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A CLASS OF SYMMETRIC FUNCTIONS FOR MULTIPLICATIVELY CONVEX FUNCTION

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Abstract. A new symmetric function, which generalizes Hamy symmetric function, is defined. Its properties, including Schur-geometric convexity, are investigated. Some analytic inequalities are also established.

1. Introduction

The unweighted arithmetic and geometric means of positive sequence $x = (x_1, x_2, ..., x_n)$ with $x_i > 0$ for $1 \leqslant i \leqslant n$, denoted by $A_n(x)$ and $G_n(x)$, respectively, are defined as follows

$$A_n(x) = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad G_n(x) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}.$$

Recently, C. P. Niculescu [1] developed a parallel theory to classical theory of convex functions, based on a change of variable formula, by replacing the arithmetic mean with the geometric one. The author defined the multiplicatively convex function, i.e., *GG-convex function*, which reveals an entire new world of beautiful inequalities. Its definition reads as follows:

DEFINITION 1.1. Suppose that I is a subinterval of $(0,\infty)$. A function $f : I \to (0,\infty)$ is called multiplicatively convex if

$$x, y \in I \text{ and } \lambda \in [0,1] \Rightarrow f(x^{1-\lambda}y^{\lambda}) \leqslant f(x)^{1-\lambda} f(y)^{\lambda}. \tag{1.1}$$

Some interesting results related to it are also established therein. In particular, under the presence of continuity and differentiability, the following theorems are proven respectively.

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THEOREM A. [1, Theorem 2.3]. *Suppose that* I *is a subinterval of* $(0,\infty)$. *A continuous function* $f : I \to [0,\infty)$ *is multiplicatively convex if and only if*

$$x, y \in I \Rightarrow f(\sqrt{xy}) \leqslant \sqrt{f(x)f(y)}, \tag{1.2}$$

or

$$x_1, \ldots, x_n \in I \Rightarrow f(\sqrt[n]{x_1 \ldots x_n}) \leqslant \sqrt[n]{f(x_1)\ldots f(x_n)}. \tag{1.3}$$

THEOREM B. [1, Proposition 4.3]. *Let* $f : I \to (0,\infty)$ *be a differential function defined on a subinterval of* $(0,\infty)$. *Then the following assertions are equivalent:*

(i) f *is multiplicatively convex;*

(ii) *The function* $\frac{xf'(x)}{f(x)}$ *is nondecreasing.*

If moreover f *is twice differentiable, then* f *is multiplicatively convex if, and only if,*

$$x[f(x)f''(x) - f'^2(x)] + f(x)f'(x) \geqslant 0 \quad \textit{for every } x > 0. \tag{1.4}$$

It is well known that the technique of majorization play a very important role in the classical study of convex functions. Schur-convex function, which preserve the ordering of majorization, has many important applications in analytic inequalities. Hardy, Littlewood, and Polya were also interested in some inequalities that are related to Schur-convex functions [3]. For more details, the interested readers can see the popular book by Marshall and Olkin [2]. X. M. Zhang [4] proposed the *Schur-geometrically-convex* theory as a parallel one to Schur-convex theory by defining logarithmical majorization and using multiplicatively convex function.

For fixed $n \geqslant 2$, let

$$x = (x_1, x_2, \ldots, x_n), \qquad y = (y_1, y_2, \ldots, y_n)$$

be two n-tuples of positive numbers. And let

$$x_{[1]} \geqslant x_{[2]} \geqslant \ldots \geqslant x_{[n]}, \quad y_{[1]} \geqslant y_{[2]} \geqslant \ldots \geqslant y_{[n]},$$

be their ordered components.

DEFINITION 1.2. [4, p. 89] The $n-$ tuple x is said to be logarithmically majorized by y (in symbols $\ln x \prec \ln y$), if

$$\prod_{i=1}^{m} x_{[i]} \leqslant \prod_{i=1}^{m} y_{[i]}, \; m = 1, 2, \ldots, n-1; \tag{1.5}$$

and

$$\prod_{i=1}^{n} x_{[i]} = \prod_{i=1}^{n} y_{[i]}. \tag{1.6}$$

DEFINITION 1.3. [4, p. 107] Assume that I is a subinterval of $(0,\infty)$. A function $f : I^n \to (0,\infty)$ is called Schur-geometrically-convex function if

$$\ln x \prec \ln y \textit{ on } I^n \Rightarrow f(x) \leqslant f(y). \tag{1.7}$$

The following theorem gives a criteria of a symmetric function on I^n being Schur-geometrically-convex one.

