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REGULARITY RESULTS FOR DEGENERATE ELLIPTIC EQUATIONS RELATED TO GAUSS MEASURE

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Abstract. In this paper we study a Dirichlet problem relative to the equation $Lu = g\varphi - (f_i\varphi)_{x_i}$, where L is a linear elliptic operator with lower-order terms whose ellipticity condition is given in terms of the function $\varphi(x) = (2\pi)^{-\frac{n}{2}} \exp\left(-|x|^2/2\right)$, the density of the Gaussian measure. We use the notion of rearrangement with respect to the Gauss measure to obtain a priory estimate of the solution u and we study the summability of u in the Lorentz-Zygmund spaces when g and f_i varies in suitable Lorent-Zygmund spaces.

1. Introduction

In this paper we study the problem

$$\begin{cases} -\left(a_{ij}(x)u_{x_i}\right)_{x_j} - \left(d_i(x)u\right)_{x_i} + b_i(x)u_{x_i} + c(x)u = g\varphi - \left(f_i\varphi\right)_{x_i} & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

where Ω is an open set of $\mathbb{R}^n$ $(n \geqslant 2)$, $\varphi(x) = (2\pi)^{-\frac{n}{2}} \exp\left(-|x|^2/2\right)$ is the density of the Gauss measure and $a_{ij}(x)$, $d_i(x)$, $b_i(x)$, $i,j = 1,..,n$, and $c(x)$ are measurable functions on Ω such that

(*i*) $a_{ij}(x)\xi_i\xi_j \geqslant \varphi(x)|\xi|^2$ for $a.e.\ x \in \Omega, \forall \xi \in \mathbb{R}^n$,

(*ii*) $\frac{a_{ij}(x)}{\varphi(x)} \in L^\infty(\Omega)$,

(*iii*) $\left(\sum b_i^2(x)\right)^{\frac{1}{2}} \leqslant b(x)\varphi(x)$, $b(x) \in L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)$,

(*iv*) $\left(\sum d_i^2(x)\right)^{\frac{1}{2}} \leqslant d(x)\varphi(x)$, $d(x) \in L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)$,

(*v*) $\frac{c(x)}{\varphi(x)} \in L^\infty(\Omega)$ and $c(x) \geqslant 0$,

(*vi*) $g(x) \in L^2(\log L)^{-\frac{1}{2}}(\varphi,\Omega)$,

(*vii*) $f_i(x) \in L^2(\varphi,\Omega)$ $i = 1,..n,$ $\sum f_i^2(x) = f^2(x)$.

Problem (1.1) is related to the generator of Ornstein-Uhlenbeck semigroup (see [11]).

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Degenerate elliptic equations has been studied when Ω is bounded (see for example [24], [31], [3] and [4]). In our case Ω can be not bounded and the ellipticity condition (i) is given in term of Gaussian density.

First of all we observe that the natural space for searching weak solution of the problem (1.1) (see Section 3 for the definition) is the weighted Sobolev space $H_0^1(\varphi,\Omega)$, that is the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{H_0^1(\varphi,\Omega)} = \left(\int_\Omega |\nabla u(x)|^2 \ \varphi(x)dx\right)^{\frac{1}{2}}.$$

We obtain a priory estimates using symmetrization techniques. It is well known that if Ω is bounded such estimates can be obtained via Schwarz symmetrization by comparing the solution of original problem with the solution of a simpler one which is defined in a ball and has spherical symmetric data (see for example [27, 29, 28, 2, 3, 4] and [5]).

In this case we use the notion of rearrangement with respect to Gauss measure (see Section 2 for the definition) and we compare the solution of problem (1.1) with the solution of a problem defined in an half-space having the same Gauss measure as Ω. More precisely, if u is the solution of problem (1.1), we prove that

$$u^\star(x) \leqslant w(x), \tag{2}$$

where $u^\star(x)$ is the rearrangement of $u(x)$ with respect to Gauss measure and $w(x)$ is the solution of the following "symmetrized" problem:

$$\begin{cases} -\left(w_{x_1}\varphi(x)\right)_{x_1} + \left(D(\Phi(x_1))w\varphi(x)\right)_{x_1} - B(\Phi(x_1))w_{x_1}\varphi(x) & \\ \qquad = g^\star(x_1)\varphi(x) - \left(F(\Phi(x_1))\,\varphi(x)\right)_{x_1} & \text{in } \Omega^\star \\ w = 0 & \text{on } \partial\Omega^\star. \end{cases} \tag{1.3}$$

Here $\Omega^\star$ is the half-space $\{x = (x_1,...,x_n) \in \mathbb{R}^n : x_1 > \lambda\}$, with $\lambda \in \mathbb{R}$ such that $\gamma_n(\Omega) = \gamma_n(\Omega^\star)$, $g^\star$ is the rearrangement with respect to Gauss measure of the function g, F^2, D^2 and B^2 are functions related to f^2, d^2 and b^2, built on the level sets of u (see Section 2 for the definition) and $\Phi(x_1) = \frac{1}{\sqrt{2\pi}}\int_{x_1}^{+\infty}\exp\left(-\frac{t^2}{2}\right)\,dt$.

Comparison (2) provides estimates of u in terms of the solution of a problem of the same type of (1.1), but simpler, because it is defined in an half-space and its coefficients depend only on one variable. Moreover we are able to prove an estimate of norm of u in $H_0^1(\varphi,\Omega)$ that gives also a sufficient condition for the existence in terms of the summability of the data.

When $d_i(x) \equiv 0$ or $b_i(x) \equiv 0$ $i = 1,...,n$, the solution of problem (1.3) can be explicitly written (see Corollary 3.1 and Corollary 3.2), then the pointwise comparison (2) gives an explicit estimate for the solution u which is the starting point to obtain regularity result.

Let us observe that by Gross inequality we have that if $u \in H_0^1(\varphi,\Omega)$ is a solution of problem (1.1) then u belongs to Lorentz-Zygmund space $L^2(\log L)^{\frac{1}{2}}(\varphi,\Omega)$ (see Section 2 for the definition). We study how the summability of u improves by improving the summability of the data f and g in Lorentz-Zygmund spaces $L^{p,q}(\log L)^{\alpha}(\varphi,\Omega)$.

Comparison results using rearrangement with respect to Gauss measure are proved in [9] when $d_i(x) \equiv b_i(x) \equiv c(x) \equiv f_i(x) \equiv 0 \quad i = 1,..,n$ and in [15] when $d_i(x) \equiv f_i(x) \equiv 0$ and $c(x) \geqslant c_0(x)\varphi(x)$. Parabolic case has been studied in [13].

2. Notations and preliminary results

In this section we recall some definitions and results which will be useful in the following. Let γ_n be the n-dimensional Gauss measure on $\mathbb{R}^n$ defined by

$$\gamma_n(dx) = \varphi(x)\,dx = (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{2}\right) dx, \quad x \in \mathbb{R}^n$$

normalized by $\gamma_n(\mathbb{R}^n) = 1$.

We will denote by $\Phi(\tau)$ the measure of the half-space $\{x \in \mathbb{R}^n : x_1 > \tau\}$, i.e.

$$\Phi(\tau) = \gamma_n(\{x \in \mathbb{R}^n : x_1 > \tau\}) = \frac{1}{\sqrt{2\pi}} \int_{\tau}^{+\infty} \exp\left(-\frac{t^2}{2}\right) dt \quad \forall \tau \in \mathbb{R} \cup \{-\infty, +\infty\}.$$

We observe that we have

$$\lim_{t \to 0^+;1^-} (2\pi)^{-\frac{1}{2}} \frac{\exp\left(-\frac{\Phi^{-1}(t)^2}{2}\right)}{t(2\log\frac{1}{t})^{\frac{1}{2}}} = 1 \tag{4}$$

One of the main tools to prove the comparison result is the isoperimetric inequality with respect to Gauss measure. Let us define the perimeter with respect to Gauss measure as

$$P(E) = (2\pi)^{-\frac{n}{2}} \int_{\partial E} \exp\left(-\frac{|x|^2}{2}\right) \mathcal{H}_{n-1}(dx),$$

where E is a $(n-1)-$rectifiable set and $\mathcal{H}_{n-1}$ denotes the $(n-1)-$dimensional Hausdorff measure. For all $\lambda \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$, we denote by $H(\xi,\lambda)$ the half-space defined by

$$H(\xi,\lambda) = \{x \in \mathbb{R}^n : (x,\xi) > \lambda\}$$

and we set $H(\xi,\lambda) = \mathbb{R}^n$ if $\lambda = -\infty$ and $H(\xi,\lambda) = \emptyset$ if $\lambda = +\infty$. It is well known (see [12], [16] and [21]) that among all measurable sets of $\mathbb{R}^n$ with prescribed Gauss measure, the half-spaces take the smallest perimeter, that is

$$P(E) \geqslant P(H(\xi,\lambda))$$

for all subsets $E \subset \mathbb{R}^n$ such that $\gamma_n(E) = \gamma_n(H(\xi,\lambda))$.

Now we give the notion of rearrangement. If u is a measurable function in Ω, we denote by

(a) u^* the usual decreasing rearrangement of u with respect to Lebesgue measure, i.e.[1]

$$u^*(s) = \inf\{t \geqslant 0 : |\{x \in \Omega : |u| > t\}| \leqslant s\} \quad s \in]0,1]$$

(b) $u^{\circledast}$ the decreasing rearrangement of u with respect to Gauss measure, i.e.

$$u^{\circledast}(s) = \inf\{t \geqslant 0 : \mu(t) \leqslant s\} \quad s \in]0,1],$$

where $\mu(t) = \gamma_n(\{x \in \Omega : |u| > t\})$ is the distribution function of u;

(c) $u^{\star}$ the rearrangement with respect to Gauss measure of u, i.e.

$$u^{\star}(x) = u^{\circledast}(\Phi(x_1)) \quad x \in \Omega^{\star},$$

where $\Omega^{\star} = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_1 > \lambda\}$ is the half-space such that $\gamma_n(\Omega^{\star}) = \gamma_n(\Omega)$.

By definition $u^{\star}$ is a function which depend only on the first variable and its level sets are half spaces (see [17]); moreover $u, u^{\circledast}$ and $u^{\star}$ have the same distribution function.

For general results about the properties of rearrangement with respect to a positive measure see, for example, [14] and [26]. We just recall that if $u(x)$, $v(x)$ are measurable functions the Hardy type inequality

$$\int_{\Omega} |u(x)v(x)| \gamma_n(dx) \leqslant \int_{\Omega^{\star}} u^{\star}(x) v^{\star}(x) \ \gamma_n(dx) = \int_0^{\gamma_n(\Omega)} u^{\circledast}(s) v^{\circledast}(s)\, ds \qquad (5)$$

holds. The L^p weighted norm is invariant under the rearrangement with respect Gauss measure:

$$\|u\|_{L^p(\varphi,\Omega)} = \|u^{\star}\|_{L^p(\varphi,\Omega)} = \|u^{\circledast}\|_{L^p(0,\gamma_n(\Omega))},$$

while for the L^p weighted norm of $|\nabla u|$ a Polya-Szëgo type inequality holds (see [30]):

$$\|\nabla u^{\star}\|_{L^p(\varphi,\Omega)} \leqslant \|\nabla u\|_{L^p(\varphi,\Omega)}. \qquad (6)$$

In what follows we will use also the notion of pseudo-rearrangement firstly introduced in [3] (see also [25]).

