Mathematical
Inequalities
& Applications
Volume 10, Number 4 (2007), 843–861

WEIGHTED INEQUALITIES OF HARDY TYPE FOR MATRIX OPERATORS: THE CASE $q < p$

RYSKUL OINAROV, CHRISTOPHER A. OKPOTI AND LARS-ERIK PERSSON

(*communicated by S. Saitoh*)

Abstract. A non-negative triangular matrix operator is considered in weighted Lebesgue spaces of sequences. Under some additional conditions on the matrix, some new weight characterizations for discrete Hardy type inequalities with matrix operator are proved for the case $1 < q < p < \infty$. Some further results are pointed out.

1. Introduction

Let $1 < p, q < \infty$ and $1/p + 1/p' = 1$. Let $f = \{f_i\}_{i=1}^{\infty}$ be an arbitrary positive sequence of real numbers. If $f_i \geqslant 0$ $(f_i > 0)$, $i \geqslant 1$, then we write $f \geqslant 0$ $(f > 0)$.

We will study inequalities of the following form

$$\|uAf\|_{l_q} \leqslant C\,\|vf\|_{l_p}, \tag{1}$$

where A is a matrix operator of the form $A = R$ or $A = K$ defined by

$$(Rf)_i = \sum_{j=1}^{i} a_{i,j} f_j, \quad i \geqslant 1 \tag{2}$$

and

$$(Kf)_j = \sum_{i=j}^{\infty} a_{i,j} f_i, \quad j \geqslant 1, \tag{3}$$

respectively. Moreover,

$$\|f\|_{l_p} = \left(\sum_{i=1}^{\infty} |f_i|^p\right)^{1/p},$$

$(a_{i,j})$ is a non-negative triangular matrix (i.e. $a_{i,j} \geqslant 0$, $i \geqslant j \geqslant 1$ and $a_{j,i} = 0$, $j < i$), and $u = \{u_i\}_{i=1}^{\infty}$, $v = \{v_i\}_{i=1}^{\infty}$ are non-negative weight sequences.

Here and in the sequel we use the conventions $a_{0,0} = 0$ and $(.)_0 = 0$ (i.e. $0 = f_0 = u_0 = v_0 = \ldots$), $\sum_{k=i}^{j} (.)_k = 0$, if $j < i$ and $\lim_{j\to\infty} \sum_{k=j}^{\infty} (.)_k = 0$.

Mathematics subject classification (2000): 26D10, 26D15.

Key words and phrases: inequalities, discrete Hardy type inequalities, weights, matrix operators.

Paper MIA-10-77

REMARK 1.1. If $a_{i,j} \equiv 1$ for $i \geqslant j \geqslant 1$, then (2) and (3) coincide with the usual Hardy's discrete operator $(Pf)_i = \sum_{j=1}^{i} f_j$ and its dual $(Qf)_j = \sum_{i=j}^{\infty} f_i$, respectively. These operators and (1) give us the weighted Hardy inequalities. Concerning the history and development of discrete Hardy type inequalities we refer to [5] and [6] and the references given there.

The inequality (1) with the operator (2) was recently studied in [10] (see also [9]) for the case $1 < p \leqslant q < \infty$, but the matrix $(a_{i,j})$ was defined as a non-negative kernel. Some scales of characterizations for the special case with product weight kernel, i.e. when $a_{i,j} = l_i h_j$, $i,j = 1,2,\dots$ and, moreover, a sufficient condition for a general kernel $(a_{i,j})$, which at least for a special case is also necessary, were proved. The first result in this direction is due to K. F. Andersen and H. P. Heinig ([1], Theorem 4.1), who proved a sufficient condition for (1) to hold for the case $1 \leqslant p \leqslant q < \infty$ with a special non-negative kernel $(a_{i,j})$ that was assumed to be non-increasing in j and non-decreasing in i.

Throughout this paper $a \ll b$, $(b \gg a)$, means that $a \leqslant \lambda b$, where $\lambda > 0$ is a constant or depends only on inessential parameters. If $b \ll a \ll b$, then we write $a \approx b$.

We need to assume that the matrix $(a_{i,j})$, $a_{i,j} \geqslant 0$ satisfies the following additional condition:

Suppose that there exist $c = \{c_i\}_{i=1}^{\infty}$, $c > 0$ and $b = \{b_i\}_{i=1}^{\infty}$, $b > 0$, such that,

$$a_{i,j} \approx \frac{a_{i,k}}{c_k} c_j + \frac{a_{k,j}}{b_k} b_i, \quad i \geqslant k \geqslant j \geqslant 1. \tag{4}$$

We note that (4) implies that

$$\frac{a_{i,k}}{c_k} \ll \frac{a_{i,j}}{c_j} \text{ and } \frac{a_{k,j}}{b_k} \ll \frac{a_{i,j}}{b_i}, \quad i \geqslant k \geqslant j \geqslant 1. \tag{5}$$

REMARK 1.2. If $b_i = b_k = c_j = c_k = 1$ in (4), then $a_{i,j} \approx a_{i,k} + a_{k,j}$, which implies that $a_{i,k} \ll a_{i,j}$ and $a_{k,j} \ll a_{i,j}$, $i \geqslant k \geqslant j \geqslant 1$.

In the paper [8], R. Oinarov and S. Kh. Shalginbayeva studied the following three weights inequality

$$\|uAf\|_{l_q} \leqslant C\left(\|vf\|_{l_p} + \|\omega Pf\|_{l_p}\right) \tag{6}$$

for all arbitrary positive sequences f for the case $1 < p \leqslant q < \infty$ and u, v, ω are non-negative weight sequences. If the weight $\omega \equiv 0$, that is $\omega_i = 0$, $\forall i \geqslant 1$, then the inequality (1) coincides with (6). Therefore (1) can be regarded as a special case of (6).

In this paper we will prove some corresponding weight characterizations for the case $1 < q < p < \infty$.

In Section 2 we state these main results (Theorems 2.1 and 2.2) together with some necessary technical lemmas, while the proofs can be found in Section 3 and some further results are given in Section 4 (see Theorems 4.1 and 4.2).

2. Main results and some lemmas

Our main results read:

THEOREM 2.1. *Let* $1 < q < p < \infty$ *and assume that the elements of the matrix* $(a_{i,j})$ *satisfy the condition* (4). *Then* (1) *with* $A = R$ *(defined by* (2)*) holds if and only if* $B := \max\{B_1, B_2\} < \infty$, *where*

$$B_1 := \left(\sum_{k=1}^{\infty} v_k^{-p'} \left(\sum_{i=k}^{\infty} a_{i,k}^q u_i^q\right)^{\frac{p}{p-q}} \left(\frac{1}{c_k^{p'}} \sum_{i=1}^{k} c_i^{p'} v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}}\right)^{\frac{p-q}{pq}}$$

and

$$B_2 := \left(\sum_{k=1}^{\infty} u_k^q \left(\sum_{i=1}^{k} a_{k,i}^{p'} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}} \left(\frac{1}{b_k^q} \sum_{i=k}^{\infty} b_i^q u_i^q\right)^{\frac{q}{p-q}}\right)^{\frac{p-q}{pq}}.$$

Moreover, for the best constant C *in* (1) *we have that* $C \approx B$.

THEOREM 2.2. *Let* $1 < q < p < \infty$ *and suppose that the elements of the matrix* $(a_{i,j})$ *satisfy the condition* (4). *Then* (1) *with* $A = K$ *(defined by* (3)*) holds if and only if* $B^* := \max\{B_1^*, B_2^*\} < \infty$, *where*

$$B_1^* := \left(\sum_{k=1}^{\infty} u_k^q \left(\sum_{i=k}^{\infty} a_{i,k}^{p'} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}} \left(\frac{1}{c_k^q} \sum_{i=1}^{k} c_i^q u_i^q\right)^{\frac{q}{p-q}}\right)^{\frac{p-q}{pq}}$$

and

$$B_2^* := \left(\sum_{k=1}^{\infty} v_k^{-p'} \left(\sum_{i=1}^{k} a_{k,i}^q u_i^q\right)^{\frac{p}{p-q}} \left(\frac{1}{b_k^{p'}} \sum_{i=k}^{\infty} b_i^{p'} v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}}\right)^{\frac{p-q}{pq}}.$$

Moreover, for the best constant C *in* (1) *we have that* $C \approx B^*$.

