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ON n–TH JAMES AND KHINTCHINE CONSTANTS OF BANACH SPACES

L. MALIGRANDA, L. I. NIKOLOVA, L.-E. PERSSON AND T. ZACHARIADES

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Abstract. For any Banach space X the n-th James constants $J_n(X)$ and the n-th Khintchine constants $K^n_{p,q}(X)$ are investigated and discussed. Some new properties of these constants are presented. The main result is an estimate of the n-th Khintchine constants $K^n_{p,q}(X)$ by the n-th James constants $J_n(X)$. In the case of $n = 2$ and $p = q = 2$ this estimate is even stronger and improvs an earlier estimate proved by Kato-Maligranda-Takahashi [25].

Introduction

Several constants of a Banach space $X = (X, \|.\|)$ are used in the description of its geometric properties. The James constant $J(X)$, the Jordan-von Neumann constant $C_{NJ}(X)$, the n-th James constants $J_n(X)$ and the n-th Khintchine constants $K^n_{p,q}(X)$ are examples of such constants. We will derive some properties of these constants and also investigate the relations among them.

Our main results are about estimates of the Jordan-von Neumann constant by the James constant and also the n-th Khintchine constants by the n-th James constants.

In Section 1 we collect and discuss some properties of the constants $J(X)$ and $C_{NJ}(X)$. Moreover, we prove a new estimate which improves the Kato-Maligranda-Takahashi [25] estimate (see Theorem 1). In Section 2 we consider the n-th James constants $J_n(X)$ and collect their properties as the measure of B-convexity of a Banach space X. We calculate them for L^p spaces by using Clarkson's inequalities (see Theorem 2). The n-th strong James constants $J^s_n(X)$ are also considered and a non-trivial estimate of $J_n(X)$ by $J^s_n(X)$ is proved (see Theorem 3). Our conjecture is that $J^s_n(X) < J_n(X)$ for $n \geqslant 3$.

In Section 3 the n-th Khintchine constants $K^n_{p,q}(X)$ are considered with their properties. We also calculate or estimate these constants for some classes of Banach spaces.

In Section 4 the main result on an estimate of the n-th Khintchine constants $K^n_{p,q}(X)$ by the n-th James constants $J_n(X)$ is presented and proved (see Theorem 4). Finally, several facts concerning the constants $J_n(X)$ and their close relation to the notion of

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type of an infinite dimensional Banach space X are pointed out (see e. g. Proposition 4). In particular, we prove the following formula: $p(X) = \sup\{\frac{\ln n}{\ln J_n(X)}, n \geqslant 2\}$, where $p(X) = \sup\{p : X \text{ is of type } p\}$ (see Theorem 5).

Finally, in the last Section 5 we investigate the relation between the n-th James and the n-th Khintchine constants for isomorphic Banach spaces (see Theorem 6).

Throughout this paper we assume that $X = (X, \|.\|)$ is a real Banach space with $\dim X \geqslant 2$. B_X will denote the closed unit ball $\{x \in X : \|x\| \leqslant 1\}$ of X and $S_X = \{x \in X : \|x\| = 1\}$ is its unit sphere.

1. An estimate of the Jordan-von Neumann constant by the James constant

The *James* non-square *constant* of a Banach space X is the number $J(X)$ defined by

$$J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : x, y \in B_X\},$$

and the *Jordan-von Neumann constant* $C_{NJ}(X)$ of X is defined by

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X \text{ not both zero}\right\}.$$

These constants have been studied by several authors (see e.g. Casini [4], Gao-Lau [11], [12], Kato-Maligranda-Takahashi [25] and Kato-Maligranda [24]). Let us collect some properties of these constants:

(*i*) $J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : x, y \in S_X\}$.

(*ii*) $\sqrt{2} \leqslant J(X) \leqslant 2$ and $1 \leqslant C_{NJ}(X) \leqslant 2$.

(*iii*) X is Hilbert space $\Longrightarrow J(X) = \sqrt{2}$ and the converse is not true; $C_{NJ}(X) = 1 \Longleftrightarrow X$ is a Hilbert space.

