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ON WILKER–TYPE INEQUALITIES

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Abstract. In this note, two Wilker-type inequalities involving hyperbolic functions are established.

1. Introduction

J. B. Wilker [1] proposed two open questions as the following statements:
(a) If $0 < x < \pi/2$, then

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \tag{1}$$

(b) There exists a largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x \tag{2}$$

for $0 < x < \pi/2$.

J. S. Sumner et al. [2] affirmed the truth of the Problems above and obtained a further results as follows

THEOREM A. *If* $0 < x < \pi/2$, *then*

$$\frac{16}{\pi^4}x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 < \frac{8}{45}x^3 \tan x. \tag{3}$$

Furthermore, $16/\pi^4$ *and* $8/45$ *are the best constants in* (3).

B. N. Guo et al. [3] gave new proofs of the inequalities (1) and (2). Recently, the author of this paper [4] showed a new simple proof of inequality (1); I. Pinelis [5] got other proof of inequalities (3) by using L'Hospital rules for monotonicity; L. Zhang and L. Zhu [6] gave a new elementary proof of double inequalities (3).

In this note, we establish two Wilker-type inequalities involving hyperbolic functions in the form of inequalities (1) and (2), and obtain the following results.

THEOREM 1. *If* $x > 0$, *then*

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2. \tag{4}$$

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THEOREM 2. *If $x > 0$, then*

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2 + \frac{8}{45}x^3 \tanh x. \tag{5}$$

Furthermore, $8/45$ is the best constant in (5).

2. Three Lemmas

LEMMA 1. ([7 − 9]) *Let $f, g : [a,b] \to \mathbb{R}$ be two continuous functions which are differentiable on (a,b). Further, let $g' \neq 0$ on (a,b). If f'/g' is increasing (or decreasing) on (a,b), then the functions $\frac{f(x)-f(b)}{g(x)-g(b)}$ and $\frac{f(x)-f(a)}{g(x)-g(a)}$ are also increasing (or decreasing) on (a,b).*

LEMMA 2. ([10−11]) *Let l_n and m_n $(n = 1, 2, \cdots)$ be real numbers, and let the power series $L(x) = \sum_{n=1}^{\infty} l_n x^n$ and $M(x) = \sum_{n=1}^{\infty} m_n x^n$ be convergent for $|x| < R$. If $m_n > 0$ for $n = 1, 2, \cdots$, and if l_n/m_n is strictly increasing (or decreasing) for $n = 1, 2, \cdots$, then the function $L(x)/M(x)$ is strictly increasing (or decreasing) on $(0, R)$.*

In the following, we show a new inequality about hyperbolic functions:

LEMMA 3. *Let $x > 0$, then*

$$\left(\frac{\sinh x}{x}\right)^p > \cosh x \tag{6}$$

holds if and only if $p \geqslant 3$.

Proof. Let $F(x) = \frac{\log \cosh x}{\log \frac{\sinh x}{x}} = \frac{f_1(x)}{g_1(x)}$, where $f_1(x) = \log \cosh x$, and $g_1(x) = \log \frac{\sinh x}{x}$. Then

$$\frac{f_1'(x)}{g_1'(x)} = \frac{x \sinh x \tanh x}{x \cosh x - \sinh x} = \frac{f_2(x)}{g_2(x)}, \tag{7}$$

where $f_2(x) = x \sinh t \tanh x$, and $g_2(x) = x \cosh x - \sinh x$.

So

$$\frac{f_2'(x)}{g_2'(x)} = \frac{\tanh x}{x} + 1 + \frac{1}{\cosh^2 x}. \tag{8}$$

Because that

$$\left(\frac{\tanh x}{x}\right)' = \frac{2x - \sinh 2x}{2x^2 \cosh^2 x} < 0$$

and

$$\left(\frac{1}{\cosh^2 x}\right)' = -2\frac{1}{\cosh^2 x} \tanh x < 0$$

for $x \in (0, +\infty)$, we obtain that the function $\frac{f_2'(x)}{g_2'(x)}$ is decreasing on $(0, +\infty)$ by (8). Then $\frac{f_1'(x)}{g_1'(x)} = \frac{f_2(x)-f_2(0)}{g_2(x)-g_2(0)}$ is also decreasing on $(0, +\infty)$ by Lemma 1. This leads to that $F(x) = \frac{f_1(x)-f_1(0)}{g_1(x)-g_1(0)}$ is decreasing on $(0, +\infty)$ by Lemma 1.

In view of $\lim_{x \to 0^+} F(x) = 3$, and $\lim_{x \to +\infty} F(x) = 0$, the proof of Lemma 3 is complete.

3. A short proof of Theorem 1

From Lemma 3 and arithmetic-geometric mean inequality, we have

$$\left(\frac{\sinh x}{x}\right)^2+\frac{\tanh x}{x}\geqslant 2\sqrt{\left(\frac{\sinh x}{x}\right)^2\frac{\tanh x}{x}}=2\sqrt{\left(\frac{\sinh x}{x}\right)^3\frac{1}{\cosh x}}>2.$$

REMARK 1. In the same way, we can obtain

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}\geqslant 2\sqrt{\left(\frac{\sin x}{x}\right)^2\frac{\tan x}{x}}=2\sqrt{\left(\frac{\sin x}{x}\right)^3\frac{1}{\cos x}}>2$$

using the inequality $\left(\frac{\sin x}{x}\right)^3>\cos x \quad (0<x<\frac{\pi}{2})$. That is, we have showed another new simple proof of inequality (1).