For the proofs we need the following lemmas:

LEMMA 2.1. *Let* $\gamma > 0$ *and let* $\{\beta_k\}$ *be a positive sequence. Then, for each* $j \in \mathbb{Z}_+$,

$$\left(\sum_{k=1}^{j} \beta_k\right)^{\gamma} \approx \sum_{k=1}^{j} \beta_k \left(\sum_{i=1}^{k} \beta_i\right)^{\gamma-1} \tag{7}$$

and, if, in addition, $\sum_k \beta_k < \infty$, *then for* $1 \leqslant j, k < N \leqslant \infty$

$$\left(\sum_{k=j}^{N} \beta_k\right)^{\gamma} \approx \sum_{k=j}^{N} \beta_k \left(\sum_{i=k}^{N} \beta_i\right)^{\gamma-1}. \tag{8}$$

REMARK 2.1. The estimates (7) and (8), due to K. F. Andersen and H. P. Heinig [[1], p. 844], have been used by many authors including K. Goswin and G. Erdmann [[3], p. 12] and G. Bennett [[2], Lemmas 2 and 3].

LEMMA 2.2. *Let* $1 < p < \infty$. *Then, with the notations from the introduction,*

$$\sum_{i=1}^{\infty} u_i^p \left(\sum_{j=1}^{i} a_{i,j} f_j \right)^p \approx \sum_{j=1}^{\infty} f_j \sum_{i=j}^{\infty} a_{i,j} u_i^p \left(\sum_{k=1}^{j} a_{i,k} f_k \right)^{p-1} \tag{9}$$

and

$$\sum_{j=1}^{\infty} v_j^{-p} \left(\sum_{i=j}^{\infty} a_{i,j} g_i \right)^p \approx \sum_{i=1}^{\infty} g_i \sum_{j=1}^{i} a_{i,j} v_j^{-p} \left(\sum_{k=i}^{\infty} a_{k,j} g_k \right)^{p-1}. \tag{10}$$

Also the following remark will be crucial in our discussions later on.

REMARK 2.2. The operator K (defined by (3)) is the conjugate of the operator R (defined by (2)). Therefore, the inequality (1) with $A = R$ (defined by (2)) holds if and only if the inequality

$$\left\| v^{-1} K g \right\|_{l_{p'}} \leqslant C \left\| u^{-1} g \right\|_{l_{q'}} \tag{11}$$

holds with the same constant C (see [4]). Here g is an arbitrary positive sequence.

3. Proofs

Proof of Lemma 9. Using the mean-value theorem and elementary estimates we find that

$$b^p - a^p \approx b^{p-1}(b-a) \tag{12}$$

for $b > a > 0$, $p > 0$. According to (12) and the Fubini theorem, we have

$$\begin{aligned}
\sum_{i=1}^{\infty} u_i^p \left(\sum_{j=1}^{i} a_{i,j} f_i \right)^p &= \sum_{i=1}^{\infty} u_i^p \sum_{j=1}^{i} \left(\left(\sum_{k=1}^{j} a_{i,k} f_k \right)^p - \left(\sum_{k=1}^{j-1} a_{i,k} f_k \right)^p \right) \\
&\approx \sum_{i=1}^{\infty} u_i^p \sum_{j=1}^{i} a_{i,j} f_j \left(\sum_{k=1}^{j} a_{i,k} f_k \right)^{p-1} \\
&= \sum_{j=1}^{\infty} f_j \sum_{i=j}^{\infty} a_{i,j} u_i^p \left(\sum_{k=1}^{j} a_{i,k} f_k \right)^{p-1}
\end{aligned}$$

and

$$\sum_{j=1}^{\infty} v_j^{-p} \left(\sum_{i=j}^{\infty} a_{i,j} g_i \right)^p = \sum_{j=1}^{\infty} v_j^{-p} \sum_{i=j}^{\infty} \left(\left(\sum_{k=i}^{\infty} a_{k,j} g_k \right)^p - \left(\sum_{k=i+1}^{\infty} a_{k,j} g_k \right)^p \right)$$
$$\approx \sum_{j=1}^{\infty} v_j^{-p} \sum_{i=j}^{\infty} a_{i,j} g_i \left(\sum_{k=i}^{\infty} a_{k,j} g_k \right)^{p-1}$$
$$= \sum_{i=1}^{\infty} g_i \sum_{j=1}^{i} a_{i,j} v_j^{-p} \left(\sum_{k=i}^{\infty} a_{k,j} g_k \right)^{p-1}.$$

The proof is complete. □

Proof of Theorem 2.1. *Necessity*. Let us assume that (1) holds for a finite constant C and, for fixed $N \in \mathbb{Z}_+$, apply the following test sequence to (1): $f_N = \left\{ \left(f_N\right)_j \right\}_{j=1}^{\infty}$, where

$$\left(f_N\right)_j := \left(\sum_{i=j}^{N} a_{i,j}^q u_i^q \right)^{\frac{1}{p-q}} \left(\frac{1}{c_j^{p'}} \sum_{i=1}^{j} c_i^{p'} v_i^{-p'} \right)^{\frac{q-1}{p-q}} v_j^{-p'}, \tag{13}$$

$j = 1, .., N,$ and $\left(f_N\right)_j = 0$ for $j > N$.

Applying (13) to the right hand side of (1), we have that

$$\|v f_N\|_{l_p} = \left(\sum_{j=1}^{N} v_j^{-p'} \left(\sum_{i=j}^{N} a_{i,j}^q u_i^q \right)^{\frac{p}{p-q}} \left(\frac{1}{c_j^{p'}} \sum_{i=1}^{j} c_i^{p'} v_i^{-p'} \right)^{\frac{p(q-1)}{p-q}} \right)^{\frac{1}{p}}. \tag{14}$$

For the left hand side of (1), we use (9) with p replaced by q and find that

$$\|uRf_N\|_{l_q}^q = \sum_{i=1}^{\infty} u_i^q \left(\sum_{j=1}^{i} a_{i,j} \left(f_N\right)_j \right)^q$$
$$\approx \sum_{j=1}^{\infty} \left(f_N\right)_j \sum_{i=j}^{\infty} a_{i,j} u_i^q \left(\sum_{k=1}^{j} a_{i,k} \left(f_N\right)_k \right)^{q-1}$$

[using (5) and the definition (13)]

$$\gg \sum_{j=1}^{N} \left(f_N\right)_j \sum_{i=j}^{N} \frac{a_{i,j}^q}{c_j^{q-1}} u_i^q \left(\sum_{k=1}^{j} c_k \left(f_N\right)_k \right)^{q-1}$$
$$= \sum_{j=1}^{N} \left(f_N\right)_j \frac{1}{c_j^{q-1}} \sum_{i=j}^{N} a_{i,j}^q u_i^q \left(\sum_{k=1}^{j} c_k v_k^{-p'} c_k^{\frac{q}{p-q}} \left(\sum_{i=k}^{N} \frac{a_{i,k}^q}{c_k^q} u_i^q \right)^{\frac{1}{p-q}} \times \right.$$
$$\left. \times \left(\frac{1}{c_k^{p'}} \sum_{i=1}^{k} c_i^{p'} v_i^{-p'} \right)^{\frac{q-1}{p-q}} \right)^{q-1},$$