THEOREM C. [4, p. 108]. *Let $f(x) = f(x_1, x_2, ..., x_n)$ be symmetric and have continuous partial derivatives on I^n, where I is a subinterval of $(0,\infty)$. Then $f : I^n \to (0,\infty)$ is a Schur-geometrically-convex function if*

$$(\ln x_1 - \ln x_2)\left(x_1 \frac{\partial f(x)}{\partial x_1} - x_2 \frac{\partial f(x)}{\partial x_2}\right) \geqslant 0 \text{ on } I^n. \tag{1.8}$$

On the other hand, all kinds of means about numbers and their inequalities have stimulated the interests of many researchers all the time (See, for example, [2, 5-8] and the references cited therein.). The Hamy symmetric function [5, 8] is defined as

$$F_n(x,r) = F_n(x_1, ..., x_n; r) = \sum_{1 \leqslant i_1 < i_2 < ... < i_r \leqslant n} \left(\prod_{j=1}^{r} x_{i_j}\right)^{\frac{1}{r}}, \ r = 1, 2, ..., n. \tag{1.9}$$

Corresponding to this is the $r-$ th order Hamy mean

$$\sigma_n(x,r) = \sigma_n(x_1, ..., x_n; r) = \frac{1}{\binom{n}{r}} \sum_{1 \leqslant i_1 < i_2 < ... < i_r \leqslant n} \left(\prod_{j=1}^{r} x_{i_j}\right)^{\frac{1}{r}}, \tag{1.10}$$

where $\binom{n}{r} = \frac{n!}{(n-r)!r!}$. T. Hara *et al.* [5] established the following refinement of the classical arithmetic and geometric means inequality:

$$G_n(x) = \sigma_n(x,n) \leqslant \sigma_n(x,n-1) \leqslant ... \leqslant \sigma_n(x,2) \leqslant \sigma_n(x,1) = A_n(x). \tag{1.11}$$

The paper [6] by H. T. Ku, M. C. Ku and X. M. Zhang contains some interesting inequalities including the fact that $(\sigma_n(x,r))^r$ is log-concave. (See also the popular book [8] by P. S. Bullen.) At present, K. Z. Guan [7] investigated further and generalized Hamy symmetric function and its mean. The Schur-convexity is proved.

Now we define the new symmetric function and its mean.

DEFINITION 1.4. Let $f(x)$ be a non-negative function defined on an interval $I \subset (0,\infty)$. The following symmetric function is defined by

$$\sum_n^r (f(x)) = \sum_{1 \leqslant i_1 < i_2 < ... < i_r \leqslant n} f\left(\prod_{j=1}^{r} x_{i_j}^{1/r}\right), \ r = 1, 2, ..., n, \tag{1.12}$$

where $x_1, x_2, ..., x_n \in I$.

Corresponding to this is the following $r-$ th order mean:

$$\sigma_n^r(f(x)) = \frac{1}{\binom{n}{r}} \sum_{1 \leqslant i_1 < i_2 < ... < i_r \leqslant n} f\left(\prod_{j=1}^{r} x_{i_j}^{1/r}\right), \ r = 1, 2, ..., n. \tag{1.13}$$

Obviously, when $f(x) = x, \ x \in (0,\infty)$, (1.12) and (1.13) reduce to (1.9) and (1.10), respectively. Thus, this symmetric function and its mean generalize the Hamy symmetric function and the $r-$ th order Hamy mean, respectively.

The main purpose of the paper is to investigate the function $\sum_n^r (f(x))$ and its mean $\sigma_n^r(f(x))$. The properties, including Schur-geometric convexity, are proven. Some analytic inequalities are established.

2. Main Results

In this section, we give our results related to $\sum_n^k(f(x))$ and $\sigma_n^r(f(x))$. The properties, including Schur-geometric-convexity, are proven.