(*iv*) $J(X) < 2 \Longleftrightarrow C_{NJ}(X) < 2 \Longleftrightarrow$ the space X is uniformly non-square, i.e., there exists a $\delta \in (0,1)$ such that for any $x, y \in S_X$ either $\|x+y\|/2 \leqslant 1-\delta$ or $\|x-y\|/2 \leqslant 1-\delta$.

(*v*) $J(X^{**}) = J(X)$, $\max\{\sqrt{2}, 2J(X)-2\} \leqslant J(X^*) \leqslant J(X)/2+1$, and there exists a two-dimensional Banach space X such that $J(X^*) \neq J(X)$, where X^* and X^{**} are dual and the second dual of X; $C_{NJ}(X^*) = C_{NJ}(X)$.

(*vi*) If $1 \leqslant p \leqslant \infty$ and $\dim L^p(\mu) \geqslant 2$, then $J(L^p(\mu)) = \max\{2^{1/p}, 2^{1-1/p}\}$ and $C_{NJ}(L^p(\mu)) = \max\{2^{2/p-1}, 2^{1-2/p}\} = 2^{|1-2/p|}$.

Kato-Maligranda-Takahashi [25] proved that

$$J(X)^2/2 \leqslant C_{NJ}(X) \leqslant \frac{J(X)^2}{(J(X)-1)^2+1}. \tag{1}$$

Moreover, if X is not uniformly non-square, then we have equalities in (1) and there exists a two-dimensional Banach space X for which $J(X)^2/2 < C_{NJ}(X)$.

To improve the second inequality in (1) we will need the following lemma.

LEMMA 1. *Let X be a normed space and $x, y \in X$ with $\|x\| \leqslant \|y\|$. Then*

$$\min(\|x+y\|, \|x-y\|) \leqslant \|y\| + [J(X)-1]\,\|x\|. \tag{2}$$

Proof. Let $x \neq 0$ (otherwise we have even equality). Then

$$\begin{aligned} \|x+y\| &= \Big\|\frac{\|x\|}{\|x\|}x + \frac{\|x\|}{\|y\|}y + \left(1-\frac{\|x\|}{\|y\|}\right)y\Big\| \\ &\leqslant \|x\|\Big\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\Big\| + \left(1-\frac{\|x\|}{\|y\|}\right)\|y\| \\ &= \|x\|\Big\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\Big\| + \|y\| - \|x\| \\ &= \|y\| + \left(\Big\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\Big\| - 1\right)\|x\|, \end{aligned}$$

and, similarly, $\|x-y\| \leqslant \|y\| + \left(\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\| - 1\right)\|x\|$, which gives

$$\begin{aligned} \min(\|x+y\|, \|x-y\|) &\leqslant \|y\| + \left[\min\left(\Big\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\Big\|, \Big\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\Big\|\right) - 1\right]\|x\| \\ &\leqslant \|y\| + [J(X)-1]\,\|x\|. \end{aligned}$$

Our improvement of (1) reads:

THEOREM 1. *For any Banach space X we have*

$$\begin{aligned} C_{NJ}(X) &\leqslant \frac{J(X)^2}{4} + 1 + \frac{J(X)}{4}\left(\sqrt{J(X)^2 - 4J(X) + 8} - 2\right) \\ &\leqslant \frac{J(X)^2}{2} + 2 - J(X) \leqslant \frac{J(X)^2}{(J(X)-1)^2 + 1.} \end{aligned} \tag{3}$$

Proof. If $J(X) = 2$, then we have equalities in (3). Therefore, we assume that $J(X) < 2$. To prove the first inequality in (3), let $x, y \in X$, not both zero and $\|x\| \leqslant \|y\|$. Then for $x' = \frac{x}{\|x\|+\|y\|}$, $y' = \frac{y}{\|x\|+\|y\|}$, we have that

$$A := \frac{\|x+y\|^2 + \|x-y\|^2}{2\left(\|x\|^2+\|y\|^2\right)} = \frac{\|x'+y'\|^2 + \|x'-y'\|^2}{2\left(\|x'\|^2+\|y'\|^2\right)}.$$