[using now (5) again and also (7) with $\gamma = (p-1)/(p-q)$ and the definition (13) again]

$$\geqslant \sum_{j=1}^{N} (f_N)_j \frac{1}{c_j^{q-1}} \left(\sum_{i=j}^{N} a_{i,j}^q u_i^q\right)^{\frac{p-1}{p-q}} \left(\frac{1}{c_j^q}\right)^{\frac{q-1}{p-q}} \times$$

$$\times \left(\sum_{k=1}^{j} c_k^{p'} v_k^{-p'} \left(\sum_{i=1}^{k} c_i^{p'} v_i^{-p'}\right)^{\frac{q-1}{p-q}}\right)^{q-1}$$

$$\approx \sum_{j=1}^{N} (f_N)_j \left(\frac{1}{c_j}\right)^{\frac{p(q-1)}{p-q}} \left(\sum_{i=j}^{N} a_{i,j}^q u_i^q\right)^{\frac{p-1}{p-q}} \left(\sum_{k=1}^{j} c_k^{p'} v_k^{-p'}\right)^{\frac{(p-1)(q-1)}{p-q}} \tag{3.4}$$

$$= \sum_{j=1}^{N} (f_N)_j \left(\frac{1}{c_j^{p'}}\right)^{\frac{(p-1)(q-1)}{p-q}} \left(\sum_{i=j}^{N} a_{i,j}^q u_i^q\right)^{\frac{p-1}{p-q}} \left(\sum_{k=1}^{j} c_k^{p'} v_k^{-p'}\right)^{\frac{(p-1)(q-1)}{p-q}}$$

$$= \sum_{j=1}^{N} v_j^{-p'} \left(\sum_{i=j}^{N} a_{i,j}^q u_i^q\right)^{\frac{p}{p-q}} \left(\frac{1}{c_j^{p'}} \sum_{k=1}^{j} c_k^{p'} v_k^{-p'}\right)^{\frac{p(q-1)}{p-q}}.$$

From (1), (14) and (3.4) it follows that

$$C \gg \left(\sum_{j=1}^{N} v_j^{-p'} \left(\sum_{i=j}^{N} a_{i,j}^q u_i^q\right)^{\frac{p}{p-q}} \left(\frac{1}{c_j^{p'}} \sum_{k=1}^{j} c_k^{p'} v_k^{-p'}\right)^{\frac{p(q-1)}{p-q}}\right)^{\frac{p-q}{pq}}$$

with a constant independent of N and, hence, as $N \longrightarrow \infty$ we obtain that

$$C \gg B_1. \tag{16}$$

Next we note that the estimate (1) implies that (11) holds for the operator K (defined by (3)). We therefore assume that (11) holds and, for fixed $N \in \mathbb{Z}_+$, apply the following test sequence to (11): $g_N = \left\{(g_N)_j\right\}_{j=1}^{\infty}$, where

$$(g_N)_j := u_j^q \left(\sum_{k=1}^{j} a_{j,k}^{p'} v_k^{-p'}\right)^{\frac{(p-1)(q-1)}{p-q}} \left(\frac{1}{b_j^q} \sum_{i=j}^{N} b_i^q u_i^q\right)^{\frac{q-1}{p-q}}, \tag{17}$$

$j = 1, .., N,$ and $(g_N)_j = 0$ for $j > N.$

Applying (17) to the right hand side of (11), we have

$$\left\|u^{-1} g_N\right\|_{l_{q'}} = \left(\sum_{j=1}^{N} u_j^q \left(\sum_{k=1}^{j} a_{j,k}^{p'} v_k^{-p'}\right)^{\frac{q(p-1)}{p-q}} \left(\frac{1}{b_j^q} \sum_{i=j}^{N} b_i^q u_i^q\right)^{\frac{q}{p-q}}\right)^{\frac{1}{q'}}. \tag{18}$$

For the left hand side of (11) we first use (10) with p replaced by p' and find that

$$\begin{aligned}
\left\|v^{-1}Kg_N\right\|_{l_{p'}}^{p'} &= \sum_{j=1}^{\infty} v_j^{-p'}\left(\sum_{i=j}^{\infty} a_{i,j}\left(g_N\right)_i\right)^{p'}\\
&\approx \sum_{i=1}^{\infty}\left(g_N\right)_i \sum_{j=1}^{i} a_{i,j}v_j^{-p'}\left(\sum_{k=i}^{\infty} a_{k,j}\left(g_N\right)_k\right)^{p'-1}\\
&\geqslant \sum_{i=1}^{N}\left(g_N\right)_i \sum_{j=1}^{i} a_{i,j}v_j^{-p'}\left(\sum_{k=i}^{N} \frac{a_{k,j}}{b_k} b_k\left(g_N\right)_k\right)^{p'-1}
\end{aligned}$$

[using (5) twice, the definition (17) and finally (8) with $\gamma = (p-1)/(p-q)$]

$$\begin{aligned}
&\gg \sum_{i=1}^{N}\left(g_N\right)_i \frac{1}{b_i^{p'-1}} \sum_{j=1}^{i} a_{i,j}^{p'}v_j^{-p'}\left(\sum_{k=i}^{N} b_k\left(g_N\right)_k\right)^{p'-1}\\
&= \sum_{i=1}^{N}\left(g_N\right)_i \frac{1}{b_i^{p'-1}} \sum_{j=1}^{i} a_{i,j}^{p'}v_j^{-p'} \times\\
&\quad\times\left(\sum_{k=i}^{N} u_k^q b_k\left(b_k^{p'}\sum_{j=1}^{k}\frac{a_{k,j}^{p'}}{b_k^{p'}}v_j^{-p'}\right)^{\frac{(p-1)(q-1)}{p-q}}\left(\frac{1}{b_k^q}\sum_{j=k}^{N} b_j^q u_j^q\right)^{\frac{q-1}{p-q}}\right)^{p'-1}\\
&\gg \sum_{i=1}^{N}\left(g_N\right)_i\left(\frac{1}{b_i}\right)^{\frac{q}{p-q}}\left(\sum_{j=1}^{i} a_{i,j}^{p'}v_j^{-p'}\right)^{\frac{p-1}{p-q}}\left(\sum_{k=i}^{N} u_k^q b_k^q\left(\sum_{j=k}^{N} u_j^q b_j^q\right)^{\frac{q-1}{p-q}}\right)^{p'-1}\\
&\approx \sum_{i=1}^{N}\left(g_N\right)_i\left(\frac{1}{b_i}\right)^{\frac{q}{p-q}}\left(\sum_{j=1}^{i} a_{i,j}^{p'}v_j^{-p'}\right)^{\frac{p-1}{p-q}}\left(\sum_{k=i}^{N} u_k^q b_k^q\right)^{\frac{1}{p-q}}\\
&= \sum_{i=1}^{N} u_i^q\left(\sum_{j=1}^{i} a_{i,j}^{p'}v_j^{-p'}\right)^{\frac{q(p-1)}{p-q}}\left(\frac{1}{b_i^q}\sum_{k=i}^{N} u_k^q b_k^q\right)^{\frac{q}{p-q}}.
\end{aligned} \tag{3.8}$$

Combining (11), (18) and (3.8) we obtain that

$$C \gg \left(\sum_{i=1}^{N} u_i^q\left(\sum_{j=1}^{i} a_{i,j}^{p'}v_j^{-p'}\right)^{\frac{q(p-1)}{p-q}}\left(\frac{1}{b_i^q}\sum_{k=i}^{N} u_k^q b_k^q\right)^{\frac{q}{p-q}}\right)^{\frac{p-q}{pq}}$$

with a constant independent of N and, hence, as $N \longrightarrow \infty$ we find that

$$C \gg B_2. \tag{20}$$

Thus, in view of (16) and (20) and our assumption, we have that $B = \max\{B_1, B_2\} < \infty$ and, moreover,

$$C \gg B. \tag{21}$$

Sufficiency. We assume that $B = \max\{B_1, B_2\} < \infty$. Applying (9) with p replaced by q and (4) to the left hand side of (1) and using the Fubini theorem we have that

$$\begin{aligned}
\|uRf\|_{l_q}^q &= \sum_{i=1}^{\infty} u_i^q \left(\sum_{j=1}^{i} a_{i,j} f_j\right)^q \\
&\approx \sum_{j=1}^{\infty} f_j \sum_{i=j}^{\infty} a_{i,j} u_i^q \left(\sum_{k=1}^{j} a_{i,k} f_k\right)^{q-1} \\
&\approx \sum_{j=1}^{\infty} f_j \sum_{i=j}^{\infty} a_{i,j} u_i^q \left(\sum_{k=1}^{j} \left(\frac{a_{i,j}}{c_j} c_k + \frac{a_{j,k}}{b_j} b_i\right) f_k\right)^{q-1} \\
&= \sum_{j=1}^{\infty} f_j \sum_{i=j}^{\infty} a_{i,j} u_i^q \left(\frac{a_{i,j}}{c_j} \sum_{k=1}^{j} c_k f_k + \frac{b_i}{b_j} \sum_{k=1}^{j} a_{j,k} f_k\right)^{q-1}.
\end{aligned}$$