Let $u : \Omega \to \mathbb{R}$ be a measurable function, $f \in L^p(\varphi, \Omega)$ with $1 \leqslant p \leqslant +\infty$, $f \geqslant 0$ and $\Omega^{\circledast} = [0, \gamma_n(\Omega)]$. We will say that a function $\widetilde{f_u} : \Omega^{\circledast} \to \mathbb{R}$ is a Gauss pseudo-rearrangement of f with respect to u if there exists a family $\mathcal{E}(u) = \{E(s)\}_{s\in\Omega^{\circledast}}$ of measurable subsets of Ω such that

$$\begin{gathered} \gamma_n(E(s)) = s, \\ s_1 \leqslant s_2 \quad \Rightarrow \quad E(s_1) \subseteq E(s_2) \\ E(s) = \{x \in \Omega : |u(x)| > u^{\circledast}(s)\} \quad \text{if} \quad s = \mu(t) \end{gathered} \qquad (2.4)$$

and

$$\widetilde{f_u}(s) = \frac{d}{ds} \int_{E(s)} f(x) \varphi(x)\, dx \quad \text{for a.e.} \quad s \in \Omega^{\circledast}.$$

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[1] We denote by $|D|$ the n-dimentional Lebesgue measure of a subset $D \subset \mathbb{R}^n$.

The function $\widetilde{f_u}$ is built on the level sets of u. The following proposition shows that $\widetilde{f_u}$ is weak limit of a sequence of functions having the same decreasing rearrangements of f. The proof is a slight modification of the analogous result for the pseudo-rearrangement obtained in [3].

PROPOSITION 2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a open set and $f \in L^p(\varphi,\Omega)$, $p > 1$. Let $u : \Omega \to \mathbb{R}$ be a measurable function and $\widetilde{f_u}$ be a Gauss pseudo-rearrangement of f with respect to u. Then there exists a sequence $\{f_h\}_{h\in\mathbb{N}} \subseteq L^p\left(\Omega^{\circledast}\right)$ such that*

$$f_h^*(s) = f^{\circledast}(s) \quad \text{and} \quad f_h \rightharpoonup \widetilde{f_u} \quad \text{in} \quad L^p(\Omega^{\circledast}).$$

For more property about pseudo-rearrangement see also [19].

We often will use the following Hardy inequalities.

PROPOSITION 2.2. *Suppose $r > 0, 1 \leqslant q \leqslant \infty$ and $-\infty < \alpha < +\infty$. Let ψ be a nonnegative measurable function on (0,1). If $1 \leqslant q < \infty$, then the inequalities*

$$\left(\int_0^1\left(t^{-r}(1-\log t)^{\alpha}\int_0^t\psi(s)ds\right)^q\frac{dt}{t}\right)^{\frac{1}{q}} \leqslant c\left(\int_0^1\left(t^{1-r}(1-\log t)^{\alpha}\psi(t)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}} \tag{8}$$

and

$$\left(\int_0^1\left(t^{r}(1-\log t)^{\alpha}\int_t^1\psi(s)ds\right)^q\frac{dt}{t}\right)^{\frac{1}{q}} \leqslant c\left(\int_0^1\left(t^{1+r}(1-\log t)^{\alpha}\psi(t)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}} \tag{9}$$

hold; while, for $q = \infty$ it holds that

$$\sup_{0<t<1}\left(t^{-r}(1-\log t)^{\alpha}\left(\int_0^t\psi(s)\,ds\right)\right) \leqslant c\sup_{0<t<1}\left(t^{1-r}(1-\log t)^{\alpha}\psi(t)\right) \tag{10}$$

and

$$\sup_{0<t<1}\left(t^{r}(1-\log t)^{\alpha}\left(\int_t^1\psi(s)ds\right)\right) \leqslant c\sup_{0<t<1}\left(t^{1+r}(1-\log t)^{\alpha}\psi(t)\right). \tag{11}$$

In all cases, the constants $c = c(r,q,\ \alpha)$ are independent of ψ .

In the limit case, $r = 0$, the exponent of the logarithmic term increases by a factor of 1.

PROPOSITION 2.3. *Suppose $1 \leqslant q \leqslant \infty$ and $\frac{1}{q} + \alpha \neq 0$. Let ψ be a nonnegative measurable function on $(0,1)$. If $1 \leqslant q < \infty$, then, it holds*

$$\left(\int_0^1\left((1-\log t)^{\alpha}\int_0^t\psi(s)\,ds\right)^q\frac{dt}{t}\right)^{\frac{1}{q}} \leqslant c\left(\int_0^1\left(t(1-\log t)^{\alpha+1}\psi(t)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}} \tag{12}$$

if $\frac{1}{q}+\alpha>0,$ *or*

$$\left(\int_0^1\left((1-\log t)^{\alpha}\int_t^1\psi(s)\,ds\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\leqslant c\left(\int_0^1\left(t(1-\log t)^{\alpha+1}\psi(t)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}} \tag{13}$$

if $\frac{1}{q}+\alpha<0$. *Moreover, for* $q=\infty,$ *it holds*

$$\sup_{0<t<1}\left((1-\log t)^{\alpha}\int_0^t\psi(s)\,ds\right)\leqslant c\sup_{0<t<1}\left(t(1-\log t)^{\alpha+1}\psi(t)\right) \tag{14}$$

when $\alpha>0,$ *or*

$$\sup_{0<t<1}\left((1-\log t)^{\alpha}\int_t^1\psi(s)\,ds\right)\leqslant c\sup_{0<t<1}\left(t(1-\log t)^{\alpha+1}\psi(t)\right) \tag{15}$$

when $\alpha<0$. *In all cases, the constants* $c=c(q,\ \alpha)$ *are independent of* ψ .

For more details we refer, for instance, to [7].

Now we want to recall the definition and the main properties of Lorentz-Zygmund spaces. Let u be any measurable function in Ω for $0<q,p\leqslant\infty$ and $-\infty<\alpha<+\infty$, we put[2]

$$||u||_{L^{p,q}(\log L)^{\alpha}(\varphi,\Omega)}=\begin{cases}\left(\int_0^{\gamma_n(\Omega)}\left[t^{\frac{1}{p}}(1-\log t)^{\alpha}u^{\circledast}(t)\right]^q\frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } 0<q<\infty,\\ \sup\limits_{t\in(0,\gamma_n(\Omega))}\left[t^{\frac{1}{p}}(1-\log t)^{\alpha}u^{\circledast}(t)\right] & \text{if } q=\infty.\end{cases} \tag{2.13}$$

We say that u belongs to the Lorentz-Zygmund space $L^{p,q}(\log L)^{\alpha}(\varphi,\Omega)$ if

$$||u||_{L^{p,q}(\log L)^{\alpha}(\varphi,\Omega)}<\infty.$$

We remark that for $1<p\leqslant\infty$, $1\leqslant q\leqslant\infty$ and $-\infty<\alpha<+\infty$, (2.13) is a quasinorm, but replacing $u^{\circledast}(t)$ with

$$u^{\circledast\circledast}(t)=\frac{1}{t}\int_0^t u^{\circledast}(s)ds$$

we obtain an equivalent norm.

It is clear from definition of Lorentz space and (2.13) that the space $L^{p,q}(\log L)^0(\varphi,\Omega)$ is just the Lorentz space $L^{p,q}(\varphi,\Omega)$. Moreover if $1<p<\infty$ the space $L^{p,p}(\log L)^{\alpha}(\varphi,\Omega)$ is the Zygmund space $L^p(\log L)^{\alpha}(\varphi,\Omega)$, while if $p=\infty$ and $\alpha\geqslant 0$ the space $L^{\infty,\infty}(\log L)^{-\alpha}(\varphi,\Omega)$ is the Zygmund space $L^{\alpha}_{\exp}(\varphi,\Omega)$.

We will remind same inclusion relations among Lorentz-Zygmund spaces.

If $0<r<p\leqslant\infty, 0<q,s\leqslant\infty$ and $-\infty<\alpha,\beta<\infty$, then we get

$$L^{p,q}(\log L)^{\alpha}(\varphi,\Omega)\subseteq L^{r,s}(\log L)^{\beta}(\varphi,\Omega).$$

[2]We will use the following ‘arithmetic’ convention: $\frac{s}{\infty}=0$ for $s>0$.

It is clear from definition (2.13) that the space $L^{p,q}(\log L)^{\alpha}(\varphi,\Omega)$ decreases as α increases. When the first exponents are the same, $0 < p \leqslant \infty$, $0 < q,s \leqslant \infty$ and $-\infty < \alpha,\beta < \infty$, the following inclusion holds

$$L^{p,q}(\log L)^{\alpha}(\varphi,\Omega) \subseteq L^{p,s}(\log L)^{\beta}(\varphi,\Omega)$$

whenever either

$$q \leqslant s \quad \text{and} \quad \alpha \geqslant \beta$$

or

$$q > s \quad \text{and} \quad \alpha + \frac{1}{q} > \beta + \frac{1}{s}.$$

Let us observe that the space $L^{p,q}(\log L)^{\alpha}(\varphi,\Omega)$ is not trivial if and only if one of the following conditions hold :

$$\begin{cases} p < \infty, \\ p = \infty \ \text{ and } \ \alpha + \dfrac{1}{q} < 0, \\ p = \infty, q = \infty \ \text{ and } \ \alpha = 0. \end{cases}$$

For more properties and for the definition of the classical Lorentz-Zygmund space $L^{p,q}(\log L)^{\alpha}(\Omega)$ we refer to [7, 8] and [23]. In what follows, for the sake of simplicity, we will denote by $||u||_{p,q;\alpha}$ the quasinorm $||u||_{L^{p,q}(\log L)^{\alpha}(\varphi,\Omega^{\star})}$.

We recall that the weighted Sobolev space $H_0^1(\varphi,\Omega)$ is the closure of $C_0^{\infty}(\Omega)$ under the norm

$$\|u\|_{H_0^1(\varphi,\Omega)} = \left(\int_{\Omega} |\nabla u(x)|^2 \, d\gamma_n(x)\right)^{\frac{1}{2}}.$$

We remark that when $\gamma_n(\Omega) < 1$, the following Poincarè type inequality holds

$$\|u\|_{L^2(\varphi,\Omega)} \leqslant C \, \|\nabla u\|_{L^2(\varphi,\Omega)},$$

where C is a constant depending on $\gamma_n(\Omega)$ (see [13, 15]).

The following imbendding theorem is a straight consequence of the Sobolev logarithmic inequalities (see [20, 1]). We give a direct proof which uses properties of rearrangement of functions.

PROPOSITION 2.4. *If* $|\nabla f| \in L^p(\varphi,\Omega)$, $1 \leqslant p < \infty$, *then* $f \in L^p(\log L)^{\frac{1}{2}}(\varphi,\Omega)$ *and*

$$\|f\|_{L^p(\log L)^{\frac{1}{2}}(\varphi,\Omega)} \leqslant C_1 \, \|\nabla f\|_{L^p(\varphi,\Omega)}. \tag{17}$$

If $|\nabla f| \in L^{\infty}(\Omega)$ *then* $f \in L^{\infty}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)$ *and*

$$\|f\|_{L^{\infty}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)} \leqslant C_2 \, \|\nabla f\|_{L^{\infty}(\Omega)}. \tag{18}$$

In both cases the constants C_1, C_2 *depend on p and* $\gamma_n(\Omega)$.

