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NEW DISTORTION THEOREMS FOR SAKAGUCHI FUNCTIONS

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Abstract. Let A be the class of functions $f(z)$ of the form $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$ that are analytic in the open unit disk $\mathbb{D}=\{z\in\mathbb{C}\mid |z|<1\}$. In 1959, K. Sakaguchi [9] has considered the subclass of A consisting of those $f(z)$ which satisfy $\operatorname{Re}\left(\frac{zf'(z)}{f(z)-f(-z)}\right)>0$, where $z\in\mathbb{D}$. We call such a function a "Sakaguchi function", and denote the class of those functions by S_s. Various authors have studied this class ([6, 7, 9, 10]). We obtain new distortion theorems, Koebe domain, k-quasiconformality, and the radius of convexity for the class S_s.

1. Introduction and definitions

Let A denote the class of all analytic functions defined on the open unit disk $\mathbb{D}=\{z\in\mathbb{C}\mid |z|<1\}$ and normalized by $f(0)=f'(0)-1=0$, and S^* be the class of starlike functions on $\mathbb{D}$. A function $f\in A$ is starlike with respect to symmetric points in $\mathbb{D}$, if for every $r<1$ close to 1 and every z_0 on $|z|=r$, the angular velocity of $f(z)$ about $f(-z_0)$ is positive at $z=z_0$ as z traverses the circle $|z|=r$ in the positive direction. This class was introduced and studied by K. Sakaguchi [9], who proved that the given defining condition is equivalent to

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)-f(-z)}\right)>0. \tag{1}$$

Let Ω be the family of functions $w(z)$ regular in $\mathbb{D}$ and satisfying the conditions $w(0)=0$ and $|w(z)|<1$ for $z\in\mathbb{D}$. Denote by P the family of Carathéodory functions

$$p(z)=1+p_1z+p_2z^2+\cdots \tag{2}$$

regular in $\mathbb{D}$ and such that $p(z)$ is in P if and only if

$$p(z)=\frac{1+w(z)}{1-w(z)}$$

for some $w(z)\in\Omega$ and every $z\in\mathbb{D}$.

The function $f(z)$ is *subordinate* to $F(z)$ in $\mathbb{D}$, denoted by $f\prec F$, if there exists an $w(z)\in\Omega$ so that $f(z)=F(w(z))$ for all $z\in\mathbb{D}$.

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For each $r > 0$, let $\mathbb{D}_r := \{z \in \mathbb{C} \mid |z| < r\}$ be the open disk of radius r. Then *the radius of convexity* of f, denoted by $r(f)$, is

$$r(f) := \sup\{r > 0 \mid f(\mathbb{D}_r) \text{ is convex}\}. \tag{3}$$

If $\mathcal{M}$ is a class of functions $f(z)$ regular in $\mathbb{D}$, then the *Koebe domain* for $\mathcal{M}$ is denoted by $K(\mathcal{M})$ and it is the collection of points w such that w is in $f(\mathbb{D})$ for every function $f(z)$ on $\mathbb{D}$, i.e.,

$$K(\mathcal{M}) = \bigcap_{f \in \mathcal{M}} f(\mathbb{D}).$$

Suppose that the set $\mathcal{M}$ is invariant under rotations, so that $e^{i\alpha}f(e^{-i\alpha}z)$ is in $\mathcal{M}$ whenever $f(z)$ is so. Then $K(\mathcal{M})$ will be either the single point $w = 0$ or an open disk $|w| < R$, in which case R is often easy to find. Indeed, suppose that we have a sharp lower bound $M(r)$ for $|f(re^{i\theta})|$ for all functions in $\mathcal{M}$, and that $\mathcal{M}$ consists of only univalent functions. Then

$$R = \lim_{r \to 1^-} M(r) \tag{4}$$

gives the disk $|w| < R$ as the Koebe domain for the set $\mathcal{M}$.

Suppose now that $f : \mathbb{D} \to \mathbb{C}$ is a sense-preserving homeomorphism (cf. [3]). For each $z \in \mathbb{D} \setminus \{\infty, f^{-1}(\infty)\}$, let

$$H(z) = \limsup_{r \to 0} \frac{L(z,r)}{l(z,r)},$$

where

$$L(z,r) = \max_{|z-w|=r} |f(z) - f(w)| \quad \text{and} \quad l(z,r) = \min_{|z-w|=r} |f(z) - f(w)|.$$

Then f is said *k-quasiconformal*, where $1 \leqslant k < \infty$, if $H(z)$ is bounded in $\mathbb{D} \setminus \{\infty, f^{-1}(\infty)\}$ and if $H(z) \leqslant k$ almost everywhere in $\mathbb{D}$ [3]. Such a number k is called the *k-quasiconformality* of f and is abbreviated by k-qc.