Using now the fact that $(\alpha+\beta)^\gamma \approx \alpha^\gamma + \beta^\gamma$, for all $\alpha \geqslant 0, \beta \geqslant 0, \gamma \geqslant 0$, we find that

$$\|uRf\|_{l_q}^q \approx I_1 + I_2, \tag{22}$$

where

$$I_1 := \sum_{j=1}^{\infty} f_j c_j \frac{1}{c_j^q} \sum_{i=j}^{\infty} a_{i,j}^q u_i^q \left(\sum_{k=1}^{j} c_k f_k\right)^{q-1}$$

and

$$I_2 := \sum_{j=1}^{\infty} f_j \sum_{i=j}^{\infty} a_{i,j} u_i^q b_i^{q-1} \left(\frac{1}{b_j} \sum_{k=1}^{j} a_{j,k} f_k\right)^{q-1}.$$

Estimate of I_1 : Let

$$\widetilde{a}_{i,j} := \max_{j \leqslant k \leqslant i} \frac{a_{i,k}}{c_k} \text{ and } \overline{a}_{i,j} := \max_{j \leqslant k \leqslant i} \frac{a_{k,j}}{b_k}, \ i \geqslant j \geqslant 1.$$

Then

$$\widetilde{a}_{i,j} \approx \frac{a_{i,j}}{c_j} \text{ and } \overline{a}_{i,j} \approx \frac{a_{i,j}}{b_i}, \ i \geqslant j \geqslant 1 \tag{23}$$

and

$$\widetilde{a}_{i,j} \leqslant \widetilde{a}_{i,k} \text{ and } \overline{a}_{i,k} \geqslant \overline{a}_{j,k}, \ i \geqslant j \geqslant k \geqslant 1. \tag{24}$$

We denote

$$\Delta^+ b_j = b_j - b_{j+1}, \quad \Delta^- b_j = b_j - b_{j-1} \tag{25}$$

and define

$$E_j := \sum_{i=1}^{j} f_i c_i \left(\sum_{k=1}^{i} c_k f_k\right)^{q-1}, \; j \geqslant 1,$$

$$D_j := \frac{1}{c_j^q}\sum_{i=j}^{\infty} a_{i,j}^q u_i^q \text{ and } \widetilde{D}_j := \sum_{i=j}^{\infty} \widetilde{a}_{i,j}^q u_i^q, \; j \geqslant 1.$$

Then, according to (23) and (24), we have $D_j \approx \widetilde{D}_j$ and $\triangle^+\widetilde{D}_j \geqslant 0$ for $i \geqslant 1$. By Abel's transformation and (7) with γ replaced by q, we obtain that

$$\begin{aligned}
I_1 &= \sum_{j=1}^{\infty} \frac{1}{c_j^q}\sum_{i=j}^{\infty} a_{i,j}^q u_i^q f_j c_j \left(\sum_{k=1}^{j} c_k f_k\right)^{q-1} \\
&= \sum_{j=1}^{\infty} D_j \triangle^- E_j \approx \sum_{j=1}^{\infty} \widetilde{D}_j \triangle^- E_j \\
&= \sum_{j=1}^{\infty} E_j \triangle^+ \widetilde{D}_j \\
&= \sum_{j=1}^{\infty} \left(\sum_{i=1}^{j} c_i f_i \left(\sum_{k=1}^{i} c_k f_k\right)^{q-1}\right) \triangle^+ \widetilde{D}_j \\
&\approx \sum_{j=1}^{\infty} \left(\sum_{i=1}^{j} c_i f_i\right)^{q} \triangle^+ \widetilde{D}_j.
\end{aligned}$$

Now letting $f_i \to c_i f_i, \; v_i \to \left(\frac{v_i}{c_i}\right)^p$ and applying the discrete Hardy inequality for the case $q<p$ (see [2] and also [6], Theorem 5 (ii)) we get that

$$I_1 \ll \left(\sum_{k=1}^{\infty} v_k^{-p'} c_k^{p'} \left(\sum_{j=k}^{\infty} \triangle^+\widetilde{D}_j\right)^{\frac{p}{p-q}} \left(\sum_{i=j}^{k} v_i^{-p'} c_i^{p'}\right)^{\frac{p(q-1)}{p-q}}\right)^{\frac{p-q}{p}} \left(\sum_{i=1}^{\infty} |v_i f_i|^p\right)^{\frac{q}{p}} \tag{3.15}$$
$$\approx B_1^q \, \|vf\|_p^q .$$

Estimate of I_2 : Here the ideas and methods of the proof of Theorem 2.15 in [4] are applied for the estimate of I_2. We define

$$\left(\overline{R}f\right)_j = \sum_{k=1}^{j} \overline{a}_{j,k} f_k. \tag{27}$$

Then by (23) and (24) we have $\left(\overline{R}f\right)_j \approx \frac{1}{b_j}\left(Rf\right)_j$ and $\triangle^-\left(\overline{R}f\right)_j \geqslant 0$ for $i \geqslant 1$.

For the estimate of I_2 we apply Hölder's inequality with exponents p and p' and find that

$$
\begin{aligned}
I_2 &= \sum_{j=1}^{\infty} f_j v_j v_j^{-1} \sum_{i=j}^{\infty} a_{i,j} u_i^q b_i^{q-1} \left(\frac{1}{b_j} \sum_{k=1}^{j} a_{j,k} f_k \right)^{q-1} \\
&\approx \sum_{j=1}^{\infty} f_j v_j v_j^{-1} \sum_{i=j}^{\infty} a_{i,j} u_i^q b_i^{q-1} \left(\left(\overline{R} f\right)_j \right)^{q-1} \\
&\leqslant \|vf\|_{l_p} \left(\sum_{j=1}^{\infty} v_j^{-p'} \left(\sum_{i=j}^{\infty} a_{i,j} b_i^{q-1} u_i^q \right)^{p'} \left(\left(\overline{R} f\right)_j \right)^{p'(q-1)} \right)^{\frac{1}{p'}}
\end{aligned}
$$

[using (25), (27), and Abel's transformation]

$$
\begin{aligned}
&= \|vf\|_{l_p} \left(\sum_{j=1}^{\infty} \left(\overline{R} f\right)_j^{p'(q-1)} \Delta^+ \left(\sum_{i=j}^{\infty} v_i^{-p'} \left(\sum_{k=i}^{\infty} a_{k,i} b_k^{q-1} u_k^q \right)^{p'} \right) \right)^{\frac{1}{p'}} \\
&= \|vf\|_{l_p} \left(\sum_{j=1}^{\infty} \left(\left(\sum_{i=j}^{\infty} v_i^{-p'} \left(\sum_{k=i}^{\infty} a_{k,i} b_k^{q-1} u_k^q \right)^{p'} \right)^{\frac{1}{p'}} \right)^{p'} \Delta^- \left(\overline{R} f\right)_j^{p'(q-1)} \right)^{\frac{1}{p'}}
\end{aligned}
$$

[now using Minkowski's inequality]

$$
\begin{aligned}
&\leqslant \|vf\|_{l_p} \left(\sum_{j=1}^{\infty} \left(\sum_{k=j}^{\infty} b_k^{q-1} u_k^q \left(\sum_{i=j}^{k} a_{k,i}^{p'} v_i^{-p'} \right)^{\frac{1}{p'}} \right)^{p'} \Delta^- \left(\overline{R} f\right)_j^{p'(q-1)} \right)^{\frac{1}{p'}} \\
&= \|vf\|_{l_p} \left(\sum_{j=1}^{\infty} \Phi_j^{p'} \Delta^- \left(\overline{R} f\right)_j^{p'(q-1)} \right)^{\frac{1}{p'}} \\
&:= \|vf\|_{l_p} I_{21}.
\end{aligned}
$$