THEOREM 2.1. *Suppose that $f : I \to (0,\infty)$ is a multiplicatively convex function, where I is a subinterval of $(0,\infty)$. Then*

$$\sigma_n^n(f(x)) \leqslant \sigma_n^{n-1}(f(x)) \leqslant \dots \leqslant \sigma_n^2(f(x)) \leqslant \sigma_n^1(f(x)). \tag{2.1}$$

Proof. It suffices to prove that

$$\sigma_n^{k+1}(f(x)) \leqslant \sigma_n^k(f(x)),\ k=1,2,\dots,n-1. \tag{2.2}$$

Since f is multiplicatively convex, using arithmetic-geometric mean inequality and Theorem A yields

$$\begin{aligned}
&\sum_{1\leqslant i_1<i_2<\dots<i_{k+1}\leqslant n} f\left((x_{i_1}x_{i_2}\dots x_{i_{k+1}})^{1/(k+1)}\right)\\
&= \sum_{1\leqslant i_1<i_2<\dots<i_{k+1}\leqslant n} \left\{f\left(\left((x_{i_2}\dots x_{i_{k+1}})^{\frac{1}{k}}(x_{i_1}x_{i_3}\dots x_{i_{k+1}})^{\frac{1}{k}}\dots(x_{i_1}x_{i_2}\dots x_{i_k})^{\frac{1}{k}}\right)^{\frac{1}{k+1}}\right)\right\}\\
&\leqslant \sum_{1\leqslant i_1<i_2<\dots<i_{k+1}\leqslant n} \{f\left((x_{i_2}\dots x_{i_{k+1}})^{1/k}\right)f\left((x_{i_1}x_{i_3}\dots x_{i_{k+1}})^{1/k}\right)\\
&\qquad \dots f\left((x_{i_1}x_{i_2}\dots x_{i_k})^{1/k}\right)\}^{1/(k+1)}\\
&\leqslant \sum_{1\leqslant i_1<i_2<\dots<i_{k+1}\leqslant n} \frac{1}{k+1}\{f\left((x_{i_2}\dots x_{i_{k+1}})^{1/k}\right)+f\left((x_{i_1}x_{i_3}\dots x_{i_{k+1}})^{1/k}\right)\\
&\qquad +\dots+f\left((x_{i_1}x_{i_2}\dots x_{i_k})^{1/k}\right)\}\\
&= \frac{n-k}{k+1}\sum_{1\leqslant i_1<i_2<\dots<i_k\leqslant n} f\left((x_{i_1}x_{i_2}\dots x_{i_k})^{1/k}\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
\sigma_n^{k+1}(f(x)) &= \frac{1}{\binom{n}{k+1}}\sum_{1\leqslant i_1<i_2<\dots<i_{k+1}\leqslant n} f\left((x_{i_1}x_{i_2}\dots x_{i_{k+1}})^{1/(k+1)}\right)\\
&\leqslant \frac{1}{\binom{n}{k+1}}\frac{n-k}{k+1}\sum_{1\leqslant i_1<i_2<\dots<i_k\leqslant n} f\left((x_{i_1}x_{i_2}\dots x_{i_k})^{1/k}\right)\\
&= \frac{1}{\binom{n}{k}}\sum_{1\leqslant i_1<i_2<\dots<i_k\leqslant n} f\left((x_{i_1}x_{i_2}\dots x_{i_k})^{1/k}\right)\\
&= \sigma_n^k(f(x)).
\end{aligned}$$

□

COROLLARY 2.2. [5]. *Suppose that* $x_i > 0,\ i = 1,2,...,n,$ *then*

$$G_n(x) = \sigma_n(x,n) \leqslant \sigma_n(x,n-1) \leqslant ... \leqslant \sigma_n(x,2) \leqslant \sigma_n(x,1) = A_n(x). \tag{2.3}$$

Proof. Let $f(x) = x, x \in (0,\infty)$. Using Theorem B, we can easily see that $f(x)$ is a multiplicatively convex function in $(0,\infty)$. By Theorem 2.1, we get (2.3) (or (1.11)) and so the proof is complete. □

THEOREM 2.3. *Suppose that* $f : I \to (0,\infty)$ *has a continuous derivative on* I. *If* $f(x)$ *is monotonic and multiplicatively convex on* I, *then* $\sum_n^r(f(x))$ *is a Schur-geometrically-convex function on* I^n, *where* I *is a subinterval of* $(0,\infty)$.

In the proof we shall use the following lemma.