REMARK 2. Following the same method as that we had used in proving Lemma 3, the following well-known result (see [12]) reappears:

$$\left(\frac{\sin x}{x}\right)^q>\cos x, 0<x<\frac{\pi}{2} \tag{9}$$

holds if and only if $q\leqslant 3$.

4. A concise proof of Theorem 2

Let $H(x)=\dfrac{\frac{\sinh^2 x\cosh x}{x^2}+\frac{\sinh x}{x}-2\cosh x}{x^3\sinh x}=\dfrac{A(x)}{B(x)}$, where $A(x)=\frac{\sinh^2 x\cosh x}{x^2}+\frac{\sinh x}{x}-2\cosh x$, and $B(x)=x^3\sinh x$.

Then

$$\begin{aligned}
A(x)&=\frac{1}{3x^2}(\sinh^3 x)'+\frac{\sinh x}{x}-2\cosh x\\
&=\frac{1}{4x^2}(\cosh 3x-\cosh x)+\frac{\sinh x}{x}-2\cosh x\\
&=\frac{1}{4}\sum_{n=0}^{\infty}\frac{3^{2n+2}-1}{(2n+2)!}x^{2n}+\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n+1)!}-2\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\\
&=\sum_{n=0}^{\infty}\left[\frac{1}{4}\frac{3^{2n+2}-1}{(2n+2)!}+\frac{1}{(2n+1)!}-\frac{2}{(2n)!}\right]x^{2n}\\
&=\sum_{n=2}^{\infty}\left[\frac{1}{4}\frac{3^{2n+2}-1}{(2n+2)!}+\frac{1}{(2n+1)!}-\frac{2}{(2n)!}\right]x^{2n}\\
&=\sum_{n=2}^{\infty}a_n x^{2n},
\end{aligned}$$

and

$$\begin{aligned} B(x) &= x^3 \sinh x \\ &= \sum_{n=0}^{\infty} \frac{x^{2n+4}}{(2n+1)!} \\ &= \sum_{n=2}^{\infty} \frac{x^{2n}}{(2n-3)!} \\ &= \sum_{n=2}^{\infty} b_n x^{2n}, \end{aligned}$$

where $a_n = \frac{1}{4}\frac{3^{2n+2}-1}{(2n+2)!} + \frac{1}{(2n+1)!} - \frac{2}{(2n)!}$, $b_n = \frac{1}{(2n-3)!}$, $n \geqslant 2$, and $n \in \mathbb{N}^+$.

So

$$\begin{aligned} c_n &= \frac{a_n}{b_n} \\ &= \left[\frac{1}{4}\frac{3^{2n+2}-1}{(2n+2)!} + \frac{1}{(2n+1)!} - \frac{2}{(2n)!}\right] / \left[\frac{1}{(2n-3)!}\right] \\ &= \frac{1}{32}\frac{9^{n+1}-32n^2-40n-9}{4n^5-5n^3+n} \\ &= \frac{1}{32}h(n), \end{aligned}$$

where $h(n) = \frac{9^{n+1}-32n^2-40n-9}{4n^5-5n^3+n}$, $n \geqslant 2$, and $n \in \mathbb{N}^+$.

we obtain results in two cases:

(1) Obviously, $h(2) < h(3)$. That is, $c_2 < c_3$.

(2) Let $h(x) = \frac{9^{x+1}-32x^2-40x-9}{4x^5-5x^3+x}$, $l(x) = \frac{9^x}{4x^5-5x^3+x}$, $m(x) = \frac{32x^2+40x+9}{4x^5-5x^3+x}$, and $x \in [3,+\infty)$. Then

$$h(x) = 9l(x) - m(x), \tag{10}$$

where $x \in [3,+\infty)$.

In the following we shall prove that the function $l(x)$ is increasing on $[3,+\infty)$ and $m(x)$ is decreasing on $[3,+\infty)$.

(i) We compute

$$l'(x) = \frac{l(x)}{4x^5-5x^3+x} n(x), \tag{11}$$

where $n(x) = (\log 9)(4x^5-5x^3+x) - (20x^4-15x^2+1)$, and $x \in [3,+\infty)$. Then $n(3) = (\log 9)(840) - 1480 > 0$, and

$$\begin{aligned} n'(x) &= (\log 9)(20x^4-15x^2+1) - (80x^3-30x), \\ n'(3) &= (\log 9)(1484) - 2070 > 0; \end{aligned}$$

$$n''(x) = (\log 9)(80x^3 - 30x) - (240x^2 - 30),$$
$$n''(3) = (\log 9)(2070) - 2130 > 0;$$
$$n^{(3)}(x) = (\log 9)(240x^2 - 30) - 480x,$$
$$n^{(3)}(3) = (\log 9)(2130) - 1440 > 0;$$
$$n^{(4)}(x) = (\log 9)480x - 480 > 0.$$

So $n(x) > 0$ and $l'(x) > 0$ for $x \in [3, +\infty)$. That is, $l(x)$ is increasing on $[3, +\infty)$.

(ii) Evidently, $m(x)$ is decreasing on $[3, +\infty)$.

Therefore, we obtain that $h(x)$ is increasing on $[3, +\infty)$ by (10).

Combining (1) and (2), we conclude that c_n is increasing for $n = 2, 3, \cdots$, and $H(x) = \frac{A(x)}{B(x)}$ is increasing on $(0, +\infty)$ by Lemma 2.

Furthermore, $\lim_{x \to 0^+} H(x) = \frac{8}{45}$, and $\lim_{x \to +\infty} H(x) = +\infty$. The proof of Theorem 3 is complete.

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