Hence

$$
I_2 \leqslant \|vf\|_{l_p} I_{21}, \tag{28}
$$

where

$$
\Phi_j := \sum_{k=j}^{\infty} b_k^{q-1} u_k^q \left(\sum_{i=j}^{k} a_{k,i}^{p'} v_i^{-p'} \right)^{\frac{1}{p'}} \quad \text{and} \quad I_{21} := \left(\sum_{j=1}^{\infty} \Phi_j^{p'} \Delta^- \left(\overline{R} f\right)_j^{p'(q-1)} \right)^{\frac{1}{p'}}.
$$

To estimate Φ_j we apply Hölder's inequality with exponents s and s' (note that $1/s + 1/s' = 1$), where $p' < s < p/(p-q)$.

$$\Phi_j = \sum_{k=j}^{\infty} \left(\left(\sum_{i=k}^{\infty} b_i^q u_i^q \right)^{\frac{1}{p}} \left(\sum_{i=j}^{k} a_{k,i}^{p'} v_i^{-p'} \right)^{\frac{1}{p'}} b_k^{-1} (b_k u_k)^{\frac{q}{s}} \right) \left(\left(\sum_{i=k}^{\infty} b_i^q u_i^q \right)^{-\frac{1}{p}} (b_k u_k)^{\frac{q}{s'}} \right)$$

$$\leqslant \left(\sum_{k=j}^{\infty} \left(\sum_{i=k}^{\infty} b_i^q u_i^q \right)^{\frac{s}{p}} \left(\sum_{i=j}^{k} a_{k,i}^{p'} v_i^{-p'} \right)^{\frac{s}{p'}} b_k^{-s} (b_k u_k)^q \right)^{\frac{1}{s}} \left(\sum_{k=j}^{\infty} b_k^q u_k^q \left(\sum_{i=k}^{\infty} b_i^q u_i^q \right)^{-\frac{s'}{p}} \right)^{\frac{1}{s'}}$$

[now using (8)]

$$\approx \left(\sum_{i=j}^{\infty} b_i^q u_i^q \right)^{\frac{1}{s'} - \frac{1}{p}} \left(\sum_{k=j}^{\infty} \left(\sum_{i=k}^{\infty} b_i^q u_i^q \right)^{\frac{s}{p}} \left(\sum_{i=j}^{k} a_{k,i}^{p'} v_i^{-p'} \right)^{\frac{s}{p'}} b_k^{q-s} u_k^q \right)^{\frac{1}{s}}$$

$$= \left(\sum_{i=j}^{\infty} b_i^q u_i^q \right)^{\frac{1}{p'} - \frac{1}{s}} \widetilde{\Phi}_j^{\frac{1}{s}}.$$

Hence

$$\Phi_j \ll \left(\sum_{i=j}^{\infty} b_i^q u_i^q \right)^{\frac{1}{p'} - \frac{1}{s}} \widetilde{\Phi}_j^{\frac{1}{s}}, \tag{29}$$

where

$$\widetilde{\Phi}_j := \sum_{k=j}^{\infty} \left(\sum_{i=k}^{\infty} b_i^q u_i^q \right)^{\frac{s}{p}} \left(\sum_{i=j}^{k} a_{k,i}^{p'} v_i^{-p'} \right)^{\frac{s}{p'}} b_k^{q-s} u_k^q. \tag{30}$$

To estimate I_{21} we use (29), (25) and (12) and obtain that

$$I_{21} = \left(\sum_{j=1}^{\infty} \Phi_j^{p'} \Delta^- \left(\overline{R} f \right)_j^{p'(q-1)} \right)^{\frac{1}{p'}}$$

$$\ll \left(\sum_{j=1}^{\infty} \left(\sum_{i=j}^{\infty} b_i^q u_i^q \right)^{1-\frac{p'}{s}} \widetilde{\Phi}_j^{\frac{p'}{s}} \Delta^- \left(\overline{R} f \right)_j^{p'(q-1)} \right)^{\frac{1}{p'}}$$

$$\approx \left(\sum_{j=1}^{\infty} \left(\sum_{i=j}^{\infty} b_i^q u_i^q \right)^{1-\frac{p'}{s}} \widetilde{\Phi}_j^{\frac{p'}{s}} \left(\overline{R} f \right)_j^{p'(q-1)-1} \Delta^- \left(\overline{R} f \right)_j \right)^{\frac{1}{p'}}$$

$$= \left(\sum_{j=1}^{\infty} \left(\widetilde{\Phi}_j^{\frac{p'}{s}} \left(\overline{R} f \right)_j^{p'(q-1)\left(\frac{1}{p} + \frac{1}{s}\right) - 1} \left(\Delta^- \left(\overline{R} f \right)_j \right)^{\frac{p'}{s}} \right) \times \right.$$

$$\left. \times \left(\left(\sum_{i=j}^{\infty} b_i^q u_i^q \right)^{\frac{s-p'}{s}} \left(\overline{R} f \right)_j^{(q-1)\left(\frac{s-p'}{s}\right)} \left(\Delta^- \left(\overline{R} f \right)_j \right)^{\frac{s-p'}{s}} \right) \right)^{\frac{1}{p'}}$$

[now applying Hölder's inequality with exponents $\frac{s}{p'}$ and $\frac{s}{s-p'}$ and again (12)]

$$\leqslant \left(\sum_{j=1}^{\infty} \widetilde{\Phi}_j \left(\overline{R}f\right)_j^{s(q-1)\left(\frac{1}{p}+\frac{1}{s}\right)-\frac{s}{p'}} \left(\Delta^-\left(\overline{R}f\right)_j\right)\right)^{\frac{1}{s}} \times$$

$$\times \left(\sum_{j=1}^{\infty}\sum_{i=j}^{\infty} b_i^q u_i^q \left(\overline{R}f\right)_j^{(q-1)} \Delta^-\left(\overline{R}f\right)_j\right)^{\frac{s-p'}{sp'}}$$

$$\approx \left(\sum_{j=1}^{\infty} \widetilde{\Phi}_j \Delta^-\left(\overline{R}f\right)_j^{q-\frac{s(p-q)}{p}}\right)^{\frac{1}{s}} \left(\sum_{j=1}^{\infty}\sum_{i=j}^{\infty} b_i^q u_i^q \Delta^-\left(\overline{R}f\right)_j^q\right)^{\frac{s-p'}{sp'}}$$

[using again Abel's transformation and $\left(\overline{R}f\right)_j \approx \frac{1}{b_j}(Rf)_j$]

$$= \left(\sum_{j=1}^{\infty} \left(\overline{R}f\right)_j^{q-\frac{s(p-q)}{p}} \Delta^+\widetilde{\Phi}_j\right)^{\frac{1}{s}} \left(\sum_{i=1}^{\infty} b_i^q u_i^q \sum_{j=1}^{i} \Delta^-\left(\overline{R}f\right)_j^q\right)^{\frac{s-p'}{sp'}}$$

$$\approx \left(\sum_{j=1}^{\infty} \left(\overline{R}f\right)_j^{q-\frac{s(p-q)}{p}} \Delta^+\widetilde{\Phi}_j\right)^{\frac{1}{s}} \left(\sum_{i=1}^{\infty} u_i^q \left(\sum_{k=1}^{i} a_{i,k}f_k\right)^q\right)^{\frac{s-p'}{sp'}}$$

$$:= I_{22}\, \|uRf\|_{l_q}^{\frac{q\left(s-p'\right)}{sp'}}.$$

Hence

$$I_{21} \leqslant I_{22}\, \|uRf\|_{l_q}^{\frac{q\left(s-p'\right)}{sp'}}, \tag{31}$$

where

$$I_{22} := \left(\sum_{j=1}^{\infty} \left(\overline{R}f\right)_j^{q-\frac{s(p-q)}{p}} \Delta^+\widetilde{\Phi}_j\right)^{\frac{1}{s}}.$$