Proof. For $1 \leqslant p < +\infty$, using (9), (4) and (6) we obtain

$$\begin{aligned} \|f\|^p_{L^p(\log L)^{\frac{1}{2}}(\varphi,\Omega)} &\leqslant c\int_0^{\gamma_n(\Omega)} \left(t(1-\log t)^{\frac{1}{2}}\left|\frac{d}{dt}f^{\circledast}(t)\right|\right)^p dt \\ &\leqslant c\int_0^{\gamma_n(\Omega)} \left|\frac{d}{dt}f^{\circledast}(t)\right|^p \exp\left(-\frac{\Phi^{-1}(t)^2}{2}p\right) dt \\ &= c\left\|\nabla f^{\star}\right\|^p_{L^p_{(\varphi,\Omega)}} \leqslant c\left\|\nabla f\right\|^p_{L^p(\varphi,\Omega)}, \end{aligned}$$

where c is a generic constant depending only by p and $\gamma_n(\Omega)$ and which may vary from line to line.

For $p = +\infty$ the inequality (18) follows in the same way with (9) replaced by (15). $\square$

Moreover we recall an inequality which will be useful in the following:

PROPOSITION 2.5. *Let $f \in L^{p,q}(\log L)^{\alpha}(\varphi,\Omega)$ with $1 \leqslant \sigma < p \leqslant \infty, \sigma \leqslant q \leqslant \infty$ e $-\infty < \alpha < +\infty$ and $F^{\sigma} = \left(\widetilde{f^{\sigma}}\right)_u$. Then $F \in L^{p,q}(\log L)^{\alpha}(\Omega^{\circledast})$ and for same positive constant C*

$$\|F\|_{L^{p,q}(\log L)^{\alpha}(\Omega^{\circledast})} \leqslant C\,\|f\|_{L^{p,q}(\log L)^{\alpha}(\varphi,\Omega)}.$$

The proof of Proposition 2.5 is a slight modification of the proof of Lemma 2.2 in [10], where Lorentz space are considered.

The following proposition is a slight modification of Gronwall lemma (see [22] for the proof of the classical one).

LEMMA 2.1. *Given the functions $\lambda,\gamma,\phi,\theta$ in $[a,+\infty)$ suppose that $\lambda \geqslant 0, \gamma \geqslant 0$ and that $\lambda\theta$ and $\lambda\phi$ belong to $L^1(a,+\infty)$. Moreover suppose that $\lambda\gamma$ belongs to $L^1(a,k)$ for each $k > a$ and*

$$\lim_{k\to\infty}\left(\int_k^{+\infty}\lambda(\tau)\phi(\tau)\,d\tau\right)\left(\exp\left[\int_a^k \lambda(s)\,\gamma(s)\,ds\right]\right) = 0. \qquad (19)$$

If for a.e. $t \geqslant a$

$$\phi(t) \leqslant \theta(t) + \gamma(t)\int_t^{\infty}\lambda(\tau)\phi(\tau)\,d\tau,$$

then for a.e. $t \geqslant a$

$$\phi(t) \leqslant \theta(t) + \gamma(t)\int_t^{\infty}\theta(\tau)\lambda(\tau)\exp\left(\int_t^{\tau}\lambda(r)\gamma(r)\,dr\right)\,d\tau.$$

3. Comparison result

In this section we will prove a comparison result between the solution of problem (1.1) and a simpler Dirichlet problem which is defined in an half-space and has coefficients depending only on the first variable. The proof uses as main tools the isoperimetric inequality with respect to Gauss measure, Coarea formula and the properties of the rearrangement of a function.

We recall that $u \in H_0^1(\varphi,\Omega)$ is a weak solution of problem (1.1), if

$$\begin{aligned}\int_\Omega (a_{ij}(x)u_{x_i}\psi_{x_j} + d_i(x)u\psi_{x_i} + b_i(x)u_{x_i}\psi + c(x)u\psi)\,dx \\ = \int_\Omega (g + f_i\psi_{x_i})\,\varphi(x)dx \qquad \forall \psi \in H_0^1(\varphi,\Omega).\end{aligned} \tag{3.1}$$

Note that in the hypotheses $(i)-(vii)$, using Proposition 2.5, Hardy-Littlewood's inequality (5), Hölder's and Poicarè's inequalities, all terms in (3.1) are well defined.

THEOREM 3.1. *Let Ω be an open set of $\mathbb{R}^n$ with $\gamma_n(\Omega) < 1$ and let $u \in H_0^1(\varphi,\Omega)$ be solution of (1.1) under the assumptions $(i)-(vii)$; moreover, suppose that either $\|b\|_{L^\infty(\log L)^{-\frac12}(\varphi,\Omega)}$ is small enough or $b \in L^{\infty,a}(\log L)^{-\frac12}(\varphi,\Omega)$ with $2 < a < \infty$. Let $w(x)=w^\star(x)$ be the solution of problem (1.3), where $\Omega^\star$ is the half-space $\{x=(x_1,...,x_n)\in\mathbb{R}^n : x_1>\lambda\}$, with $\lambda\in\mathbb{R}$ such that $\gamma_n(\Omega)=\gamma_n(\Omega^\star)$ and F, B and D are functions such that $F^2=\left(\widetilde{f^2}\right)_u$, $B^2=\left(\widetilde{b^2}\right)_u$ and $D^2=\left(\widetilde{d^2}\right)_u$.*

Then

$$u^\star(x_1) \leqslant w^\star(x_1) = w(x) \quad \text{for a.e.} \quad x\in\Omega^\star. \tag{21}$$

Proof. Let be $h>0$ and $t\in[0,\sup|u|[$. If we take

$$\psi(x) = \begin{cases} h\,\mathrm{sign}\,u & \text{if } |u|>t+h \\ (|u|-t)\,\mathrm{sign}\,u & \text{if } t<|u|\leqslant t+h \\ 0 & \text{otherwise} \end{cases}$$

in (3.1), then we get

$$\begin{aligned}&\frac1h\int_{t<|u|\leqslant t+h} a_{ij}(x)\,u_{x_i}u_{x_j}\,dx + \frac1h\int_{t<|u|\leqslant t+h} d_i(x)uu_{x_i}\,dx \\ &\quad+\frac1h\int_{t<|u|\leqslant t+h} b_i(x)u_{x_i}\,(|u|-t)\,\mathrm{sign}\,u\,dx + \int_{|u|>t+h} b_i(x)\,u_{x_i}\,\mathrm{sign}\,u\,dx \\ &\quad+\frac1h\int_{t<|u|\leqslant t+h} c(x)u(x)\,(|u|-t)\,\mathrm{sign}\;u\,dx + \int_{|u|>t+h} c(x)\,u\,(x)\,\mathrm{sign}\;u\,dx \\ &=\frac1h\int_{t<|u|\leqslant t+h} f_i\varphi(x)u_{x_i}\,dx + \frac1h\int_{t<|u|\leqslant t+h} g(x)\varphi(x)\,(|u|-t)\,\mathrm{sign}\;u\,dx \\ &\quad+\int_{|u|>t+h} g(x)\varphi(x)\,\mathrm{sign}\;u\,dx\end{aligned}$$

Under the conditions (i), (iii), (iv) and (v), letting $h\to0$ and by Cauchy-

Schwarz inequality, we obtain

$$\begin{aligned} &-\frac{d}{dt}\int_{|u|>t}|\nabla u|^2\,\varphi(x)\,dx \\ &\leqslant \int_t^{+\infty}\left(-\frac{d}{ds}\int_{|u|>s}|\nabla u|^2\,\varphi(x)\;dx\right)^{\frac12}\left(-\frac{d}{ds}\int_{|u|>s}b^2(x)\varphi(x)\;dx\right)^{\frac12}ds \\ &\quad+\left(-\frac{d}{dt}\int_{|u|>t}|\nabla u|^2\,\varphi(x)\,dx\right)^{\frac12} t\left(-\frac{d}{dt}\int_{|u|>t}d^2(x)\varphi(x)\;dx\right)^{\frac12} \\ &\quad+\left(-\frac{d}{dt}\int_{|u|>t}f^2\varphi(x)\,dx\right)^{\frac12}\left(-\frac{d}{dt}\int_{|u|>t}|\nabla u|^2\,\varphi(x)\,dx\right)^{\frac12} \\ &\quad+\int_{|u|>t}g(x)\varphi(x)\mathrm{sign}\,u\,dx. \end{aligned} \tag{3.3}$$

On the other hand, Coarea formula (see [18]) and isoperimetric inequality with respect to the Gauss measure give

$$-\frac{d}{dt}\int_{|u|>t}|\nabla u|\,\varphi(x)dx \geqslant \int_{\partial\{|u|>t\}^\star}\varphi(x)\mathcal{H}_{n-1}(dx)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\Phi^{-1}(\mu(t))^2}{2}\right) \tag{23}$$

where $\{|u|>t\}^\star$ is the half space having Gauss measure $\mu(t)$.

Then, using (23) and Hölder inequality, we get

$$1\leqslant(2\pi)^{\frac12}\exp\left(\frac{\Phi^{-1}(\mu(t))^2}{2}\right)\left(-\mu'(t)\right)^{\frac12}\left(-\frac{d}{dt}\int_{|u|>t}|\nabla u|^2\,\varphi(x)\,dx\right)^{\frac12}. \tag{24}$$

Moreover from (2.4), we have

$$\left(-\frac{d}{dt}\int_{|u|>t}f^2(x)\varphi(x)\,dx\right)^{\frac12}=\left(-\frac{d}{dt}\int_0^{\mu(t)}F^2(s)\;ds\right)^{\frac12}=F(\mu(t))\left(-\mu'(t)\right)^{\frac12}, \tag{25}$$

$$\left(-\frac{d}{dt}\int_{|u|>t}b^2(x)\varphi(x)\,dx\right)^{\frac12}=\left(-\frac{d}{dt}\int_0^{\mu(t)}B^2(s)\;ds\right)^{\frac12}=B(\mu(t))\left(-\mu'(t)\right)^{\frac12}, \tag{26}$$

and

$$\left(-\frac{d}{dt}\int_{|u|>t}d^2(x)\varphi(x)\,dx\right)^{\frac12}=\left(-\frac{d}{dt}\int_0^{\mu(t)}D^2(s)\;ds\right)^{\frac12}=D(\mu(t))\left(-\mu'(t)\right)^{\frac12}, \tag{27}$$

where F,B and D are functions such that $F^2=\left(\widetilde{f^2}\right)_u$, $B^2=\left(\widetilde{b^2}\right)_u$ and $D^2=\left(\widetilde{d^2}\right)_u$.

