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A RELATION BETWEEN TWO CLASSES OF INDEFINITE WEIGHTS IN SINGULAR ONE–DIMENSIONAL p–LAPLACIAN PROBLEMS

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Abstract. We introduce several types of classes of an indefinite weight h in singular one-dimensional p-Laplacian problems

$$\varphi_p(u'(t))' + h(t)f(u(t)) = 0,$$

where $\varphi_p(x) = |x|^{p-2}x, p > 1$ and $h \in C((0,1),[0,\infty))$ may be singular at 0 and/or 1 and $f \in C(\mathbb{R},\mathbb{R})$. We show a relation among them according to p employing Minkowski inequality and integral transformations.

1. Introduction

Recently, singular one-dimensional p-Laplacian problems have been extensively studied with various kinds of boundary conditions:

$$\varphi_p(u'(t))' + h(t)f(u(t)) = 0, \tag{1.1}$$

where $\varphi_p(x) = |x|^{p-2}x, p > 1$ and $h \in C((0,1),[0,\infty))$ may be singular at 0 and/or 1 and $f \in C(\mathbb{R},\mathbb{R})$. We call such a function h as an indefinite weight. Radial problems for partial differential equations on exterior domains can be converted to problem (1.1).

The purpose of this paper is to clarify a relation among several types of classes of indefinite weights h. As an immediate consequence, using the relation of these classes, we obtain the existence results of positive solutions (see Corollary 2.3).

Let us list classes of indefinite weights h. The simplest case of h is of class $L^1(0,1)$. Thus our first class of indefinite weights is

$$\mathscr{A} \triangleq \left\{ h \in C((0,1),[0,\infty)) : \int_0^1 h(s)ds < \infty \right\}.$$

In [3, 6, 7, 9, 10, 12, 13], they proved the existence results of eigenvalues and positive and sign-changing solutions.

The next class of h

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$$\mathscr{B} \triangleq \left\{ h \in C((0,1),[0,\infty)) : \int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} h(\tau)d\tau\right) ds + \int_{\frac{1}{2}}^1 \varphi_p^{-1}\left(\int_{\frac{1}{2}}^s h(\tau)d\tau\right) ds < \infty \right\}$$

comes from the study of the existence of positive solutions for p-Laplacian problems. It is obvious that $\mathscr{A} \subset \mathscr{B}$. Wang [11] initiated and Kong-Wang [5] and recently, Agarwal-Lü-O'Regan [1] studied the existence of positive solutions for problem (1.1) subject to Dirichlet boundary condition (for short, (D)). Meanwhile, there has been few results about eigenvalues and sign-changing solutions of (1.1) $+$(D) under this condition.

In semilinear problems, (that is, $p = 2$ in (1.1)), $\mathscr{B}$ becomes the following type of class

$$\mathscr{C}' \triangleq \left\{ h \in C((0,1),[0,\infty)) : \int_0^1 s(1-s)h(s)ds < \infty \right\}.$$

Here Green's function plays a key role to get an operator and there are vast amount of literatures [2, 4] and references therein. It is very natural to extend class $\mathscr{C}'$ to the following class for p-Laplacian problem

$$\mathscr{C} \triangleq \left\{ h \in C((0,1),[0,\infty)) : \int_0^1 s^{p-1}(1-s)^{p-1}h(s)ds < \infty \right\}.$$

The advantage of this condition allows us to use a variational method.

Let us recall that there was a class like a bridge to go from $\mathscr{A}$ to $\mathscr{B}$ or $\mathscr{C}$ which is

$$\mathscr{D} \triangleq \left\{ h \in C((0,1),[0,\infty)) : \text{there are } \alpha,\beta > 0 \text{ such that } \alpha,\beta < p-1 \text{ and } \int_0^1 s^{\alpha}(1-s)^{\beta}h(s)ds < \infty \right\}.$$

It is obvious to see that $\mathscr{D} \subset \mathscr{C}$, for all $p > 1$. Also, we can prove easily $\mathscr{D} \subset \mathscr{B}$. In fact, (without loss of generality, we assume that h has singularity at 0)

$$\begin{aligned} \int_0^1 \varphi_p^{-1}\left(\int_s^1 h(\tau)d\tau\right) ds &\leqslant \int_0^1 \varphi_p^{-1}\left(\int_s^1 \frac{\tau^{\alpha}}{s^{\alpha}} h(\tau)d\tau\right) ds \\ &\leqslant \varphi_p^{-1}\left(\int_0^1 \tau^{\alpha} h(\tau)d\tau\right) \int_0^1 \varphi_p^{-1}(s^{-\alpha})ds < \infty. \end{aligned}$$

However, so far, it is not known about a relation between $\mathscr{B}$ and $\mathscr{C}$, for any $p > 1$. Of course, it is well-known and easy to see that when $p = 2$, $\mathscr{B}$ and $\mathscr{C}$ are equivalent. As mentioned, the goal of this paper is to give a relation between $\mathscr{B}$ and $\mathscr{C}$, for any $p > 1$. The authors believe that this is the first results about it.