To estimate I_{22} we use (27), (30), (25), $\left(\overline{R}f\right)_j \approx \frac{1}{b_j}(Rf)_j$ and Hölder's inequality with exponents $\frac{pq}{s(p-q)}$ and $\frac{pq}{pq-s(p-q)}$.

$$I_{22} = \left(\sum_{j=1}^{\infty} \left(\overline{R}f\right)_j^{q-\frac{s(p-q)}{p}} \Delta^+\widetilde{\Phi}_j\right)^{\frac{1}{s}}$$

$$= \left(\sum_{j=1}^{\infty} \left(\overline{R}f\right)_j^{q-\frac{s(p-q)}{p}} \Delta^+ \left(\sum_{k=j}^{\infty}\left(\sum_{i=k}^{\infty} b_i^q u_i^q\right)^{\frac{s}{p}} \left(\sum_{i=j}^{k} a_{k,i}^{p'} v_i^{-p'}\right)^{\frac{s}{p'}} b_k^{q-s} u_k^q\right)\right)^{\frac{1}{s}}$$

$$\ll \left(\sum_{j=1}^{\infty} \left(\frac{1}{b_j}\sum_{k=1}^{j} a_{j,k}f_k\right)^{q-\frac{s(p-q)}{p}} \left(\sum_{i=j}^{\infty} b_i^q u_i^q\right)^{\frac{s}{p}} \left(\sum_{i=1}^{j} a_{j,i}^{p'} v_i^{-p'}\right)^{\frac{s}{p'}} b_j^{q-s} u_j^q\right)^{\frac{1}{s}}$$

$$= \left(\sum_{j=1}^{\infty} \left(u_j \sum_{k=1}^{j} a_{j,k} f_k \right)^{q - \frac{s(p-q)}{p}} \left(\frac{1}{b_j^q} \sum_{i=j}^{\infty} b_i^q u_i^q \right)^{\frac{s}{p}} \left(\sum_{i=1}^{j} a_{j,i}^{p'} v_i^{-p'} \right)^{\frac{s}{p'}} u_j^{\frac{s(p-q)}{p}} \right)^{\frac{1}{s}}$$

$$\leqslant \left(\sum_{j=1}^{\infty} u_j^q \left(\sum_{k=1}^{j} a_{j,k} f_k \right)^q \right)^{\frac{pq - s(p-q)}{spq}} \times$$

$$\times \left(\sum_{j=1}^{\infty} \left(\frac{1}{b_j^q} \sum_{i=j}^{\infty} b_i^q u_i^q \right)^{\frac{q}{p-q}} \left(\sum_{i=1}^{j} a_{j,i}^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} u_j^q \right)^{\frac{p-q}{pq}} .$$

Hence

$$I_{22} \leqslant B_2 \|uRf\|_{l_q}^{\frac{pq - s(p-q)}{sp}} . \tag{32}$$

Combining (28), (31) and (32) we have that

$$I_2 \ll B_2 \|vf\|_{l_p} \|uRf\|_{l_q}^{q-1} , \tag{33}$$

and from (22), (3.15) and (33) we obtain that

$$\|uRf\|_{l_q}^q \leqslant C_1^q B_1^q \|vf\|_{l_p}^q + C_2 B_2 \|vf\|_{l_p} \|uRf\|_{l_q}^{q-1} . \tag{34}$$

Applying now Young's inequality to the second term on the right hand side of (34), we get that

$$\begin{aligned} \|uRf\|_{l_q}^q &\leqslant C_1^q B_1^q \|vf\|_{l_p}^q + \frac{1}{q} C_2^q B_2^q \|vf\|_{l_p}^q + \frac{1}{q'} \|uRf\|_{l_q}^{(q-1)q'} \\ &= \left(C_1^q B_1^q + \frac{1}{q} C_2^q B_2^q \right) \|vf\|_{l_p}^q + \frac{1}{q'} \|uRf\|_{l_q}^q , \end{aligned}$$

that is,

$$\|uRf\|_{l_q}^q \leqslant \left(q C_1^q B_1^q + C_2^q B_2^q \right) \|vf\|_{l_p}^q$$

which implies

$$\|uRf\|_{l_q} \ll B \|vf\|_{l_p} .$$

Hence, (1) with $A = R$ holds with a constant $C \ll B < \infty$. By using also (21) we see that $C \approx B$ and the proof is complete. □

Proof of Theorem 2.2. The operator K is the conjugate of the operator R. Hence, according to Remark 2.2 the proof follows by using Theorem 2.1 with $1/u$, $1/v$, p' and q' replaced by v, u, q and p, respectively. By using these substitutions we find that B in Theorem 2.1 will be replaced by B^* and, moreover $C \approx B^*$ so the proof is complete. □

4. Further results

Our main aim in this Section is to point out the fact that our main results can be used to derive other inequalities of interest. Partly inspired by the inequality (6) we first consider an additive estimate of the form

$$\|uRf\|_{l_q} \leqslant C\left(\|vf\|_{l_p} + \|wP_cf\|_{l_p}\right),\ f \geqslant 0, \tag{35}$$

where $(P_cf)_j = \sum_{i=1}^{j} c_if_i$ and $u,\ v,\ w$ and c are non-negative weight sequences. From (35) we have (1) when $w = 0$ and

$$\|uRf\|_{l_q} \leqslant C\,\|wP_cf\|_{l_p},\ f \geqslant 0, \tag{36}$$

when $v = 0$.

In this Section we will derive necessary and sufficient conditions for (35) to hold under the assumption (4) and, in addition,

$$u > 0, v > 0 \text{ and } w \in l_p \text{ (that is } \sum_{i=1}^{\infty} w_i^p < \infty), \tag{37}$$

see Theorem 4.1. In particular, by using this result with $w = 0$ we get a version of Theorem 2.1 (the conditions are formally different but of course equivalent). We also note that this result can be used to obtain a characterization of (36) (because of the assumption $v > 0$). Moreover, in this Section we also include a characterization of (36) (see Theorem 4.2).

For $n \geqslant 1$ and $u,\ v$ and w satisfying (37) we define

$$\varphi_n := \left\{\min_{1\leqslant k\leqslant n}\left[\left(\sum_{i=k}^{n} c_i^{p'} v_i^{-p'}\right)^{-\frac{1}{p'}} + \left(\sum_{i=k}^{\infty} w_i^p\right)^{\frac{1}{p}}\right]\right\}^{-1} \tag{38}$$

and

$$U_n := \left(\sum_{i=n}^{\infty} b_i^q u_i^q\right)^{\frac{1}{q}}. \tag{39}$$

Then $\triangle^-\varphi_j \geqslant 0$ and $\triangle^+U_j \geqslant 0$ for $i \geqslant 1$, where $\triangle^-$ and $\triangle^+$ are defined by (25).

Let

$$G_1 := \left(\sum_{j=1}^{\infty}\left[\frac{1}{c_j}\left(\sum_{i=j}^{\infty} a_{i,j}^q u_i^q\right)^{\frac{1}{q}}\right]^{\frac{pq}{p-q}} \triangle^-\varphi_j^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}} \tag{40}$$

and

$$G_2 := \left(\sum_{j=1}^{\infty}\left[\frac{1}{b_j}\left(\sum_{i=1}^{j} \frac{a_{j,i}^{p'}}{c_i^{p'}}\triangle^-\varphi_i^{p'}\right)^{\frac{1}{p'}}\right]^{\frac{pq}{p-q}} \triangle^+U_j^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}}. \tag{41}$$

THEOREM 4.1. *Let* $1 < q < p < \infty$, *let* u, v *and* w *satisfy* (37) *and assume that the elements of the matrix* $(a_{i,j})$ *are non-negative and satisfy the condition* (4). *Then the estimate* (35) *holds if and only if* $G := \max\{G_1, G_2\} < \infty$, *where* G_1 *and* G_2 *are defined by* (38) – (41). *Moreover,* $C \approx G$, *where* C *is the best constant in* (35).