Since $\|x' \pm y'\| = \frac{\|x\pm y\|}{\|x\|+\|y\|} \leqslant 1$ it follows that

$$A \leqslant \frac{1+\|x'-y'\|^2}{2\left(\|x'\|^2+\|y'\|^2\right)} \quad \text{and} \quad A \leqslant \frac{\|x'+y'\|^2+1}{2\left(\|x'\|^2+\|y'\|^2\right)}.$$

Thus,

$$A \leqslant \frac{1+\min\left(\|x'+y'\|, \|x'-y'\|\right)^2}{2\left(\|x'\|^2+\|y'\|^2\right)}$$

and, hence, by Lemma 1,

$$A \leqslant \frac{1 + [\,\|y'\| + (J(X) - 1)\,\|x'\|\,]^2}{2\,(\|x'\|^2 + \|y'\|^2)}.$$

Note that $\|x'\| + \|y'\| = 1$ and $\|x'\| = \frac{\|x\|}{\|x\|+\|y\|} \leqslant \frac{1}{2}$, which gives that

$$A \leqslant \frac{1 + [\,1 + (J(X) - 2)\,\|x'\|\,]^2}{2\left[\|x'\|^2 + (1 - \|x'\|)^2\right]}.$$

Now, consider the function

$$f\,(u) = \frac{1 + [1 + (J-2)\,u]^2}{u^2 + (1-u)^2} \quad \text{for} \quad u \in [0, \frac{1}{2}] \quad \text{with} \quad J < 2.$$

Note that the derivative

$$f'(u) = \frac{2J\left[(2-J)u^2 - (4-J)u + 1\right]}{[u^2 + (1-u)^2]^2}$$

is zero at $u_1 = \frac{4-J-\sqrt{J^2-4J+8}}{2(2-J)} \in (0, \frac{1}{2})$ and $u_2 = \frac{4-J+\sqrt{J^2-4J+8}}{2(2-J)} > 1$. Therefore, for $u \in [0, \frac{1}{2}]$, we have that

$$f\,(u) \leqslant f\,(u_1) = \frac{J^2}{2} + 2 + \frac{J}{2}\sqrt{J^2 - 4J + 8} - J.$$

The second estimate in (3) follows easily from the following equivalences:

$$\begin{aligned}
\frac{J^2}{4} + 1 + \frac{J}{4}\left(\sqrt{J^2 - 4J + 8} - 2\right) \quad &\leqslant \quad \frac{J^2}{2} + 2 - J \\
&\iff J\sqrt{J^2 - 4J + 8} \leqslant J^2 - 2J + 4 \\
&\iff J^2(J^2 - 4J + 8) \leqslant (J^2 - 4J + 4)^2 \\
&\iff 16J \leqslant 4J^2 + 16 \\
&\iff 0 \leqslant 4(J-2)^2.
\end{aligned}$$

To prove the third estimate in (3) consider the function

$$g(t) = \frac{t^2}{2} + 2 - t - \frac{t^2}{(t-1)^2 + 1} \quad \text{for} \quad t \in [\sqrt{2}, 2]$$

and observe that the derivative $g'(t) = t - 1 + 2\frac{(t-1)t}{[(t-1)^2+1]^2} > 0$ and, thus, g is increasing with $g(t) \leqslant g(2) = 0$. The proof of (3) is complete.

The estimates (3) were proved independently in 2003 by Maligranda [31, Theorem 1] and Nikolova-Persson-Zachariades [37, p. 8] (see also [38]) as an improvement of the upper Kato-Maligranda-Takahashi estimate in (1). Maligranda even formulated the following conjecture (cf. [31] and [32]):

$$\textit{The estimate } C_{NJ}(X) \leqslant \frac{J(X)^2}{4} + 1 \textit{ holds for any Banach space } X. \tag{4}$$

Recently Saejung [42] published a paper with a contribution to the proof of this Maligranda conjecture but his "proof" contains only a proof of the first estimate in (3) and, thus, the conjecture is not really proved. Also Takahashi [43] announced the estimate (3). Up to now we were able only to prove estimates (3) and therefore we can still ask to prove or disprove the Maligranda conjecture.