Applying (24), Hölder inequality, (5), (25), (26), and (27) it follows

$$\begin{aligned}\left(-\frac{d}{dt}\int_{|u|>t}|\nabla u|^2\varphi(x)\,dx\right)^{\frac12} &\leqslant (2\pi)^{\frac12}\exp\left(\frac{\Phi^{-1}(\mu(t))^2}{2}\right)\left(-\mu'(t)\right)^{\frac12}\times\\ &\times\int_t^{+\infty}\left(-\frac{d}{ds}\int_{|u|>s}|\nabla u|^2\varphi(x)\,dx\right)^{\frac12}B(\mu(s))\left(-\mu'(s)\right)^{\frac12}ds\\ &+tD(\mu(t))\left(-\mu'(t)\right)^{\frac12}+F(\mu(t))\left(-\mu'(t)\right)^{\frac12}\\ &+(2\pi)^{\frac12}\exp\left(\frac{\Phi^{-1}(\mu(t))^2}{2}\right)\left(-\mu'(t)\right)^{\frac12}\int_0^{\mu(t)}g^{\circledast}(s)ds.\end{aligned}\tag{3.9}$$

Now we want use Gronwall lemma with

$$\phi(t)=\exp\left(-\frac{\Phi^{-1}(\mu(t))^2}{2}\right)\left(-\mu'(t)\right)^{-\frac12}\left(-\frac{d}{dt}\int_{|u|>t}|\nabla u|^2\varphi(x)\,dx\right)^{\frac12}.$$

We remark that the condition (19) holds if $\|b\|_{L^\infty(\log L)^{-\frac12}(\varphi,\Omega)}$ is small enough or $b\in L^{\infty,a}(\log L)^{-\frac12}(\varphi,\Omega)$ with $2<a<\infty$. For comfort of the reader we bring back the details in Appendix.

Applying Gronwall lemma, by (3.9) we have

$$\begin{aligned}\phi(t)\leqslant{}&(tD(\mu(t))+F(\mu(t)))\exp\left(-\frac{\Phi^{-1}(\mu(t))^2}{2}\right)+(2\pi)^{\frac12}\int_0^{\mu(t)}g^{\circledast}(s)ds\\ &+(2\pi)^{\frac12}\int_t^{+\infty}\exp\left[\int_t^s(2\pi)^{\frac12}B(\mu(r))\exp\left(\frac{\Phi^{-1}(\mu(r))^2}{2}\right)\left(-\mu'(r)\right)dr\right]\times\\ &\times\Big[(sD(\mu(s))+F(\mu(s)))B(\mu(s))\left(-\mu'(s)\right)\\ &+(2\pi)^{\frac12}\left(-\mu'(s)\right)B(\mu(s))\exp\left(\frac{\Phi^{-1}(\mu(s))^2}{2}\right)\int_0^{\mu(s)}g^{\circledast}(z)dz\Big]ds,\end{aligned}\tag{3.10}$$

hence setting $\mu(r)=\tau$ and $\mu(s)=\sigma$ we get

$$\begin{aligned}&\phi(t)\leqslant(tD(\mu(t))+F(\mu(t)))\exp\left(-\frac{\Phi^{-1}(\mu(t))^2}{2}\right)+(2\pi)^{\frac12}\int_0^{\mu(t)}g^{\circledast}(s)ds\\ &+(2\pi)^{\frac12}\int_0^{\mu(t)}\exp\left[\int_\sigma^{\mu(t)}(2\pi)^{\frac12}B(\tau)\exp\left(\frac{\Phi^{-1}(\tau)^2}{2}\right)d\tau\right]\times\\ &\times\left[u^{\circledast}(\sigma)D(\sigma)B(\sigma)+F(\sigma)B(\sigma)+(2\pi)^{\frac12}\exp\left(\frac{\Phi^{-1}(\sigma)^2}{2}\right)B(\sigma)\int_0^\sigma g^{\circledast}(z)dz\right]d\sigma.\end{aligned}$$

Using (24) we have

$$\left(-\mu'(t)\right)^{-1} \leqslant (2\pi)^{\frac{1}{2}}\left(tD\left(\mu(t)\right)+F\left(\mu(t)\right)\right)\exp\left(\frac{\Phi^{-1}\left(\mu(t)\right)^{2}}{2}\right)$$
$$+(2\pi)\exp\left(\Phi^{-1}\left(\mu(t)\right)^{2}\right)\int_{0}^{\mu(t)} g^{\circledast}(s)ds$$
$$+(2\pi)\exp\left(\Phi^{-1}\left(\mu(t)\right)^{2}\right)\int_{0}^{\mu(t)}\exp\left[\int_{\sigma}^{\mu(t)}(2\pi)^{\frac{1}{2}}B(\tau)\exp\left(\frac{\Phi^{-1}(\tau)^{2}}{2}\right)d\tau\right]\times$$
$$\times\left[u^{\circledast}(\sigma)D(\sigma)B(\sigma)+F(\sigma)B(\sigma)+(2\pi)^{\frac{1}{2}}\exp\left(\frac{\Phi^{-1}(\sigma)^{2}}{2}\right)B(\sigma)\int_{0}^{\sigma}g^{\circledast}(z)dz\right]d\sigma.$$

Consequently, setting $\mu(t)=s$ and integrating by part we get

$$\begin{aligned}
-\left(u^{\circledast}(s)\right)' \leqslant\ & (2\pi)\exp\left(\Phi^{-1}(s)^{2}\right)\times\\
&\times\int_{0}^{s}\exp\left[\int_{\sigma}^{s}(2\pi)^{\frac{1}{2}}B(\tau)\exp\left(\frac{\Phi^{-1}(\tau)^{2}}{2}\right)d\tau\right]g^{\circledast}(\sigma)d\sigma\\
&+(2\pi)\exp\left(\Phi^{-1}(s)^{2}\right)\int_{0}^{s}\left[u^{\circledast}(\sigma)D(\sigma)B(\sigma)+F(\sigma)B(\sigma)\right]\times\\
&\times\exp\left[\int_{\sigma}^{s}(2\pi)^{\frac{1}{2}}B(\tau)\exp\left(\frac{\Phi^{-1}(\tau)^{2}}{2}\right)d\tau\right]d\sigma\\
&+(2\pi)^{\frac{1}{2}}\exp\left(\frac{\Phi^{-1}(s)^{2}}{2}\right)\left[u^{\circledast}(s)D(s)+F(s)\right].
\end{aligned} \tag{3.11}$$

If $w(x)=w^{\star}(x)$ is solution of (1.3) we obtain

$$\begin{aligned}
-\left(w^{\circledast}(s)\right)' =\ & (2\pi)\exp\left(\Phi^{-1}(s)^{2}\right)\times\\
&\times\int_{0}^{s}\exp\left[\int_{\sigma}^{s}(2\pi)^{\frac{1}{2}}B(\tau)\exp\left(\frac{\Phi^{-1}(\tau)^{2}}{2}\right)d\tau\right]g^{\circledast}(\sigma)d\sigma\\
&+(2\pi)\exp\left(\Phi^{-1}(s)^{2}\right)\int_{0}^{s}\left[w^{\circledast}(\sigma)D(\sigma)B(\sigma)+F(\sigma)B(\sigma)\right]\times\\
&\times\exp\left[\int_{\sigma}^{s}(2\pi)^{\frac{1}{2}}B(\tau)\exp\left(\frac{\Phi^{-1}(\tau)^{2}}{2}\right)d\tau\right]d\sigma\\
&+(2\pi)^{\frac{1}{2}}\exp\left(\frac{\Phi^{-1}(s)^{2}}{2}\right)\left[w^{\circledast}(s)D(s)+F(s)\right].
\end{aligned} \tag{3.12}$$

indeed (3.12) can be deduced in the same way as (3.11), starting from problem (1.3) and observing that in this case all the inequalities are equalities.

To prove the comparison (21) we argue as in [2] (see Theorem 2)

From (3.11) and (3.12) writing

$$V(s)=\int_0^s\left[\left(u^{\circledast}(\sigma)-w^{\circledast}(\sigma)\right)D(\sigma)B(\sigma)\right]\times \\ \times\exp\left[\int_\sigma^{\gamma_n(\Omega)}(2\pi)^{\frac{1}{2}}B(\tau)\exp\left(\frac{\Phi^{-1}(\tau)^2}{2}\right)d\tau\right]d\sigma,$$

we have

$$\begin{cases} -\left(A(s)Q(s)D^{-1}(s)B^{-1}(s)V'(s)\right)'\leqslant(2\pi)\exp\left(\Phi^{-1}(s)^2\right)Q(s)A(s)V(s) \\ V(0)=V'(\gamma_n(\Omega))=0, \end{cases} \tag{3.13}$$

where

$$A(s)=\exp\left[\int_{\gamma_n(\Omega)}^s(2\pi)^{\frac{1}{2}}\exp\left(\frac{\Phi^{-1}(\sigma)^2}{2}\right)D(\sigma)\,d\sigma\right]$$

and

$$Q(s)=\exp\left[\int_{\gamma_n(\Omega)}^s(2\pi)^{\frac{1}{2}}\exp\left(\frac{\Phi^{-1}(\sigma)^2}{2}\right)B(\sigma)\,d\sigma\right].$$

The existence of a solution $w^{\star}(x)$ of problem (1.3) and the equality (3.12), guarantees that the problem

$$\begin{cases} -\left(A(s)Q(s)D^{-1}(s)B^{-1}(s)Z'(s)\right)'=(2\pi)\exp\left(\Phi^{-1}(s)^2\right)Q(s)A(s)Z(s) \\ \quad+(2\pi)^{\frac{1}{2}}\exp\left(\frac{\Phi^{-1}(s)^2}{2}\right)A(s)F(s)+(2\pi)\exp\left(\Phi^{-1}(s)^2\right)A(s)\times \\ \qquad\times\int_0^s\exp\left[\int_\sigma^s(2\pi)^{\frac{1}{2}}B(\tau)\exp\left(\frac{\Phi^{-1}(\tau)^2}{2}\right)d\tau\right]g^{\circledast}(\sigma)d\sigma \\ \quad+(2\pi)\exp\left(\Phi^{-1}(s)^2\right)A(s)\times \\ \qquad\times\int_0^sF(\sigma)B(\sigma)\exp\left[\int_\sigma^s(2\pi)^{\frac{1}{2}}B(\tau)\exp\left(\frac{\Phi^{-1}(\tau)^2}{2}\right)d\tau\right]d\sigma \\ Z(0)=Z'(\gamma_n(\Omega))=0 \end{cases}$$

have the following positive solution

$$Z(s)=\int_0^s w^{\circledast}(\sigma)D(\sigma)B(\sigma)\exp\left[\int_\sigma^{\gamma_n(\Omega)}(2\pi)^{\frac{1}{2}}B(\tau)\exp\left(\frac{\Phi^{-1}(\tau)^2}{2}\right)d\tau\right]d\sigma.$$

This allow us to state (see [6]) that the problem

$$\begin{cases} -\left(A(s)Q(s)D^{-1}(s)B^{-1}(s)\Psi'(s)\right)'=\lambda\,(2\pi)\exp\left(\Phi^{-1}(s)^2\right)A(s)Q(s)\Psi(s) \\ \Psi(0)=\Psi'(\gamma_n(\Omega))=0, \end{cases}$$

has the first eigenvalue $\lambda_1 > 1$, and consequently (see again [6]) in (3.13) we have $V(s) \leqslant 0$ and $V'(s) \leqslant 0$, i.e. (21). □

In the next two Corollaries we examine separately the cases $b_i(x) \equiv 0 \;\; i = 1,..,n$ and $d_i(x) \equiv 0 \;\; i = 1,..,n$. As a matter of the fact, under this assumptions, comparison (21) can be easily proved and the solution $w(x)$ of problem (1.3) can been written giving an explicit estimate of $u^\star(x_1)$. Moreover an estimate of the norm of $|\nabla u|$ can be proven.