LEMMA 2.4. *Assume that* $f : I \to (0,\infty)$ *has a continuous derivative on* I, *where* I *is a subinterval of* $(0,\infty)$. *If* $f(x)$ *is monotonic and multiplicatively convex on* I, *then*

$$(\ln x_1 - \ln x_2)(x_1 f'(x_1) - x_2 f'(x_2)) \geqslant 0,\ \forall\, x_1, x_2 \in I. \tag{2.4}$$

Proof. We assume, without loss of generality, that $f(x)$ is increasing and multiplicatively convex on I. The case where $f(x)$ is decreasing and multiplicatively convex is similar and so is omitted. From Theorem B, it follows that

$$(\ln x_1 - \ln x_2)\left(\frac{x_1 f'(x_1)}{f(x_1)} - \frac{x_2 f'(x_2)}{f(x_2)}\right) \geqslant 0.$$

This implies that

$$(\ln x_1 - \ln x_2)\left(x_1 f'(x_1) - x_2 f'(x_2)\frac{f(x_1)}{f(x_2)}\right) \geqslant 0. \tag{2.5}$$

Thus, combining (2.5) with the increasing nature of $f(x)$, we obtain

$$\begin{aligned}
&(\ln x_1 - \ln x_2)(x_1 f'(x_1) - x_2 f'(x_2)) \\
&\quad = (\ln x_1 - \ln x_2)\left(x_1 f'(x_1) - x_2 f'(x_2)\frac{f(x_1)}{f(x_2)}\right) \\
&\qquad + x_2 f'(x_2)(\ln x_1 - \ln x_2)\left(\frac{f(x_1)}{f(x_2)} - 1\right) \\
&\quad = (\ln x_1 - \ln x_2)\left(x_1 f'(x_1) - x_2 f'(x_2)\frac{f(x_1)}{f(x_2)}\right) \\
&\qquad + \frac{x_2 f'(x_2)}{f(x_2)}(\ln x_1 - \ln x_2)(f(x_1) - f(x_2)) \\
&\quad \geqslant 0.
\end{aligned}$$

□

Proof of Theorem 2.3. It is obvious that the function $\sum_n^r(f(x))$ is symmetric and has a continuous partial derivative on I^n. By Theorem C, we only need to prove that

$$(\ln x_1 - \ln x_2)\left(x_1\frac{\partial \sum_n^r(f(x))}{\partial x_1} - x_2\frac{\partial \sum_n^r(f(x))}{\partial x_2}\right) \geqslant 0 \text{ on } I^n. \tag{2.6}$$

To this end, we consider the following three possible cases for r.

Case 1. When $r = 1$. It is clear that $\sum_n^1(f(x)) = \sum_{i=1}^n f(x_i)$. Differentiating it with respect to x_i $(i = 1, 2)$ yields

$$\frac{\partial \sum_n^1(f(x))}{\partial x_i} = f'(x_i),\ i = 1, 2.$$

From Lemma 2.4 it follows that

$$(\ln x_1 - \ln x_2)\left(x_1\frac{\partial \sum_n^1(f(x))}{\partial x_1} - x_2\frac{\partial \sum_n^1(f(x))}{\partial x_2}\right)$$
$$= (\ln x_1 - \ln x_2)(x_1 f'(x_1) - x_2 f'(x_2)) \geqslant 0.$$

Case 2. When $r = 2$.
If $n = 2$, it is obvious that

$$\sum_2^2(f(x)) = f(\sqrt{x_1x_2}).$$

Differentiating the above with respect to x_1 and x_2 and setting $u = \sqrt{x_1x_2}$, we have

$$\frac{\partial \sum_2^2(f(x))}{\partial x_1} = \frac{f'(u)}{2}\sqrt{\frac{x_2}{x_1}}, \qquad \frac{\partial \sum_2^2(f(x))}{\partial x_2} = \frac{f'(u)}{2}\sqrt{\frac{x_1}{x_2}}.$$

One can easily find that (2.6) holds.
If $n \geqslant 3$, we can easily derive that

$$\sum_n^2(f(x)) = \sum_{1\leqslant i<j\leqslant n} f(\sqrt{x_ix_j}) = \sum_{j=2}^n f(\sqrt{x_1x_j}) + \sum_{2\leqslant i<j\leqslant n} f(\sqrt{x_ix_j}). \tag{2.7}$$

Differentiating (2.7) with respect to x_1, we obtain

$$\frac{\partial \sum_n^2(f(x))}{\partial x_1} = \frac{1}{2}\sum_{j=2}^n f'(\sqrt{x_1x_j})\sqrt{\frac{x_j}{x_1}}$$
$$= \frac{1}{2}\left[f'(\sqrt{x_1x_2})\sqrt{\frac{x_2}{x_1}} + \sum_{j=3}^n f'(\sqrt{x_1x_j})\sqrt{\frac{x_j}{x_1}}\right].$$