Proof. Let $c_i f_i = g_i$, $i \geqslant 1$ in (35). Then (35) is equivalent to

$$\left\|uRc^{-1}g\right\|_{l_q} \leqslant C\left(\left\|vc^{-1}g\right\|_{l_p} + \|wPg\|_{l_p}\right), \quad g \geqslant 0, \tag{42}$$

where $c^{-1}g := \left\{c_i^{-1}g_i\right\}_{i=1}^{\infty}$.

By (23) we have $\left(Rc^{-1}g\right)_i \approx (\widetilde{R}g)_i := \sum\limits_{j=1}^{i} \widetilde{a}_{i,j}g_j$ for $i \geqslant 1$ and the inequality

$$\left\|u\widetilde{R}g\right\|_{l_q} \leqslant C_1\left(\left\|vc^{-1}g\right\|_{l_p} + \|wPg\|_{l_p}\right), \quad g \geqslant 0, \tag{43}$$

is equivalent to (42). Moreover,

$$C \approx C_1, \tag{44}$$

where C and C_1 are the best constants in (42) and (43), respectively.

Since (24) holds, in view of Theorem 2 in [7] we obtain that the inequality

$$\left(\sum_{i=1}^{\infty} u_i\left(\sum_{j=1}^{i}\widetilde{a}_{i,j}g_j\right)^q\right)^{\frac{1}{q}} \leqslant C_2\left(\sum_{i=1}^{\infty}\left(g_i\left(\triangle^-\varphi_i^{p'}\right)^{-\frac{1}{p'}}\right)^p\right)^{\frac{1}{p}}, \; g \geqslant 0, \tag{45}$$

is equivalent to the inequality (43). Moreover,

$$C \approx C_2, \tag{46}$$

where C_2 is the best constant in (45).

By (23) the elements of matrix $(\widetilde{a}_{i,j})$ satisfy the condition (4) with $c_i = 1$, $i \geqslant 1$, i.e.

$$\widetilde{a}_{i,j} \approx \widetilde{a}_{i,k} + \frac{\widetilde{a}_{k,j}}{b_k}b_i, \quad i \geqslant k \geqslant j \geqslant 1. \tag{47}$$

Therefore, the matrix $(\widetilde{a}_{i,j})$ satisfies the conditions in Theorem 2.1 and, hence, by this Theorem the estimate (45) (and, thus, (43), (42) and (35)) holds if and only if

$$\widetilde{G}_1 := \left(\sum_{j=1}^{\infty}\left(\sum_{i=j}^{\infty}\widetilde{a}_{i,j}^{q}u_i^q\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{j}\triangle^-\varphi_i^{p'}\right)^{\frac{p(q-1)}{p-q}}\triangle^-\varphi_j^{p'}\right)^{\frac{p-q}{pq}} < \infty \tag{48}$$

and

$$\widetilde{G}_2 := \left(\sum_{j=1}^{\infty}u_j^q\left(\sum_{i=1}^{j}\widetilde{a}_{j,i}^{p'}\triangle^-\varphi_i^{p'}\right)^{\frac{q(p-1)}{p-q}}\left(\frac{1}{b_j^q}\sum_{i=j}^{\infty}b_i^q u_i^q\right)^{\frac{q}{p-q}}\right)^{\frac{p-q}{pq}} < \infty. \tag{49}$$

Moreover,

$$C_2 \approx \widetilde{G}, \tag{50}$$

where $\widetilde{G} := \max\left\{\widetilde{G}_1, \widetilde{G}_2\right\}$.

By letting $\varphi_0 = 0$ and using (25) and (12) with $\gamma = q(p-1)/(p-q)$ we get that

$$\sum_{i=1}^{j} \triangle^{-} \varphi_i^{p'} = \varphi_j^{p'}, \tag{51}$$

$$\left(\varphi_j^{p'}\right)^{\frac{p(q-1)}{p-q}} \triangle^{-} \varphi_j^{p'} \approx \triangle^{-} \varphi_j^{\frac{pq}{p-q}} \tag{52}$$

and

$$\begin{aligned} \left(\frac{1}{b_j^q}\sum_{i=j}^{\infty} b_i^q u_i^q\right)^{\frac{q}{p-q}} u_j^q &= \left(\frac{1}{b_j}\right)^{\frac{pq}{p-q}} \left(U_j^q\right)^{\frac{q}{p-q}} \triangle^{+} U_j^q \\ &\approx \left(\frac{1}{b_j}\right)^{\frac{pq}{p-q}} \triangle^{+} U_j^{\frac{pq}{p-q}}, \end{aligned} \tag{4.19}$$

for all $j \geqslant 1$.

From (23), (51), (52) and (4.19) it follows that $\widetilde{G}_1 \approx G_1$ and $\widetilde{G}_2 \approx G_2$, where $\widetilde{G}_1$ and $\widetilde{G}_2$ are defined by (48) and (49), respectively. Then, according to (44), (46) and (50), we have that $G \approx C$. The proof is complete. □

Now we consider the inequality (36). Let $W_k := \left(\sum_{i=k}^{\infty} w_i^p\right)^{-\frac{1}{p}}$, $k \geqslant 1$ and define

$$F_1 := \left(\sum_{j=1}^{\infty}\left(\frac{1}{c_j}\left(\sum_{i=j}^{\infty} a_{i,j}^q u_i^q\right)^{\frac{1}{q}}\right)^{\frac{pq}{p-q}} \triangle^{-} W_j^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}} \tag{54}$$

and

$$F_2 := \left(\sum_{j=1}^{\infty}\left(\frac{1}{b_j}\left(\sum_{i=1}^{j} \frac{a_{j,i}^{p'}}{c_i^{p'}} \triangle^{-} W_i^{p'}\right)^{\frac{1}{p'}}\right)^{\frac{pq}{p-q}} \triangle^{+} U_j^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}}. \tag{55}$$

THEOREM 4.2. *Let $1 < q < p < \infty$ and assume that the elements of the matrix $(a_{i,j})$ satisfy the condition (4). Then the estimate (36) with operator R holds if and only if $F := \max\{F_1, F_2\} < \infty$, where F_1 and F_2 are defined by (54) and (55), respectively. Moreover, $F \approx C$, where C is the best constant in (36).*

Proof. Let $\varepsilon > 0$. Setting $v_i = \varepsilon$, $\forall i \geqslant 1$ in (38), (40) and (41) we get, respectively:

$$\varphi_n(\varepsilon) := \left\{\min_{1\leqslant k\leqslant n}\left[\varepsilon\left(\sum_{i=k}^{n} c_i^{p'}\right)^{-\frac{1}{p'}} + \left(\sum_{i=k}^{\infty} w_i^p\right)^{\frac{1}{p}}\right]\right\}^{-1} \tag{56}$$

$$
= \max_{1\leqslant k\leqslant n} \frac{1}{\varepsilon \left(\sum_{i=k}^{n} c_i^{p'}\right)^{-\frac{1}{p'}} + W_k^{-1}}
= \max_{1\leqslant k\leqslant n} \frac{W_k}{\varepsilon W_k \left(\sum_{i=k}^{n} c_i^{p'}\right)^{-\frac{1}{p'}} + 1},
$$

$$
G_1(\varepsilon) := \left(\sum_{j=1}^{\infty} \left[\frac{1}{c_j}\left(\sum_{i=j}^{\infty} a_{i,j}^q u_i^q\right)^{\frac{1}{q}}\right]^{\frac{pq}{p-q}} \triangle^- \left[\varphi_j(\varepsilon)\right]^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}}
$$

and

$$
G_2(\varepsilon) := \left(\sum_{j=1}^{\infty} \left[\frac{1}{b_j}\left(\sum_{i=1}^{j} \frac{a_{j,i}^{p'}}{c_i^{p'}} \triangle^- \left[\varphi_i(\varepsilon)\right]^{p'}\right)^{\frac{1}{p'}}\right]^{\frac{pq}{p-q}} \triangle^+ U_j^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}}.
$$