COROLLARY 3.2. *Under the assumptions of Theorem* 3.1*, if* $d_i(x) \equiv 0 \;\; i = 1,...,n$ *then comparison* (21) *holds with*

$$\begin{aligned} w(x_1) = &\int_\lambda^{x_1} \exp\left(\frac{\tau^2}{2}\right) \int_\tau^{+\infty} g^\star(\sigma) \exp\left(\int_\tau^\sigma B(\Phi(r))dr - \frac{\sigma^2}{2}\right) d\sigma\, d\tau \\ &+ \int_\lambda^{x_1} F(\Phi(\tau))\, d\tau \\ &+ \int_\lambda^{x_1} \exp\left(\frac{\tau^2}{2}\right) \int_\tau^{+\infty} F(\Phi(\sigma))\, B(\Phi(\sigma)) \exp\left(\int_\tau^\sigma B(\Phi(r))dr - \frac{\sigma^2}{2}\right) d\sigma\, d\tau. \end{aligned} \tag{3.14}$$

Moreover

$$\int_\Omega |\nabla u|^q \varphi(x)dx \leqslant \int_{\Omega^\star} |\nabla w|^q \varphi(x)dx \quad \textit{for all} \quad 0 < q \leqslant 2 \tag{34}$$

holds.

Proof. Let us observe that in this case, $d_i(x) \equiv 0 \;\; i = 1,...,n$, (21) can be obtained by (3.11) and (3.12).

We prove (34). Using Hölder inequality and (3.10) we have

$$\begin{aligned} -\frac{d}{dt}\int_{|u|>t} |\nabla u|^q \varphi(x)dx \leqslant &\left(-\mu'(t)\right)^{1-\frac{q}{2}} \left\{ F(\mu(t)) + (2\pi)^{\frac{1}{2}} \exp\left(\frac{\Phi^{-1}(\mu(t))^2}{2}\right) \times \right. \\ &\times \left(\int_0^{\mu(t)} g^\circledast(s)ds\right) + (2\pi)^{\frac{1}{2}} \exp\left(\frac{\Phi^{-1}(\mu(t))^2}{2}\right) \times \\ &\times \int_t^\infty \left(F(\mu(\tau)) + (2\pi)^{\frac{1}{2}} \times \right. \\ &\times \exp\left(\frac{\Phi^{-1}(\mu(\tau))^2}{2}\right) \left. \int_0^{\mu(\tau)} g^\circledast(s)ds\right) B(\mu(\tau)) \left(-\mu'(\tau)\right) \times \\ &\times \exp\left[(2\pi)^{\frac{1}{2}} \int_t^\tau B(\mu(r)) \exp\left(\frac{\Phi^{-1}(\mu(r))^2}{2}\right) \left(-\mu'(r)\right) dr\right] d\tau \left. \vphantom{\int} \right\}^q. \end{aligned}$$

Integrating between 0 and $+\infty$ the last inequality becomes

$$\int_\Omega |\nabla u|^q \varphi(x)dx \leqslant \int_0^{\gamma_n(\Omega)} \left\{ F(s) + (2\pi)^{\frac{1}{2}} \exp\left(\frac{\Phi^{-1}(s)^2}{2}\right) \times \right.$$
$$\times \left(\int_0^s g^{\circledast}(\tau)d\tau\right) + (2\pi)^{\frac{1}{2}} \exp\left(\frac{\Phi^{-1}(s)^2}{2}\right) \times$$
$$\times \int_0^s \left(F(\tau) + (2\pi)^{\frac{1}{2}} \exp\left(\frac{\Phi^{-1}(\tau)^2}{2}\right)\int_0^\tau g^{\circledast}(\sigma)d\sigma\right) \times$$
$$\left. \times B(\tau)\exp\left[(2\pi)^{\frac{1}{2}} \int_\tau^s B(\sigma)\exp\left(\frac{\Phi^{-1}(\sigma)^2}{2}\right)d\sigma\right] d\tau \right\}^q ds,$$

and integrating by parts (34) follows. □

The next Corollary examine the case $b_i(x) \equiv 0 \ \ i = 1, ..., n$. The condition on $d_i(x)$, $i = 1, ..., n$, are needed to write explicitly the solution $w(x)$ of problem (1.3).

COROLLARY 3.3. *Under the assumptions of Theorem* 3.1*, if* $b_i(x) \equiv 0 \ \ i = 1, ...n$*, and either* $\|d\|_{L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$ *is small enough or* $d \in L^{\infty,a}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)$ *with* $2 < a < \infty$*, then comparison* (21) *holds with*

$$\begin{aligned} w(x_1) = &\int_\lambda^{x_1} F(\Phi(s))\exp\left(\int_s^{x_1} D(\Phi(r))\,dr\right)ds \\ &+ \int_\lambda^{x_1} (2\pi)^{\frac{1}{2}} \exp\left(\int_s^{x_1} D(\Phi(r))\,dr + \frac{s^2}{2}\right)\int_s^{+\infty} g^\star(z)\exp\left(-\frac{z^2}{2}\right)dzds. \end{aligned} \tag{3.16}$$

Moreover

$$\int_\Omega |\nabla u|^q \varphi(x)dx \leqslant \int_{\Omega^\star} |\nabla w|^q \varphi(x)dx \ \textit{ for all } \ 0 < q \leqslant 2 \tag{36}$$

holds.

Proof. By (3.11) integrating between s and $\gamma_n(\Omega)$, it follows that

$$u^{\circledast}(s) \leqslant \int_s^{\gamma_n(\Omega)} (2\pi)^{\frac{1}{2}} \exp\left(\frac{\Phi^{-1}(\tau)^2}{2}\right)\left(u^{\circledast}(\tau)D(\tau) + F(\tau)\right)d\tau$$
$$+ \int_s^{\gamma_n(\Omega)} (2\pi)\exp\left(\Phi^{-1}(\tau)^2\right)\int_0^\tau g^{\circledast}(z)dzd\tau.$$

We want to apply Gronwall lemma with $\phi(s) = u^{\circledast}(s)$. Condition (19) can be verified arguing as in Lemma 5.1 in the Appendix. Integrating by parts we get

$$u^{\circledast}(s) \leqslant \int_s^{\gamma_n(\Omega)} \left(F(r) + (2\pi)^{\frac{1}{2}} \exp\left(\frac{\Phi^{-1}(r)^2}{2}\right)\int_0^r g^{\circledast}(z)dz\right) \times$$
$$\times (2\pi)^{\frac{1}{2}} \exp\left(\frac{\Phi^{-1}(r)^2}{2}\right)\exp\left(\int_s^r (2\pi)^{\frac{1}{2}} \exp\left(\frac{\Phi^{-1}(\sigma)^2}{2}\right)D(\sigma)\,d\sigma\right)dr.$$

Now putting $s=\Phi(x_1)$ we have (21).

Using (3.9) with $B(t)\equiv 0$ we have

$$-\frac{d}{dt}\int_{|u|>t}|\nabla u|^q\,\varphi(x)\,dx \leqslant \left(-\mu'(t)\right)\Big\{tD\left(\mu\left(t\right)\right)+F\left(\mu\left(t\right)\right) + (2\pi)^{\frac{1}{2}}\exp\left(\frac{\Phi^{-1}\left(\mu\left(t\right)\right)^2}{2}\right)\int_0^{\mu(t)} g^{\circledast}(s)ds\Big\}^q.$$

Integrating between 0 and $+\infty$ the last inequality becomes

$$\int_\Omega |\nabla u|^q\,\varphi(x)\,dx \leqslant \int_\lambda^{+\infty}\Big\{u^\star(x)D\left(\Phi(x)\right)+F(\Phi(x)) + (2\pi)^{\frac{1}{2}}\exp\left(\frac{x^2}{2}\right)\int_x^{+\infty} g^\star(r)\exp\left(-\frac{r^2}{2}\right)dr\Big\}^q\varphi(x)dx.$$

Using (21) we have (36).

4. Regularity results

In this section we study how the summability of the solution u of problem (1.1) improves by improving the summability of the data in the Lorentz-Zygmund spaces. First of all we prove an apriori estimate for the norm of u in $H_0^1(\varphi,\Omega)$ that gives also a sufficient condition for the existence in terms of the summability of the data.

PROPOSITION 4.1. *Let $u\in H_0^1(\varphi,\Omega)$ be a solution of problem (1.1) under the assumptions $(i)-(vii)$. If $\|b\|_{L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$ and $\|d\|_{L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$ are small enough, the inequality*

$$\|\nabla u\|_{L^2(\varphi,\Omega)} \leqslant C_1\ \|g\|_{L^2(\log L)^{-\frac{1}{2}}(\varphi,\Omega)} + C_2\ \|f\|_{L^2(\varphi,\Omega)}, \tag{37}$$

holds for some positive constants C_1, C_2 depending only on $\gamma_n(\Omega)$, $\|b\|$ and $\|d\|$.

We can avoid the assumption on smallness of one of two norms $\|b\|$ or $\|d\|$ taking it in the smaller space $L^{\infty,a}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)$ with $2<a<\infty$.

Proof. We can apply Gronwall lemma to (3.9) when $\|b\|_{L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$ is small enough or $b\in L^{\infty,a}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)$ with $2<a<\infty$. Condition (19) can be verified arguing as in Lemma 5.1 in the Appendix.

We have

$$\left(-\frac{d}{dt}\int_{|u|>t}|\nabla u|^2\,\varphi(x)\,dx\right)^{\frac{1}{2}} \leqslant (2\pi)^{\frac{1}{2}}\exp\left(\frac{\Phi^{-1}\left(\mu\left(t\right)\right)^2}{2}\right)\left(-\mu'(t)\right)^{\frac{1}{2}}\int_t^{+\infty}\Bigg(sD\left(\mu\left(s\right)\right)+F\left(\mu\left(s\right)\right) + (2\pi)^{\frac{1}{2}}\exp\left(\frac{\Phi^{-1}\left(\mu\left(s\right)\right)^2}{2}\right)\int_0^{\mu(s)} g^{\circledast}(z)dz\Bigg)B\left(\mu\left(s\right)\right)\left(-\mu'\left(s\right)\right)\times$$

$$\times \exp\left[\int_t^s (2\pi)^{\frac{1}{2}} \exp\left(\frac{\Phi^{-1}(\mu(r))^2}{2}\right) B(\mu(r))\left(-\mu'(r)\right) dr\right] ds$$
$$+ tD(\mu(t))\left(-\mu'(t)\right)^{\frac{1}{2}} + F(\mu(t))\left(-\mu'(t)\right)^{\frac{1}{2}}$$
$$+ (2\pi)^{\frac{1}{2}} \exp\left(\frac{\Phi^{-1}(\mu(t))^2}{2}\right)\left(-\mu'(t)\right)^{\frac{1}{2}} \int_0^{\mu(t)} g^{\circledast}(z)dz.$$