2. B-convexity and the n-th James constants

Let us start with the notion of the uniformly non-l_n^1 and B-convexity of a Banach space X. These notions were introduced by James [15] and Beck [1].

For every natural number $n \geqslant 2$ we say, as in Giesy-James [14], that a Banach space X is *uniformly non-*l_n^1 if there exists a $\delta \in (0,1)$ such that for every $x_1, \ldots, x_n \in B_X$ it holds that $\| \sum_{k=1}^n \theta_k x_k \| \leqslant n(1-\delta)$ for some choice of signs $\theta_1, \theta_2, \ldots, \theta_n$.

A Banach space X is called *B-convex* if it is uniformly non-l_n^1 for some $n \geqslant 2$.

In the connection to these two notions we consider *the* n*-th James constants* (or *the measure of uniformly non-*l_n^1, or sometimes called *the measure of B-convexity*). For given $n \in \mathbb{N}$ *the* n*-th James constant* $J_n(X)$ of a Banach space X is defined by

$$J_n(X) := \sup \left\{ \min_{\theta_k = \pm 1} \left\| \sum_{k=1}^n \theta_k x_k \right\| : \ x_1, \ldots, x_n \in B_X \right\}.$$

Note that $J_1(X) = 1$, $J_2(X)$ is just the James constant $J(X)$ discussed in Section 1 and $J_n(l^1) = J_n(l_m^1) = n$ for $m \geqslant n$ (which can be seen by considering unit vectors). We have also equality $J_n(X) = \inf\{C > 0 : \min_{\theta_k = \pm 1} \| \sum_{k=1}^n \theta_k x_k \| \leqslant C \max_{k=1,2,\ldots,n} \|x_k\|$ for all $x_1, x_2, \ldots, x_n \in X\}$.

It is clear that X is uniformly non-l_n^1 if and only if $J_n(X) < n$ and X is B-convex if and only if $J_n(X) < n$ for some $n \geqslant 2$.

The n-th James constants were studied by several authors e.g. Giesy [13, p. 117], Pisier [40, pp. 1-3], Woyczyński [46, pp. 340-343], Kalton [20, p. 248], Kalton-Peck-Roberts [22, pp. 98-99], Kadets-Kadets [18, pp. 83-84], [19, p. 69] and Diestel-Jarchow-Tonge [6, pp. 261-266]. It seems that these constants for the first time explicitely appeared in 1966, in the paper by Giesy [13, p. 117], where he investigated the numbers $G_n(X) = \sup\left\{ \min_{\theta_k = \pm 1} \frac{1}{n} \left\| \sum_{k=1}^n \theta_k x_k \right\| : \ x_1, \ldots, x_n \in B_X \right\}$.

Let us collect some properties of n-th James constants:

(i) $1 \leqslant J_n(X) \leqslant n$; if $\dim X = \infty$, then $J_n(X) \geqslant \sqrt{n}$.

(ii) $J_n(X)$ is *increasing* in n and *subadditive* in n, that is, $J_{m+n}(X) \leqslant J_m(X) + J_n(X)$ for all $m, n \in \mathbb{N}$; in particular, $J_{n+1}(X) \leqslant J_n(X) + 1$.

(*iii*) $J_n(X)$ is *submultiplicative* sequence, i.e., $J_{mn}(X) \leqslant J_m(X)J_n(X)$ for all $m, n \in \mathbb{N}$.

(*iv*) If X is a Hilbert space and $\dim X \geqslant n$, then $J_n(X) = \sqrt{n}$; the converse is not true in general.

Before we give the proof of these properties let us prove a useful result concerning n-James constants for finitely representable spaces. The notion of finitely representable spaces was introduced by James in [16].