The following lemma, due to I. S. Jack [5], plays a crucial rôle in our investigation.

LEMMA 1.1. *Let $w(z)$ be regular in the open unit disk $\mathbb{D}$, with $w(0) = 0$. Then, if $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0, one has $z_0 w'(z_0) = k w(z_0)$ for some real $k \geqslant 1$.*

2. Main results

We will give now new distortion theorems, Koebe domain, k-qc, and the radius of convexity for the class S_s.

THEOREM 2.1. *If $f \in S_s$, then the odd starlike function*

$$F(z) = f(z) - f(-z) = 2\left(z + \sum_{k=1}^{\infty} a_{2k+1} z^{2k+1}\right) \tag{5}$$

satisfies

$$\left(\frac{zf'(z)}{f(z)-f(-z)}+\frac{zf'(-z)}{f(z)-f(-z)}-1\right)\prec\frac{2z^2}{1-z^2}, \tag{6}$$

and the result is sharp as the function $f(z)-f(-z)=\frac{2z}{1-z^2}$.

Proof. The function $\phi(z)=\frac{2z}{1-z}$ maps $|z|=r$ onto the disk centered at $c(r)=\frac{2r^2}{1-r^2}$ with radius $\rho(r)=\frac{2r}{1-r^2}$. Define the function $w(z)$ by

$$\frac{f(z)-f(-z)}{2z}=(1-w(z))^{-2}, \tag{7}$$

where $(1-w(z))^{-2}$ has the value 1 at the origin. Then $w(z)$ is analytic on $\mathbb{D}$, $w(0)=0$, and

$$z\frac{F'(z)}{F(z)}=\frac{zf'(z)}{f(z)-f(-z)}+\frac{zf'(-z)}{f(z)-f(-z)}-1=\frac{2zw'(z)}{1-w(z)}. \tag{8}$$

Now, the subordination (2.2) is equivalent to $|w(z)|<1$ for all $z\in\mathbb{D}$. Indeed, assume, on the contrary, that there exists a $z_0\in\mathbb{D}$, $\max_{|z|=|z_0|}$, such that $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at the point z_0, i.e., $|w(z_0)|=1$. Then, by Lemma 1.1, we have $z_0w'(z_0)=kw(z_0)$ for some $k\geqslant 1$, which implies that

$$\begin{aligned} z_0\frac{F'(z_0)}{F(z_0)}&=\frac{z_0f'(z_0)}{f(z_0)-f(-z_0)}+\frac{z_0f'(-z_0)}{f(z_0)-f(-z_0)}-1\\ &=\frac{2kw(z_0)}{1-w(z_0)}\\ &=k\phi(w(z_0))\notin\phi(\mathbb{D}) \end{aligned} \tag{2.5}$$

since $|w(z_0)|=1$ and $k\geqslant 1$, contradicting (2.2). Hence $|w(z)|<1$ for all $z\in\mathbb{D}$. On the other hand,

$$z\frac{F'(z)}{F(z)}=\frac{zf'(z)}{f(z)-f(-z)}+\frac{zf'(-z)}{f(z)-f(-z)}=\frac{1+w(z)}{1-w(z)}$$

implies that

$$z\frac{F'(z)}{F(z)}=\frac{zf'(z)}{f(z)-f(-z)}+\frac{zf'(-z)}{f(z)-f(-z)}-1\prec\frac{2z}{1-z},$$

which is required. The sharpness of the result follows from the fact that for $F(z)=f(z)-f(-z)=\frac{2z}{1-z^2}$, we get

$$z\frac{F'(z)}{F(z)}=\frac{zf'(z)}{f(z)-f(-z)}+\frac{zf'(-z)}{f(z)-f(-z)}-1=\frac{2z^2}{1-z^2}. \qquad \square$$