Raising to the power 2, integrating between 0 and $+\infty$, making a variable change we get

$$\int_\Omega |\nabla u|^2 \varphi(x)\, dx \leqslant C \int_0^{\gamma_n(\Omega)} \exp\left(\Phi^{-1}(t)^2\right) \left[\left(\int_0^t \left(u^{\circledast}(s)D(s) + F(s)\right) B(s)\times\right.\right.$$
$$\left.\times \exp\left[\int_s^t (2\pi)^{\frac{1}{2}} \exp\left(\frac{\Phi^{-1}(r)^2}{2}\right) B(r)\, dr\right] ds\right)^2$$
$$+ C\left(\int_0^t B(s) \exp\left(\frac{\Phi^{-1}(s)^2}{2}\right) \int_0^s g^{\circledast}(z)dz\times\right.$$
$$\left.\left.\times \exp\left[\int_s^t (2\pi)^{\frac{1}{2}} \exp\left(\frac{\Phi^{-1}(r)^2}{2}\right) B(r)\, dr\right] ds\right)^2\right] dt$$
$$+ C\int_0^{\gamma_n(\Omega)} u^{\circledast 2}(t)D^2(t)\, dt + C\int_0^{\gamma_n(\Omega)} F^2(t)\, dt$$
$$+ C\int_0^{\gamma_n(\Omega)} \exp\left(\Phi^{-1}(t)^2\right)\left(\int_0^t g^{\circledast}(z)dz\right)^2 dt.$$

We observe that if $b \in L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)$ and $\|b\|_{L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$ is sufficiently small, integrating by parts and using Proposition 2.5 we have

$$\begin{aligned}\int_s^t \frac{B(r)}{r(1-\log r)^{\frac{1}{2}}} dr &\leqslant C\|B\|_{L^\infty(\log L)^{-\frac{1}{2}}(\Omega^{\circledast})} + C\int_s^t \frac{\int_0^r B(\tau)\,d\tau}{r^2(1-\log r)^{\frac{1}{2}}} dr \\ &\leqslant C\|b\|_{L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)} + \|b\|_{L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)} \log\left(\frac{t}{s}\right),\end{aligned} \tag{4.2}$$

for some positive constant C.

If $b \in L^{\infty,a}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)$, $2 < a < \infty$, Hölder's and Young's inequalities give

$$\begin{aligned}\int_s^t \frac{B(r)}{r(1-\log r)^{\frac{1}{2}}} dr &\leqslant C(\varepsilon)\int_0^{t-s}\left(\frac{B^*(r)}{(1-\log r)^{\frac{1}{2}}}\right)^a \frac{dr}{r} + \varepsilon\int_s^t \frac{1}{r}dr \\ &\leqslant C(\varepsilon)\|b\|^a_{L^{\infty,a}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)} + \varepsilon\log\left(\frac{t}{s}\right),\end{aligned} \tag{4.3}$$

where ε can be arbitrary small and $C(\epsilon)$ is a suitable constant depending on ε.

Then using (4.2) or (4.3) we get

$$\int_\Omega |\nabla u|^2 \varphi(x)\,dx \leqslant C\int_0^{\gamma_n(\Omega)} \exp\left(\Phi^{-1}(t)^2\right)\left[\left(\int_0^t \left(u^{\circledast}(s)D(s)+F(s)\right)B(s)\left(\frac{t}{s}\right)^{\beta} ds\right)^2 \right.$$
$$\left. + \left(\int_0^t \exp\left(\frac{\Phi^{-1}(s)^2}{2}\right)B(s)\left(\frac{t}{s}\right)^{\beta}\int_0^s g^{\circledast}(z)dzds\right)^2\right] dt$$
$$+ C\int_0^{\gamma_n(\Omega)} u^{\circledast 2}(t)D^2(t)\,dt$$
$$+ C\int_0^{\gamma_n(\Omega)} F^2(t)\,dt + C\int_0^{\gamma_n(\Omega)} \exp\left(\Phi^{-1}(t)^2\right)\left(\int_0^t g^{\circledast}(z)dz\right)^2 dt.$$

where $\beta = C\,\|b\|_{L^\infty(\log L)^{-\frac12}(\varphi,\Omega)}$ if we use (4.2) and $\beta = C\varepsilon$ if we use (4.3).

Here and in what follows C will be a positive constant, depending only on $\gamma_n(\Omega)$ and $\|b\|$, which may vary from line to line.

Using (4), (5) we can apply (8) if β is sufficiently small obtaining

$$\int_\Omega |\nabla u|^2 \varphi(x)\,dx \leqslant C\int_0^{\gamma_n(\Omega)} \frac{1}{(1-\log t)}\left(u^{\circledast}(t)D^*(t)B^*(t)\right)^2 dt$$
$$+ C\int_0^{\gamma_n(\Omega)} \frac{1}{(1-\log t)}\left(B^*(t)F^*(t)\right)^2 dt$$
$$+ C\int_0^{\gamma_n(\Omega)} \frac{1}{(1-\log t)}\left(B^*(t)\frac{1}{(1-\log t)^{\frac12}}g^{\circledast\circledast}(t)\right)^2 dt$$
$$+ C\int_0^{\gamma_n(\Omega)} u^{\circledast 2}(t)D^{*2}(t)\,dt + C\int_0^{\gamma_n(\Omega)} F^{*2}(t)\,dt$$
$$+ C\int_0^{\gamma_n(\Omega)} \frac{1}{t^2(1-\log t)}\left(\int_0^t g^{\circledast}(z)dz\right)^2 dt.$$

and then by (17) we have

$$\|\nabla u\|^2_{L^2(\varphi,\Omega)} \leqslant C\,\|d\|^2_{L^\infty(\log L)^{-\frac12}(\varphi,\Omega)}\left(1+\|b\|^2_{L^\infty(\log L)^{-\frac12}(\varphi,\Omega)}\right)\|\nabla u\|^2_{L^2(\varphi,\Omega)}$$
$$+ C\left(1+\|b\|^2_{L^\infty(\log L)^{-\frac12}(\varphi,\Omega)}\right)\left(\|f\|^2_{L^2(\varphi,\Omega)}+\|g\|^2_{L^2(\log L)^{-\frac12}(\varphi,\Omega)}\right). \tag{4.4}$$

Then we have proved (37) if $\|d\|_{L^\infty(\log L)^{-\frac12}(\varphi,\Omega)}$ and $\|b\|_{L^\infty(\log L)^{-\frac12}(\varphi,\Omega)}$ are small enough or $\|d\|_{L^\infty(\log L)^{-\frac12}(\varphi,\Omega)}$ is small enough and $b \in L^{\infty,a}(\log L)^{-\frac12}(\varphi,\Omega)$. We observe that in this last case (4.4) can be obtained with $\|d\|_{L^{\infty,a}(\log L)^{-\frac12}(\varphi,\Omega)}$.

To prove (37) without smallness assumption on $\|d\|$ we assume $d \in L^{\infty,a}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)$ and we argue as in the Theorem 3.1. As a matter of the fact instead of (3.3) we have

$$-\frac{d}{dt}\int_{|u|>t}|\nabla u|^2\,\varphi(x)\,dx \leqslant \int_{|u|>t}|b(x)|\,|\nabla u|\,\varphi(x)\,dx + \left(-\frac{d}{dt}\int_{|u|>t}|\nabla u|^2\,\varphi(x)\,dx\right)^{\frac{1}{2}} \times$$
$$\times\, t\left(-\frac{d}{ds}\int_{|u|>t} d^2(x)\varphi(x)\,dx\right)^{\frac{1}{2}} + \left(-\frac{d}{dt}\int_{|u|>t} f^2\varphi(x)\,dx\right)^{\frac{1}{2}} \times$$
$$\times \left(-\frac{d}{dt}\int_{|u|>t}|\nabla u|^2\,\varphi(x)\,dx\right)^{\frac{1}{2}} + \int_{|u|>t} g(x)\varphi(x)\,\mathrm{sign}\,u\,dx.$$

Applying (24), Hölder inequality, (5), (25) and (27) we get

$$\begin{aligned}&\left(-\frac{d}{dt}\int_{|u|>t}|\nabla u|^2\,\varphi(x)\,dx\right)^{\frac{1}{2}}\\ &\quad\leqslant (2\pi)^{\frac{1}{2}}\exp\left(\frac{\Phi^{-1}(\mu(t))^2}{2}\right)\left(-\mu'(t)\right)^{\frac{1}{2}}\int_{|u|>t}|b(x)|\,|\nabla u|\,\varphi(x)\,dx\\ &\qquad + tD(\mu(t))\left(-\mu'(t)\right)^{\frac{1}{2}} + F(\mu(t))\left(-\mu'(t)\right)^{\frac{1}{2}}\\ &\qquad + (2\pi)^{\frac{1}{2}}\exp\left(\frac{\Phi^{-1}(\mu(t))^2}{2}\right)\left(-\mu'(t)\right)^{\frac{1}{2}}\int_0^{\mu(t)} g^{\circledast}(s)ds.\end{aligned} \tag{4.5}$$

Hence

$$\begin{aligned}&-\frac{d}{dt}\int_{|u|>t}|\nabla u|^2\,\varphi(x)\,dx\\ &\quad\leqslant \left\{(2\pi)^{\frac{1}{2}}\exp\left(\frac{\Phi^{-1}(\mu(t))^2}{2}\right)\left(-\mu'(t)\right)^{\frac{1}{2}}\int_{|u|>t}|b(x)|\,|\nabla u|\,\varphi(x)\,dx\right.\\ &\qquad + tD(\mu(t))\left(-\mu'(t)\right)^{\frac{1}{2}} + F(\mu(t))\left(-\mu'(t)\right)^{\frac{1}{2}}\\ &\qquad \left.+ (2\pi)^{\frac{1}{2}}\exp\left(\frac{\Phi^{-1}(\mu(t))^2}{2}\right)\left(-\mu'(t)\right)^{\frac{1}{2}}\int_0^{\mu(t)} g^{\circledast}(s)ds\right\}^2.\end{aligned}$$

Integrating between 0 and $+\infty$, by a variable change we have

$$\begin{aligned}\int_\Omega|\nabla u|^2\,\varphi(x)\,dx &\leqslant C\int_0^{\gamma_n(\Omega)}\exp\left(\Phi^{-1}(s)^2\right)\left(\int_{|u|>u^{\circledast}(s)}|b(x)|\,|\nabla u|\,\varphi(x)\,dx\right)^2 ds\\ &\quad + C\int_0^{\gamma_n(\Omega)} u^{\circledast 2}(s)D^2(s)\,ds + \int_0^{\gamma_n(\Omega)} F^2(s)\,ds +\\ &\quad + C\int_0^{\gamma_n(\Omega)}\exp\left(\Phi^{-1}(s)^2\right)\left(\int_0^s g^{\circledast}(z)dz\right)^2 ds.\end{aligned} \tag{4.6}$$