2. Main results and counterexamples

In this section, we state the main theorems to show inclusions among classes. We give the proof with counterexamples to show that inclusions are proper. The first result is as follow and the proof is obvious.

THEOREM 2.1. *For any* $p > 1$, *if the classes* $\mathscr{B}, \mathscr{C}$ *and* $\mathscr{D}$ *are restricted to the following form* $h(t) = t^{-\gamma}(1-t)^{-\delta}$, *for some* $1 < \gamma, \delta < p$, *then we have*

$$\mathscr{B} = \mathscr{C} = \mathscr{D}.$$

The main result of this paper can be described as this theorems which is an inclusion between $\mathscr{B}$ and $\mathscr{C}$ according to p. We shall employ Minkowski inequality and integral transformations.

THEOREM 2.2. *For* $1 < p < 2$, *we have* $\mathscr{C} \subsetneq \mathscr{B}$ *and for* $p > 2$, *we have* $\mathscr{B} \subsetneq \mathscr{C}$.

Proof. Without loss of generality, we may assume that h is in $C(0,1]$. The above two classes is equivalent the following classes:

$$\mathscr{B} \triangleq \left\{ h \in C((0,1],[0,\infty)) : \int_0^1 \varphi_p^{-1} \left(\int_s^1 h(\tau) d\tau \right) ds < \infty \right\},$$

$$\mathscr{C} \triangleq \left\{ h \in C((0,1],[0,\infty)) : \int_0^1 s^{p-1} h(s) ds < \infty \right\}.$$

Recall that the following integral version of Minkowski inequality for $1 \leqslant q < \infty$.

$$\left[\int \left| \int f(t,s) dt \right|^q ds \right]^{1/q} \leqslant \int \left[\int |f(t,s)|^q ds \right]^{1/q} dt.$$

We show the inclusion for the case $1 < p < 2$. We will apply Minkowski inequality by $f(t,s) = \chi_{[s,1]}(t) h(t)$ and $q = \frac{1}{p-1} > 1$, where χ is the characteristic function for $[s,1]$. The left integral is

$$\left[\int \left| \int f(t,s) dt \right|^q ds \right]^{1/q} = \left[\int_0^1 \left| \int_s^1 h(t) dt \right|^{\frac{1}{p-1}} ds \right]^{p-1}.$$

The right integral is

$$\int \left[\int |f(t,s)|^q ds \right]^{1/q} dt = \int_0^1 \left[\int_0^1 \left| \chi_{[s,1]}(t) h(t) \right|^q ds \right]^{1/q} dt.$$

Since $\chi_{[s,1]}(t) = \chi_{[0,t]}(s)$,

$$\int_0^1 \left[\int_0^1 \left| \chi_{[s,1]}(t) h(t) \right|^q ds \right]^{1/q} dt = \int_0^1 \left[\int_0^t |h(t)|^q ds \right]^{1/q} dt = \int_0^1 t^{p-1} h(t) dt.$$

Therefore, the inclusion is done for $1 < p < 2$.

Next, we show the inclusion for case $p > 2$. We will apply the Minkowski inequality again. Let $q = p-1$ and $f(t,s) = \chi_{[0,s]}(t) h(s)^{1/q}$. Then the left side of

Minkowski inequality is

$$\begin{aligned}\left[\int\left|\int f(t,s)dt\right|^q ds\right]^{1/q} &= \left[\int_0^1\left|\int_0^1 \chi_{[0,s]}(t)h(s)^{1/q}dt\right|^q ds\right]^{1/q}\\ &= \left[\int_0^1\left|\int_0^s h(s)^{1/q}dt\right|^q ds\right]^{1/q}\\ &= \left[\int_0^1 s^q h(s)ds\right]^{1/q}.\end{aligned}$$

The right side of Minkowski inequality is

$$\begin{aligned}\int\left[\int |f(t,s)|^q ds\right]^{1/q} dt &= \int_0^1\left[\int_0^1 \left|\chi_{[0,s]}(t)h(s)^{1/q}\right|^q ds\right]^{1/q} dt\\ &= \int_0^1\left[\int_t^1 h(s)ds\right]^{1/q} dt.\end{aligned}$$

The last equality comes from $\chi_{[0,s]}(t) = \chi_{[t,1]}(s)$. This shows the inclusion.