Using (23) and Abel's transformation we obtain that

$$
\begin{aligned}
G_1(\varepsilon) &\approx \left(\sum_{j=1}^{\infty} \left[\left(\sum_{i=j}^{\infty} \widetilde{a}_{i,j}^q u_i^q\right)^{\frac{1}{q}}\right]^{\frac{pq}{p-q}} \triangle^- \left[\varphi_j(\varepsilon)\right]^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}} \\
&= \left(\sum_{j=1}^{\infty} \left[\varphi_j(\varepsilon)\right]^{\frac{pq}{p-q}} \triangle^+ \left[\left(\sum_{i=j}^{\infty} \widetilde{a}_{i,j}^q u_i^q\right)^{\frac{1}{q}}\right]^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}} := \widehat{G}_1(\varepsilon)
\end{aligned}
\tag{4.23}
$$

and

$$
\begin{aligned}
G_2(\varepsilon) &\approx \left(\sum_{j=1}^{\infty} \left[\frac{1}{b_j}\left(\sum_{i=1}^{j} \widetilde{a}_{j,i}^{p'} \triangle^- \left[\varphi_i(\varepsilon)\right]^{p'}\right)^{\frac{1}{p'}}\right]^{\frac{pq}{p-q}} \triangle^+ U_j^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}} \\
&= \left(\sum_{j=1}^{\infty} \left[\frac{1}{b_j}\left(\sum_{i=1}^{j} \left[\varphi_i(\varepsilon)\right]^{p'} \triangle_{(i)}^+ \left[\widetilde{a}_{j,i}^{p'}\right]\right)^{\frac{1}{p'}}\right]^{\frac{pq}{p-q}} \triangle^+ U_j^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}} := \widehat{G}_2(\varepsilon).
\end{aligned}
\tag{4.24}
$$

Now by using the monotonicity of the functions (56), (4.23) and (4.24) at $\varepsilon > 0$ and again Abel's transformation and (23) we have that

$$
\lim_{\varepsilon\to 0} \varphi_n(\varepsilon) = \sup_{\varepsilon>0} \varphi_n(\varepsilon) = \max_{1\leqslant k\leqslant n} \sup_{\varepsilon>0} \frac{W_k}{\varepsilon W_k \left(\sum_{i=k}^{n} c_i^{p'}\right)^{-\frac{1}{p'}} + 1} = \max_{1\leqslant k\leqslant n} W_k = W_n
$$

and

$$
\begin{aligned}
\sup_{\varepsilon>0}\widehat{G}_1(\varepsilon) &= \lim_{\varepsilon\to 0}\widehat{G}_1(\varepsilon)\\
&= \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\sum_{j=1}^{n}\left[\varphi_j(\varepsilon)\right]^{\frac{pq}{p-q}}\triangle^{+}\left[\left(\sum_{i=j}^{\infty}\widetilde{a}_{i,j}^{q}u_i^{q}\right)^{\frac{1}{q}}\right]^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}}\\
&= \lim_{n\to\infty}\lim_{\varepsilon\to 0}\left(\sum_{j=1}^{n}\left[\varphi_j(\varepsilon)\right]^{\frac{pq}{p-q}}\triangle^{+}\left[\left(\sum_{i=j}^{\infty}\widetilde{a}_{i,j}^{q}u_i^{q}\right)^{\frac{1}{q}}\right]^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}} \qquad (4.25)\\
&= \lim_{n\to\infty}\left(\sum_{j=1}^{n}W_j^{\frac{pq}{p-q}}\triangle^{+}\left[\left(\sum_{i=j}^{\infty}\widetilde{a}_{i,j}^{q}u_i^{q}\right)^{\frac{1}{q}}\right]^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}}\\
&= \left(\sum_{j=1}^{\infty}\left[\left(\sum_{i=j}^{\infty}\widetilde{a}_{i,j}^{q}u_i^{q}\right)^{\frac{1}{q}}\right]^{\frac{pq}{p-q}}\triangle^{-}\left[W_j\right]^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}} \approx F_1.
\end{aligned}
$$

Similarly, we find that

$$\sup_{\varepsilon>0}\widehat{G}_2(\varepsilon) \approx F_2. \qquad (60)$$

Using (4.23), (4.24), (4.25), (60) and Theorem 4.1, we obtain the best constant C in (36) as

$$
\begin{aligned}
C &= \sup_{f\geqslant 0}\frac{\|uRf\|_{l_q}}{\|wP_cf\|_{l_p}} = \sup_{f\geqslant 0}\sup_{\varepsilon>0}\frac{\|uRf\|_{l_q}}{\varepsilon\|f\|_{l_p}+\|wP_cf\|_{l_p}}\\
&= \sup_{\varepsilon>0}\sup_{f\geqslant 0}\frac{\|uRf\|_{l_q}}{\varepsilon\|f\|_{l_p}+\|wP_cf\|_{l_p}}\\
&\approx \sup_{\varepsilon>0}G(\varepsilon) = \sup_{\varepsilon>0}\max\{G_1(\varepsilon), G_2(\varepsilon)\}\\
&\approx \max\left\{\sup_{\varepsilon>0}\widehat{G}_1(\varepsilon),\ \sup_{\varepsilon>0}\widehat{G}_2(\varepsilon)\right\}\\
&\approx \max\{F_1, F_2\} = F.
\end{aligned}
$$

The proof is complete. □

Acknowledgements. The first author wants to thank the Royal Swedish Academy of Sciences for the grant, which made this cooperation possible. The second author acknowledges the Government of Ghana and ISP (International Science Programme, Sweden) for grant and support for a three-year visit at Luleå University of Technology, Sweden. Both these authors also thank Luleå University of Technology for giving them perfect conditions during their research stay in Sweden.

REFERENCES

[1] K. F. ANDERSEN, H. HEINIG, *Weighted norm inequalities for certain integral operators*, SIAM J. Math. **14** , (1983), 834–844.
[2] G. BENNETT, *Some elementary inequalities III*, Quart. J. Math. Oxford Ser., **42**, (2) (1991), 149–174.
[3] K. GOSWIN, G. ERDMANN, *The Blocking Technique, Weighted Mean Operators and Hardy's Inequality*, Springer-Verlag, 1998.
[4] A. KUFNER, L.-E. PERSSON, *Weighted Inequalities of Hardy type,* World Scientific Publishing Co, Singapore/New Jersey/London/Hong Kong, 2003.
[5] A. KUFNER, L. MALIGRANDA AND L.-E. PERSSON, *The prehistory of the Hardy inequality,* Amer. Math. Monthly, **113**, (8) (2006), 715–732.
[6] A. KUFNER, L. MALIGRANDA AND L.-E. PERSSON, *The Hardy inequality - About its history and some related results*, Vydavatelsky Servis Publishing House, Pilsen, 2007.
[7] R. OINAROV, *A dual inequality at the additive estimate of matrix operator* (Russian). Proc. Intern. conf. "Achievement and development of mathematics in framework of program- Kazakhstan in third millennium" (Almaty, October 26–28, 2000), Almaty, 2001, 111–115.
[8] R. OINAROV, S. KH. SHALGINBAYEVA, *Weighted additive estimate of one class of matrix operators,* Izvestiya NAN RK, serial Phys.-Mat., (1) (2004), 39–49.
[9] C. A. OKPOTI, *Weight characterizations of discrete Hardy and Carleman type inequalities,* Licentiate Thesis, Department of Mathematics, Luleå University of Technology, Sweden, (45) 2005 (81 pages).
[10] C. A. OKPOTI, L.-E. PERSSON AND A. WEDESTIG, *Weight characterizations of discrete Hardy inequality with kernel*, J. Inequal. Appl., **2006**, (2006), Article ID 18030.

(Received July 21, 2006)

Ryskul Oinarov
Faculty of Mathematics and Information Technologies
Eurasian National University
Munaitpasov st., 5
Astana 010008
Kazakhstan
e-mail: o_ryskul@mail.ru

Christopher A. Okpoti and Lars-Erik Persson
Department of Mathematics
Luleå University of Technology
SE-971 87 Luleå
Sweden
e-mail: cao@ltu.se
e-mail: larserik@sm.luth.se