Now we evaluate the first integral. Putting $h(x) = |b(x)|\,|\nabla u|$ and using (4), (5) and Proposition 2.5, we obtain

$$\begin{aligned}
&\int_0^{\gamma_n(\Omega)} \exp\left(\Phi^{-1}(s)^2\right)\left(\int_{|u|>u^{\circledast}(s)} |b(x)|\,|\nabla u|\,\varphi(x)\;dx\right)^2 ds \\
&\leqslant C\int_0^{\gamma_n(\Omega)} \frac{1}{s^2(1-\log s)}\left(\int_0^s \widetilde{h}_u^{\circledast}(t)dt\right)^2 ds \leqslant \left\|\widetilde{h}_u\right\|^2_{L^2(\log L)^{-\frac{1}{2}}(0,\gamma_n(\Omega))} \qquad (4.7)\\
&\leqslant C\,\|h\|^2_{L^2(\log L)^{-\frac{1}{2}}(\varphi,\Omega)} \leqslant C\,\|b\|^2_{L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}\,\|\nabla u\|^2_{L^2(\varphi,\Omega)}.
\end{aligned}$$

Using (4.7) and (4), Hölder inequality and Proposition 2.5 the inequality (4.6) becomes

$$\begin{aligned}
\|\nabla u\|^2_{L^2(\varphi,\Omega)} \leqslant C\Big\{&\|b\|^2_{L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}\,\|\nabla u\|^2_{L^2(\varphi,\Omega)} \\
&+\|d\|^2_{L^{\infty,a}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}\,\|u\|^2_{L^2(\log L)^{\frac{1}{2}}(\varphi,\Omega)} \qquad (4.8)\\
&+\|f\|^2_{L^2(\varphi,\Omega)}+\|g\|^2_{L^2(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}\Big\}.
\end{aligned}$$

Now we want to evaluate $\|u\|_{L^2(\log L)^{\frac{1}{2}}(\varphi,\Omega)}$. We will prove that

$$\begin{aligned}
\|u\|_{L^2(\log L)^{\frac{1}{2}}(\varphi,\Omega)} \leqslant C\,\|f\|_{L^2(\varphi,\Omega)} &+ C\,\|g\|_{L^2(\log L)^{-\frac{1}{2}}(\varphi,\Omega)} \\
&+C\,\|b\|_{L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}\,\|\nabla u\|_{L^2(\varphi,\Omega)}, \qquad (4.9)
\end{aligned}$$

and, putting (4.9) in (4.8), we obtain

$$\begin{aligned}
\|\nabla u\|^2_{L^2(\varphi,\Omega)} \leqslant{}& C\,\|b\|^2_{L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}\left(1+\|d\|^2_{L^{\infty,a}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}\right)\|\nabla u\|^2_{L^2(\varphi,\Omega)} \\
&+C\left(1+\|d\|^2_{L^{\infty,a}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}\right)\left(\|f\|^2_{L^2(\varphi,\Omega)}+\|g\|^2_{L^2(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}\right). \qquad (4.10)
\end{aligned}$$

The inequality (4.10) implies the assertion (37) if $\|b\|_{L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$ is sufficiently small.

Now we prove (4.9). By (4.5) and using (24) we have

$$\begin{aligned}
\frac{1}{(-\mu'(t))} \leqslant C\Bigg\{&(2\pi)\exp\left(\Phi^{-1}(\mu(t))^2\right)\times\int_{|u|>t}|b(x)|\,|\nabla u|\,\varphi(x)\;dx \\
&+(2\pi)^{\frac{1}{2}}\exp\left(\frac{\Phi^{-1}(\mu(t))^2}{2}\right)[tD(\mu(t))+F(\mu(t))] \\
&+(2\pi)\exp\left(\Phi^{-1}(\mu(t))^2\right)\int_0^{\mu(t)} g^{\circledast}(s)ds\Bigg\}.
\end{aligned}$$

Putting $\mu(t) = s$ and integrating between s and $\gamma_n(\Omega)$ we obtain

$$u^{\circledast}(s) \leqslant C \int_s^{\gamma_n(\Omega)} (2\pi) \exp\left(\Phi^{-1}(z)^2\right) \left(\int_{|u|>u^{\circledast}(z)} |b(x)| \, |\nabla u| \, \varphi(x) \, dx + \int_0^z g^{\circledast}(r) dr \right) dz$$
$$+ \int_s^{\gamma_n(\Omega)} (2\pi)^{\frac{1}{2}} \exp\left(\frac{\Phi^{-1}(z)^2}{2}\right) \left(u^{\circledast}(z) D(z) + F(z)\right) dz.$$

We want to apply Gronwall lemma with $\phi(s) = u^{\circledast}(s)$. Condition (19) can be verified arguing as in Lemma 5.1 in the Appendix. Integrating by parts we have

$$\begin{aligned} u^{\circledast}(s) \leqslant C \int_s^{\gamma_n(\Omega)} & \left(\exp\left(\frac{\Phi^{-1}(z)^2}{2}\right) F(z) + \exp\left(\Phi^{-1}(z)^2\right) \times \right. \\ & \times \left(\int_{|u|>u^{\circledast}(z)} |b(x)| \, |\nabla u| \, \varphi(x) \, dx + \int_0^z g^{\circledast}(r) dr \right) \Bigg) \times \qquad (4.11) \\ & \times \exp\left(\int_s^z \exp\left(\frac{\Phi^{-1}(\tau)^2}{2}\right) D(\tau) \, d\tau \right) dz. \end{aligned}$$

Using (4.3) with D replaced by B we get

$$\exp\left(\int_s^z \exp\left(\frac{\Phi^{-1}(\tau)^2}{2}\right) D(\tau) \, d\tau \right) \leqslant C_1 \left(\frac{z}{s}\right)^{C\varepsilon},$$

where C_1 is a constant depending on $\|d\|$. In (4.11) for a suitable ε we can use (9) and by (4.7) we obtain (4.9). □

Now we study the summability of the solution u in the Lorentz-Zygmund space We first observe that under the assumption of Proposition 4.1 if $g \in L^2(\log L)^{-\frac{1}{2}}(\varphi, \Omega)$ and $f \in L^2(\varphi, \Omega)$, then problem (1.1) has a solution $u \in H_0^1(\varphi, \Omega)$. Moreover by Gross inequality we have that the solution $u \in L^2(\log L)^{\frac{1}{2}}$.

We will use estimates obtained in Corollaries 3.2 and 3.3 considering $d_i(x) \equiv 0$ or $b_i(x) \equiv 0$. In each case, for the convenient of the reader we will examine separately $f_i(x) \equiv 0, i = 1, ..., n$ or $g(x) \equiv 0$.

THEOREM 4.1. *Under the assumptions of Corollary* 3.2, *when* $f_i(x) \equiv 0, i = 1, ..., n$ *and* $g \in L^{p,q}(\log L)^{\alpha}(\varphi, \Omega)$, *the following results hold:*

(*a*) *if*

$$p = 2 \text{ and either } 1 \leqslant q \leqslant 2 \text{ and } \alpha \geqslant -\frac{1}{2} \text{ or } 2 < q \leqslant \infty \text{ and } \alpha > -\frac{1}{q}$$

or

$$2 < p < \infty, \ 1 \leqslant q \leqslant \infty \text{ and } -\infty < \alpha < +\infty,$$

then $u \in L^{p,q}(\log L)^{\alpha+1}(\varphi, \Omega)$. *Besides*

$$\|u\|_{L^{p,q}(\log L)^{\alpha+1}(\varphi,\Omega)} \leqslant C_1 \, \|g\|_{L^{p,q}(\log L)^{\alpha}(\varphi,\Omega)}; \qquad (48)$$

(b) *if*

$$p=\infty, 1 \leqslant q \leqslant \infty, -\infty < \alpha < +\infty \ \textit{and} \ \alpha+\frac{1}{q}<0,$$

then $u \in L^{\infty,q}(\log L)^{\alpha}(\varphi,\Omega)$. *Besides*

$$\|u\|_{L^{\infty,q}(\log L)^{\alpha}(\varphi,\Omega)} \leqslant C_2 \ \|g\|_{L^{\infty,q}(\log L)^{\alpha}(\varphi,\Omega)}. \tag{49}$$

The constants C_1, C_2 *depend on* $p,q,\alpha,\gamma_n(\Omega)$ *and* $\|b\|_{L^{\infty,a}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$.

Proof. If $w \in H_0^1(\varphi,\Omega^\star)$ is the weak solution of problem (1.3) with $f_i \equiv 0, i = 1,...,n$ by (4) and (4.2) or (4.3) we have

$$w(t) \leqslant C\int_t^{\gamma_n(\Omega)} \frac{1}{\sigma^2(1-\log\sigma)}\int_0^{\sigma}\left(\frac{\sigma}{s}\right)^{\beta} g^{\circledast}(s)ds\, d\sigma,$$

where $\beta = C\|b\|_{L^{\infty}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$ if we use (4.2) and $\beta = C\varepsilon$ if we use (4.3). Here and in what follows C will be a positive constant, depending only on $\gamma_n(\Omega)$ and $\|b\|$, which may vary from line to line. We prove result (a) when $1 \leqslant q < \infty$. By definition (2.13), (9) and (8) with $\frac{1}{p}-1+\beta<0$, we have

$$\begin{aligned}
&||w||_{p,q;\alpha+1}^q \\
&\leqslant C\int_0^{\gamma_n(\Omega)}\left(t^{\frac{1}{p}}(1-\log t)^{\alpha+1}\int_t^{\gamma_n(\Omega)}\frac{\sigma^{\beta}}{\sigma^2(1-\log\sigma)}\int_0^{\sigma}\left(\frac{1}{s}\right)^{\beta}g^{\circledast}(s)ds\,d\sigma\right)^q\frac{dt}{t}\\
&\leqslant C\int_0^{\gamma_n(\Omega)}\left(t^{\frac{1}{p}-1+\beta}(1-\log t)^{\alpha}\int_0^t\left(\frac{1}{s}\right)^{\beta}g^{\circledast}(s)ds\right)^q\frac{dt}{t}\\
&\leqslant C\int_0^{\gamma_n(\Omega)}\left(t^{\frac{1}{p}}(1-\log t)^{\alpha}g^{\circledast}(t)\right)^q\frac{dt}{t} = C||g||_{p,q;\alpha}^q.
\end{aligned}$$

The last inequality and (21) prove (48).

When $q=\infty$ the inequality (48) follows by the same method as before with (9) replaced by (11). The same proof still goes in the case (b) if we replace (9) by (13) when $1 \leqslant q < \infty$ and by (15) when $q=\infty$. □

THEOREM 4.2. *Under the assumptions of Corollary* 3.2*, when* $g(x) \equiv 0$ *and* $f_i \in L^{p,q}(\log L)^{\alpha}(\varphi,\Omega)$, $i=1,....n$*, the following results hold:*

(a) *if*

$$2<p<\infty,\ 2\leqslant q \leqslant \infty \ \textit{and} \ -\infty<\alpha<+\infty,$$

then $u \in L^{p,q}(\log L)^{\alpha+\frac{1}{2}}(\varphi,\Omega)$.*Besides*

$$\|u\|_{L^{p,q}(\log L)^{\alpha+\frac{1}{2}}(\varphi,\Omega)} \leqslant C_1 \ \|f\|_{L^{p,q}(\log L)^{\alpha}(\varphi,\Omega)}; \tag{50}$$

(b) *if*

$$p=\infty, 1 \leqslant q < \infty, -\infty < \alpha < +\infty \ \textit{and} \ \alpha+\frac{1}{q}<0$$

or

$$p = \infty, q = \infty, \quad and \quad \alpha \leqslant 0,$$

then $u \in L^{\infty,q}(\log L)^{\alpha-\frac{1}{2}}(\varphi,\Omega)$. *Besides*

$$\|u\|_{L^{\infty,q}(\log L)^{\alpha-\frac{1}{2}}(\varphi,\Omega)} \leqslant C_2 \ \|f\|_{L^{\infty,q}(\log L)^{\alpha}(\varphi,\Omega)}. \tag{51}$$

The constants C_1, C_2 *depends on* $p, q, \alpha, \gamma_n(\Omega)$ *and* $\|b\|_{L^{\infty,a}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$.