Similarly, we can get

$$\frac{\partial \sum_n^2(f(x))}{\partial x_2} = \frac{1}{2}\left[f'(\sqrt{x_1x_2})\sqrt{\frac{x_1}{x_2}} + \sum_{j=3}^{n} f'(\sqrt{x_2x_j})\sqrt{\frac{x_j}{x_2}}\right].$$

Set $u = \sqrt{x_1x_j}$, $v = \sqrt{x_2x_j}$. Obviously, $u, v \in I$. Therefore, it follows from Lemma 2.4 that

$$\begin{aligned}
&(\ln x_1 - \ln x_2)\left(x_1\frac{\partial \sum_n^2(f(x))}{\partial x_1} - x_2\frac{\partial \sum_n^2(f(x))}{\partial x_2}\right)\\
&\qquad = \frac{(\ln x_1 - \ln x_2)}{2}\sum_{j=3}^{n}(uf'(u) - vf'(v))\\
&\qquad = \sum_{j=3}^{n}(uf'(u) - vf'(v))(\ln u - \ln v)\\
&\qquad \geqslant 0.
\end{aligned}$$

Case 3. When $3 \leqslant r \leqslant n$.

Similar to the argument of Case 2, we have

$$\begin{aligned}
\frac{\partial \sum_n^r(f(x))}{\partial x_1} = &\sum_{3\leqslant i_1<\ldots<i_{r-1}\leqslant n} f'(\sqrt[r]{x_1x_{i_1}\ldots x_{i_{r-1}}})\frac{\sqrt[r]{x_1x_{i_1}\ldots x_{i_{r-1}}}}{rx_1}\\
&+\sum_{3\leqslant i_1<\ldots<i_{r-2}\leqslant n} f'(\sqrt[r]{x_1x_2x_{i_1}\ldots x_{i_{r-2}}})\frac{\sqrt[r]{x_1x_2x_{i_1}\ldots x_{i_{r-2}}}}{rx_1},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \sum_n^r(f(x))}{\partial x_2} = &\sum_{3\leqslant i_1<\ldots<i_{r-1}\leqslant n} f'(\sqrt[r]{x_2x_{i_1}\ldots x_{i_{r-1}}})\frac{\sqrt[r]{x_2x_{i_1}\ldots x_{i_{r-1}}}}{rx_2}\\
&+\sum_{3\leqslant i_1<\ldots<i_{r-2}\leqslant n} f'(\sqrt[r]{x_1x_2x_{i_1}\ldots x_{i_{r-2}}})\frac{\sqrt[r]{x_1x_2x_{i_1}\ldots x_{i_{r-2}}}}{rx_2}.
\end{aligned}$$

Put $u_* = \sqrt[r]{x_1x_{i_1}\ldots x_{i_{r-1}}}, v_* = \sqrt[r]{x_2x_{i_1}\ldots x_{i_{r-1}}}$, it is clear that $u_*, v_* \in I$. Thus, by Lemma 2.4, we can find that

$$\begin{aligned}
&(\ln x_1 - \ln x_2)\left(x_1\frac{\partial \sum_n^r(f(x))}{\partial x_1} - x_2\frac{\partial \sum_n^r(f(x))}{\partial x_2}\right)\\
&\qquad = \frac{(\ln x_1 - \ln x_2)}{r}\sum_{3\leqslant i_1<\ldots<i_{r-1}\leqslant n}(u_*f'(u_*) - v_*f'(v_*))\\
&\qquad = \sum_{3\leqslant i_1<\ldots<i_{r-1}\leqslant n}(u_*f'(u_*) - v_*f'(v_*))(\ln u_* - \ln v_*)\\
&\qquad \geqslant 0.
\end{aligned}$$

Combining the cases 1-3, we have completed the proof of the theorem. □

COROLLARY 2.5. *The Hamy symmetric function* $F_n(x,r) = F_n(x_1,x_2,...,x_n;r)$ *is Schur-geometrically-convex in* R_+^n, *where* $R_+ = (0,\infty)$.

Proof. Let $f(x) = x,\ x \in (0,\infty)$. One can easily see that $f(x)$ is increasing and multiplicatively convex in R_+. Using Theorem 2.3, we have completed the proof. □

3. Applications

In this section, some analytic inequalities are established by use of the results in section 2.

