.11), we get

$$n^{n+p} V(Q)^p V_p(K,Q^*)^{n+p} = \omega_n^p \Omega_{p,-p}(K,\Lambda_p^* Q)^{n+p} = \Omega_{p,-p}(K,L)^{n+p},$$

and

$$n^{n+p} V(Q)^p V_p(L,Q^*)^{n+p} = \omega_n^p \Omega_{p,-p}(L,\Lambda_p^* Q)^{n+p} = \Omega_{p,-p}(L,L)^{n+p} = \Omega_p(L)^{n+p}.$$

Hence

$$\Omega_{p,-p}(K,L) \leqslant \Omega_p(L).$$

The desired result is now seen to be a direct consequence of inequality (1.12).

Proof of Theorem 4.2. Since $L \in \mathcal{W}_p^n$, by (3.7) there exists a $Q \in \mathcal{K}^n$, such that $L = \Lambda_p^* \Pi_p^* Q$. From (3.5), we have

$$h_{\Pi_p K} \leqslant h_{\Pi_p L}.$$

Since $S_p(K,\cdot)$ is a positive measure, it follows from (1.5), that

$$V_p(Q,\Pi_p K) \leqslant V_p(Q,\Pi_p L).$$

Hence, by (2.4),

$$V_p(K,\Pi_p Q) \leqslant V_p(L,\Pi_p Q).$$

As in the proof of Theorem 4.1, rewrite this, by using (1.19), as

$$\Omega_{p,-p}(K,\Lambda_p^*\Pi_p^* Q) \leqslant \Omega_{p,-p}(L,\Lambda_p^*\Pi_p^* Q).$$

Recall that $L = \Lambda_p^* \Pi_p^* Q$, use (1.11) and last inequality becomes

$$\Omega_{p,-p}(K,L) \leqslant \Omega_p(L).$$

Inequality (1.12) immediately gives (3.6).

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