COROLLARY 2.2. *If* $f\in S_s$, *then*

$$\frac{r}{1+r^2}\leqslant|f(z)-f(-z)|\leqslant\frac{r}{1-r^2} \tag{10}$$

for $|z| = r$*, and this result is sharp since the extremal function is*

$$f_*(z) = \frac{z}{1-z^2}.$$

Proof. If $F(z)$ is an odd starlike function, then [8]

$$\frac{r}{1+r^2} \leqslant |F(z)| \leqslant \frac{r}{1-r^2}$$

for $|z| = r$, so, by Theorem 2.1, we obtain (2.6). □

COROLLARY 2.3. *k-qc of the class S_s is $\frac{1}{2}$.*

Poof. Using the inequality (2.6) for $|z| = r$, we get

$$\frac{1}{2(1+r^2)} \leqslant \left|\frac{f(z)-f(-z)}{z-(-z)}\right| \leqslant \frac{1}{2(1-r^2)},$$

and taking limit as $r \to 0$ gives

$$H(z) \leqslant \frac{1}{2}. \qquad \square$$

COROLLARY 2.4. *If $f \in S_s$, then*

$$\frac{1-r}{(1+r^2)(1+r)} \leqslant |f'(z)| \leqslant \frac{1}{(1-r)^2} \tag{11}$$

for $|z| = r$.

Proof. By the definition of Sakaguchi and Carathéodory functions, we have

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)-f(-z)}\right) > 0 \implies zf'(z) = (f(z)-f(-z))p(z) \tag{12}$$

for some $p(z) \in P$. On the other hand, the well-known Carathéodory's inequality [2]

$$\frac{1-r}{1+r} \leqslant |p(z)| \leqslant \frac{1+r}{1-r},$$

together with (2.8), yields (2.7) after simple calculations. □

COROLLARY 2.5. *If $f \in S_s$, then*

$$\log\frac{1+r}{\sqrt{1+r^2}} \leqslant |f(z)| \leqslant \frac{r}{1-r}.$$

Proof. Integrating (2.7) in Corollary 2.4 by [4, Theorem 7, p. 67], the result follows. □

COROLLARY 2.6. *The Koebe domain for the class S_s is the disk*

$$|w| < \log\sqrt{2},$$

where $w \in f(\mathbb{D})$ for $f \in S_s$.

Proof. Follows immediately from (1.4) and Corollary 2.5. □

THEOREM 2.7. *The radius of convexity of the class S_s is the only root r_s of the equation*

$$r^4 - 2r^3 - 2r^2 - 2r + 1 = 0$$

over the interval $0 < r < 1$.

Proof. Let $P^{(k)}$ denote the class of functions $p(z)$ having k-fold symmetry, i.e., functions of the form

$$p(z) = 1 + p_k z^k + p_{2k} z^{2k} + \cdots = 1 + \sum_{n=1}^{\infty} p_{nk} z^{nk}$$

for which $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$. Clearly, $P^{(1)} = P$ and

$$P^{(1)} \supset P^{(2)} \supset P^{(3)} \supset \cdots .$$

It is well-known that for $p(z) \in P^{(k)}$, one has

$$\left| p(z) - \frac{1 + r^{2k}}{1 - r^{2k}} \right| \leqslant \frac{2r^k}{1 - r^{2k}},$$

which, for $k = 2$, implies

$$\left| p(z) - \frac{1 + r^4}{1 - r^4} \right| \leqslant \frac{2r^2}{1 - r^4}. \tag{13}$$

On the other hand, taking logarithmic derivative in (2.8) gives

$$1 + z\frac{f''(z)}{f'(z)} = \frac{zf'(z)}{f(z) - f(-z)} + \frac{zf'(-z)}{f(z) - f(-z)} + z\frac{p'(z)}{p(z)}, \tag{14}$$

and combining Theorem 2.1 and (2.9), we get

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z) - f(-z)} + \frac{zf'(-z)}{f(z) - f(-z)} \right) \geqslant \frac{1 - r^2}{1 + r^2}. \tag{15}$$

Moreover, if $p(z) \in P^{(1)}$, one has [1]

$$\operatorname{Re}\left(z\frac{p'(z)}{p(z)} \right) \geqslant -\frac{2r}{1 - r^2}. \tag{16}$$

Considering now (2.10), (2.11) and (2.12), we get

$$\operatorname{Re}\left(1 + z\frac{f''(z)}{f'(z)} \right) \geqslant \frac{r^4 - 2r^3 - 2r^2 - 2r + 1}{1 - r^4}.$$

Since the polynomial $h(r) = r^4 - 2r^3 - 2r^2 - 2r + 1$ satisfies $h(0) = 1 > 0$ and $h(1) = -4 < 0$, it has a root r_s over $(0, 1)$, and a straightforward Mean Value

Theorem argument shows that this is the only root of it over this interval. Thus, the inequality

$$\operatorname{Re}\left(1+z\frac{f''(z)}{f'(z)}\right)>0$$

is valid for $|z|=r<r_s$, and taking into account (1.3), we conclude that the radius of convexity for the class S_s is r_s. □

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