To complete the proof, we find a function $h_1 \notin \mathscr{C}$ and $h_1 \in \mathscr{B}$, for $1 < p < 2$ and a function $h_2 \notin \mathscr{B}$ and $h_1 \in \mathscr{C}$, for $p > 2$. In order to construct examples, we change these integrals to different form. First, we define two quantities,

$$\|h\|_{\mathscr{B}} = \int_0^1 \left(\int_s^1 h(\tau)d\tau\right)^{\frac{1}{p-1}} ds$$

and

$$\|h\|_{\mathscr{C}} = \int_0^1 s^{p-1}h(s)ds.$$

We define a function v by

$$v(s) = \int_s^1 h(t)dt.$$

We assume that h is positive. So v is a C^1 strictly decreasing function on $(0,1]$ with $v(1) = 0$. Since v is a 1-1 correspondence, there is an inverse function w from $[0,\infty)$ to $(0,1]$. The function w is of C^1 with $w(0) = 1$. Also w is a decreasing function.

We will change the above quantities using this function w. First, let us put $t = w(s)$, then we have

$$\frac{dw(s)}{ds} = \frac{1}{v'(w(s))} = \frac{1}{-h(w(s))}.$$

Thus, by change of variables, we obtain that

$$\begin{aligned}\|h\|_{\mathscr{C}} &= \int_0^1 t^{p-1}h(t)dt = \int_\infty^0 w(s)^{p-1}h(w(s))dw(s)\\ &= \int_0^\infty w(s)^{p-1}ds.\end{aligned}$$

Second, we will compute explicitly the inverse a function of $v(s)^{\frac{1}{p-1}}$. In fact, if $y = v(s)^{\frac{1}{p-1}}$, then $w(y^{p-1}) = s$. Hence

$$\begin{aligned} \|h\|_{\mathscr{B}} &= \int_0^1 v(s)^{\frac{1}{p-1}} ds = \int_0^\infty w(y^{p-1}) dy \\ &= \int_0^\infty w(s^{p-1}) ds. \end{aligned}$$

For $1 < p < 2$, let $q = p-1$. Define a function w_1 by

$$w_1(s) = \frac{1}{(s \log s)^{1/q}},$$

for $s \geqslant 2$. We can patch w_1 on the interval $[0,2]$ satisfying w_1 is decreasing and $w_1(0) = 1$. By the same calculation, we can obtain h_1 by taking derivative of w_1. Therefore,

$$\|h_1\|_{\mathscr{B}} \leqslant C \int_2^\infty \frac{1}{s(\log s)^{1/q}} < \infty.$$

But

$$\|h_1\|_{\mathscr{C}} \geqslant C \int_2^\infty \frac{1}{s \log s} ds = \infty.$$

Let $p > 2$ and $q = p-1$. Define a function w_2 by

$$w_2(s) = \frac{1}{s^{1/q} \log s},$$

for $s \geqslant 2$. For $0 \leqslant s \leqslant 2$, we can define a function $w_2(s)$ which is of C^1 and decreasing using patching technique. Note that we can get h_2 by the same argument. Then

$$\|h_2\|_{\mathscr{B}} = \int_0^\infty w_2(s^q) ds$$

and there exists a constant C such that

$$\|h_2\|_{\mathscr{B}} \geqslant C \int_2^\infty \frac{1}{qs \log s} ds.$$

This is infinite. But

$$\|h_2\|_{\mathscr{C}} = \int_0^\infty w_2(s)^q ds \leqslant C \int_2^\infty \frac{1}{s(\log s)^q} ds < \infty.$$

The above two examples complete the proof. □

Lü-O'Regan (Example 1, [8]) showed the existence of positive solutions for the following problem using a upper and lower method under condition $\mathscr{B}$, for all $p > 1$. Thus, we have the following corollary.

COROLLARY 2.3. *Assume $h \in \mathscr{C}$. For $1 < p < 2$, let us consider the eigenvalue problem*

$$\begin{cases} \varphi_p(u'(t))' + \lambda f(t, u(t)) = 0, & t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \qquad (QP_\lambda)$$

where $f:(0,1)\times[0,\infty)\to[0,\infty)$ *is continuous and for each given* $\eta>0$ *there is a positive constant* M_η *satisfying*

$$|f(t,u)|\leqslant M_\eta h(t), \forall(t,u)\in(0,1)\times[0,\eta].$$

Then, there exits a positive constant λ^* *such that* (QP_λ) *has at least a positive solution for* $0<\lambda<\lambda^*$.

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