THEOREM 3.1. *Let* $x_i > 0, i = 1,2,...,n$, *and set* $G_r = \prod_{1 \leqslant i_1 < ... < i_r \leqslant n} (x_{i_1} \dots x_{i_r})^{1/r}$. *The following statements are true.*

(i) *If* $x_i \in (0,1), i = 1,2,...,n$, *then*

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n}\frac{1}{1-x_i} &\geqslant \frac{1}{\binom{n}{2}}\sum_{1\leqslant i_1<i_2\leqslant n}\frac{1}{1-G_2} \geqslant \dots \\ &\geqslant \frac{1}{\binom{n}{n-1}}\sum_{1\leqslant i_1\dots<i_{n-1}\leqslant n}\frac{1}{1-G_{n-1}} \geqslant \frac{1}{1-G_n(x)}. \end{aligned} \tag{3.1}$$

(ii) *If* $x_i \in (1,\infty), i = 1,2,...,n$, *then*

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i-1} &\geqslant \frac{1}{\binom{n}{2}}\sum_{1\leqslant i_1<i_2\leqslant n}\frac{1}{G_2-1} \geqslant \dots \\ &\geqslant \frac{1}{\binom{n}{n-1}}\sum_{1\leqslant i_1\dots<i_{n-1}\leqslant n}\frac{1}{G_{n-1}-1} \geqslant \frac{1}{G_n(x)-1}. \end{aligned} \tag{3.2}$$

(iii) [9, p. 5] *If* $a_i > 1, i = 1,2,...,n$, *then*

$$\frac{A_n(a)}{A_n(a-1)} \leqslant \frac{G_n(a)}{G_n(a-1)}. \tag{3.3}$$

(iv) *If* $a_i > 0, i = 1,2,...,n$, *then*

$$\frac{G_n(a)}{G_n(1+a)} \leqslant \frac{A_n(a)}{A_n(1+a)}. \tag{3.4}$$

Proof. (i) Let $f(x) = \frac{1}{1-x}, x \in (0,1)$. Simply calculation shows that

$$x[f(x)f''(x) - f'^2(x)] + f(x)f'(x) = \frac{1}{(1-x)^4} \geqslant 0. \tag{3.5}$$

By Theorem B and Theorem 2.1, direct calculating arrives at (3.1).

(ii) Set $f(x) = \frac{1}{x-1}, x \in (1,\infty)$. Direct and standard computing leads to

$$x[f(x)f''(x) - f'^2(x)] + f(x)f'(x) = \frac{2x-1}{(1-x)^4} \geqslant 0. \tag{3.6}$$

Using Theorem B and Theorem 2.1 again, one can easily find that (3.2) holds.

(iii) Replacing x_i of (3.1) by $1-\frac{1}{a_i}$ $(a_i>1)$ and noticing $\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1-x_i}\geqslant\frac{1}{1-G_n(x)}$, one can find that

$$\frac{A_n(a)}{A_n(a-1)}\leqslant\frac{G_n(a)}{G_n(a-1)}.$$

(iv) Let $x_i=1+\frac{1}{a_i}, i=1,2,...,n$. By (3.2), one immediately obtain (3.4). □

THEOREM 3.2. *Assume that* $f:I\to(0,\infty)$ *has a continuous derivative on* I. *If* $f(x)$ *is monotonic and multiplicatively convex on* I, *where* I *is a subinterval of* $(0,\infty)$, *then for every* $x_i\in I(1\leqslant i\leqslant n)$ *we have*

$$f\left(\prod_{i=1}^{n}x_i^{1/n}\right)\leqslant\frac{1}{\binom{n}{r}}\sum_{1\leqslant i_1<\ldots<i_r\leqslant n}f\left(\prod_{j=1}^{r}(x_{i_j})^{1/r}\right),\quad r=1,2,...,n. \tag{3.7}$$

Proof. Let $s=\left(\left(\prod_{i=1}^{n}x_i\right)^{1/n},\left(\prod_{i=1}^{n}x_i\right)^{1/n},...,\left(\prod_{i=1}^{n}x_i\right)^{1/n}\right)$ and $x=(x_1,x_2,...,x_n)$. From Lemma 7.1 of [4, p. 97], it follows that $\ln s\prec\ln x$. This together with Theorem 2.3 leads to (3.7) and the proof is complete. □

REMARK. Let $f(x)=x, x\in(0,\infty)$, and $r=1$, we get the arithmetic-geometric mean inequality: $G_n(x)\leqslant A_n(x)$.

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