Proof. Using (4) we have

$$w^{\circledast}(t) \leqslant w_1(t) + w_2(t),$$

where

$$w_1(t) = (2\pi)^{\frac{1}{2}} \int_t^{\gamma_n(\Omega)} \frac{F(\sigma)}{\sigma(1-\log\sigma)^{\frac{1}{2}}} d\sigma \tag{52}$$

and

$$\begin{aligned} w_2(t) = (2\pi) \int_t^{\gamma_n(\Omega)} \frac{1}{\sigma^2(1-\log\sigma)} \times \\ \times \int_0^{\sigma} \exp\left[(2\pi)^{\frac{1}{2}} \int_r^{\sigma} \frac{B(r)}{\sigma(1-\log\sigma)^{\frac{1}{2}}} d\tau \right] B(r)F(r)\, dr\, d\sigma. \end{aligned} \tag{4.17}$$

Case (a) with $2 \leqslant q < \infty$. In what follows C will be a positive constant. Let us observe that integrating by parts, using Hardy-Littlewood inequality (5) we get:

$$w_1(t) \leqslant C \frac{||F||_{L^1(\Omega^{\circledast})}}{\gamma_n(\Omega)\,(1-\log\gamma_n(\Omega))^{\frac{1}{2}}} + C \int_t^{\gamma_n(\Omega)} \frac{\int_0^{\sigma} F^*(s)\, ds}{\sigma^2\,(1-\log\sigma)^{\frac{1}{2}}} d\sigma \tag{54}$$

Using (54), (9) and Proposition 2.5 we have

$$\begin{aligned} \|w_1\|_{L^{p,q}(\log L)^{\alpha+\frac{1}{2}}(\varphi,\Omega)} \leqslant C||F||_{L^1(\Omega^{\circledast})} \left(\int_0^{\gamma_n(\Omega)} t^{\frac{q}{p}} (1-\log t)^{(\alpha+\frac{1}{2})q} \frac{dt}{t} \right)^{\frac{1}{q}} \\ + C \left(\int_0^{\gamma_n(\Omega)} t^{\frac{q}{p}} \ (1-\log t)^{(\alpha+\frac{1}{2})q} \left(\int_t^{\gamma_n(\Omega)} \frac{\int_0^{\sigma} F^*(s)\, ds}{\sigma^2\,(1-\log\sigma)^{\frac{1}{2}}} d\sigma \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ \leqslant C\, \|F\|_{L^{p,q}(\log L)^{\alpha}(\Omega^{\circledast})} \leqslant C\, \|f\|_{p,q;\alpha}. \end{aligned} \tag{4.19}$$

To evaluate the norm of w_2 in $L^{p,q}(\log L)^{\alpha+\frac{1}{2}}(\varphi,\Omega)$ it's possible to argue as in Theorem 4.1, obtaining

$$||w_2||_{p,q;\alpha+\frac{1}{2}} \leqslant C||f||_{p,q;\alpha}. \tag{56}$$

By (4.19) and (56) we obtain (50).

The proof of other cases is analogues and run as before. □

In a similar way it's possible to prove the following theorems.

THEOREM 4.3. *Under the assumptions of Corollary* 3.3*, when* $f_i(x) \equiv 0, i = 1, ..., n$ *and* $g \in L^{p,q}(\log L)^{\alpha}(\varphi, \Omega)$*, the following results hold:*

(*a*) *if*

$$p = 2 \text{ and either } 1 \leqslant q \leqslant 2 \text{ and } \alpha \geqslant -\frac{1}{2} \text{ or } 2 < q \leqslant \infty \text{ and } \alpha > -\frac{1}{q}$$

or

$$2 < p < \infty, 1 \leqslant q \leqslant \infty \text{ and } -\infty < \alpha < +\infty,$$

then $u \in L^{p,q}(\log L)^{\alpha+1}(\varphi, \Omega)$*. Besides*

$$\|u\|_{L^{p,q}(\log L)^{\alpha+1}(\varphi,\Omega)} \leqslant C_1 \ \|g\|_{L^{p,q}(\log L)^{\alpha}(\varphi,\Omega)};$$

(*b*) *if*

$$p = \infty, 1 \leqslant q \leqslant \infty, -\infty < \alpha < +\infty \text{ and } \alpha + \frac{1}{q} < 0,$$

then $u \in L^{\infty,q}(\log L)^{\alpha}(\varphi, \Omega)$*. Besides*

$$\|u\|_{L^{\infty,q}(\log L)^{\alpha}(\varphi,\Omega)} \leqslant C_2 \ \|g\|_{L^{\infty,q}(\log L)^{\alpha}(\varphi,\Omega)}.$$

The constants C_1, C_2 *depend on* $p, q, \alpha, \gamma_n(\Omega)$ *and* $\|d\|_{L^{\infty,a}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$.

THEOREM 4.4. *Under the assumptions of Corollary* 3.3*, when* $g \equiv 0$ *and* $f_i \in L^{p,q}(\log L)^{\alpha}(\varphi, \Omega), i = 1, ..., n$*, the following results hold:*

(*a*) *if*

$$2 < p < \infty, \ 2 \leqslant q \leqslant \infty \text{ and } -\infty < \alpha < +\infty,$$

then $u \in L^{p,q}(\log L)^{\alpha+\frac{1}{2}}(\varphi, \Omega)$*. Besides*

$$\|u\|_{L^{p,q}(\log L)^{\alpha+\frac{1}{2}}(\varphi,\Omega)} \leqslant C_1 \ \|f\|_{L^{p,q}(\log L)^{\alpha}(\varphi,\Omega)};$$

(*b*) *if*

$$p = \infty, 2 \leqslant q < \infty, -\infty < \alpha < +\infty \text{ and } \alpha + \frac{1}{q} < 0$$

or

$$p = \infty, q = \infty \text{ and } \alpha \leqslant 0,$$

then $u \in L^{\infty,q}(\log L)^{\alpha-\frac{1}{2}}(\varphi, \Omega)$*. Besides*

$$\|u\|_{L^{\infty,q}(\log L)^{\alpha-\frac{1}{2}}(\varphi,\Omega)} \leqslant C_2 \ \|f\|_{L^{\infty,q}(\log L)^{\alpha}(\varphi,\Omega)}.$$

The constants C_1, C_2 *depends on* $p, q, \alpha, \gamma_n(\Omega)$ *and* $\|d\|_{L^{\infty,a}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$.

REMARK. We remark that even if we improve the summability of the data, we cannot prove that the solution of problem (1.1) must be bounded as the following example shows.

Let us consider the problem

$$\begin{cases} -\left(w_{x_1}\varphi(x)\right)_{x_1} = g^{\star}(x_1)\varphi(x) & \text{in } \Omega^{\star} \\ w = 0 & \text{on } \partial\Omega^{\star}; \end{cases}$$

we have

$$w(x) = \int_{\Phi(x_1)}^{\gamma_n(\Omega)} \exp(\Phi^{-1}(\sigma)^2) \int_0^{\sigma} g^{\circledast}(t)dt\, d\sigma,$$

and then

$$\|w\|_{L^\infty(\Omega)} = w^{\circledast}(0) = \int_0^{\gamma_n(\Omega)} \exp(\Phi^{-1}(\sigma)^2) \int_0^{\sigma} g^{\circledast}(t)dt\, d\sigma. \tag{57}$$

Using (4) it is easy to see that (57) is finite only if $g \equiv 0$.

5. Appendix

In this Appendix we want verify that Gronwall lemma can be applied to (3.9).

LEMMA 5.1. *With reference to* (3.9) *let* $\gamma \equiv 1$,

$$\lambda(t) = B(\mu(t))\left(-\mu'(t)\right)\exp\left(\frac{\Phi^{-1}(\mu(t))^2}{2}\right) \tag{58}$$

and

$$\phi(t) = \left(-\frac{d}{dt}\int_{|u|>t} |\nabla u|^2 \varphi(x)dx\right)^{\frac{1}{2}} \exp\left(-\frac{\Phi^{-1}(\mu(t))^2}{2}\right)\left(-\mu'(t)\right)^{-\frac{1}{2}}. \tag{59}$$

If either $\|b\|_{L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}$ *is small enough or* $b \in L^{\infty,a}(\log L)^{-\frac{1}{2}}(\varphi,\Omega)$, $2 < a < \infty$*, then* (19) *holds.*

Proof. Under position (58) and (59) the condition (19) to verify becomes

$$\begin{aligned} \lim_{k\to\infty} & \left[\int_k^{+\infty} B(\mu(t))\left(-\mu'(t)\right)^{\frac{1}{2}} \left(-\frac{d}{dt}\int_{|u|>t} |\nabla u|^2 \varphi(x)dx\right)^{\frac{1}{2}} dt\right] \times \\ & \times \exp\left((2\pi)^{\frac{1}{2}} \int_0^k \exp\left(\frac{\Phi^{-1}(\mu(t))^2}{2}\right) B(\mu(t))\left(-\mu'(t)\right)dt\right) = 0. \end{aligned} \tag{5.3}$$

Using Hölder's inequality, Hardy's inequality, (4) and the Proposition 2.5, we have

$$\begin{aligned}&\left[\int_k^{+\infty} B\left(\mu\left(t\right)\right)\left(-\mu'\left(t\right)\right)^{\frac{1}{2}}\left(-\frac{d}{dt}\int_{|u|>t}\left|\nabla u\right|^2\varphi(x)dx\right)^{\frac{1}{2}}dt\right]\times\\&\qquad\times\exp\left(\left(2\pi\right)^{\frac{1}{2}}\int_0^k\exp\left(\frac{\Phi^{-1}\left(\mu\left(t\right)\right)^2}{2}\right)B\left(\mu\left(t\right)\right)\left(-\mu'\left(t\right)\right)dt\right)\\&\leqslant C_1\left\|b\right\|_{L^\infty(\log L)^{-\frac{1}{2}}(\varphi,\Omega)}\left(\int_0^{\mu(k)}\left(1-\log s\right)ds\right)^{\frac{1}{2}}\left(\int_{|u|>k}\left|\nabla u\right|^2\varphi dx\right)^{\frac{1}{2}}\times\\&\qquad\times\exp\left(\left(2\pi\right)^{\frac{1}{2}}C_2\int_{\mu(k)}^{\gamma_n(\Omega)}\frac{B\left(s\right)}{s\left(1-\log s\right)^{\frac{1}{2}}}ds\right),\end{aligned}\tag{5.4}$$

for some positive constants C_1 and C_2.

Using (4.2) if $b\in L^\infty\left(\log L\right)^{-\frac{1}{2}}\left(\varphi,\Omega\right)$ or (4.3) if $b\in L^{\infty,a}\left(\log L\right)^{-\frac{1}{2}}\left(\varphi,\Omega\right)$, $2<a<\infty$, condition (5.3) holds.

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