A Banach space X is said to be *finitely representable* in a Banach space Y if, for every $\varepsilon > 0$ and for every finite-dimensional subspace X_0 of X, there exists a subspace Y_0 of Y and an isomorphism T from X_0 onto Y_0 such that

$$\frac{1}{1+\varepsilon}\|x\| \leqslant \|Tx\| \leqslant (1+\varepsilon)\|x\| \text{ for every } x \in X_0.$$

It is well-known that an infinite dimensional Banach space X is B-convex if and only if l^1 is finitely representable in X (see e.g. [6, p. 262] or [18, p. 69]).

PROPOSITION 1. *If X is finitely representable in Y, then $J_n(X) \leqslant J_n(Y)$ for every $n \geqslant 2$.*

Proof. Let $\varepsilon > 0$. Then there exists $x_1, \ldots, x_n \in B_X$ such that $J_n(X) - \varepsilon \leqslant \|\sum_{k=1}^n \theta_k x_k\|$ for every $\theta_k = \pm 1$. Let $X_0 = [(x_k)_{k=1}^n]$. Since X is finitely representable in Y, there exists a linear one-to-one operator T from X_0 into Y such that $\frac{1}{1+\varepsilon}\|x\| \leqslant \|Tx\| \leqslant (1+\varepsilon)\|x\|$ for every $x \in X_0$.

We put $y_k = T(\frac{1}{1+\varepsilon}x_k)$ for $k = 1, \ldots, n$. Then $y_k \in B_Y$ for every $k = 1, \ldots, n$ and for each $\theta_k = \pm 1$ we have that

$$\left\|\sum_{k=1}^n \theta_k y_k\right\| \geqslant \frac{1}{(1+\varepsilon)^2}\left\|\sum_{k=1}^n \theta_k x_k\right\| \geqslant \frac{1}{(1+\varepsilon)^2}(J_n(X) - \varepsilon).$$

Thus, $J_n(Y) \geqslant \frac{1}{(1+\varepsilon)^2}(J_n(X) - \varepsilon)$ for every $\varepsilon > 0$ and, hence, $J_n(Y) \geqslant J_n(X)$.

Proof. (*i*) The first part is clear. If $\dim X = \infty$ then, according to Dvoretzky's theorem (see e.g. [7], [44]), it yields that l^2 is finitely representable in X. Hence, from Proposition 1 we obtain that

$$J_n(X) \geqslant J_n(l^2) \geqslant \min_{\theta_k \pm 1}\left\|\sum_{k=1}^n \theta_k e_k\right\|_2 = \sqrt{n},$$

where e_k are unit vectors.

(*ii*) The first part is easy to prove by just taking zero as the $(n+1)$-element. For the proof of the second part we assume that $m, n \in \mathbb{N}$ and then for arbitrary $\varepsilon > 0$ there exist $x_1^0, \ldots, x_{m+n}^0 \in B_X$ such that $\min_{\theta_k = \pm 1}\|\sum_{k=1}^{m+n} \theta_k x_k^0\| > J_{m+n}(X) - \varepsilon$, i.e. for any choice of signs $\theta_k = \pm 1$ we have

$$\left\|\sum_{k=1}^m \theta_k x_k^0\right\| + \left\|\sum_{k=m+1}^{m+n} \theta_k x_k^0\right\| > J_{m+n}(X) - \varepsilon.$$

This means that

$$\min_{\theta_k=\pm1}\left\|\sum_{k=1}^{m}\theta_k x_k^0\right\| > J_{m+n}(X) - \min_{\theta_k=\pm1}\left\|\sum_{k=m+1}^{m+n}\theta_k x_k^0\right\| - \varepsilon,$$

or

$$J_m(X) \geqslant \min_{\theta_k=\pm1}\left\|\sum_{k=1}^{m}\theta_k x_k^0\right\| > J_{m+n}(X) - \min_{\theta_k=\pm1}\left\|\sum_{k=m+1}^{m+n} > \theta_k x_k^0\right\| - \varepsilon.$$

We have

$$J_m(X) \geqslant \min_{\theta_k=\pm1}\left\|\sum_{k=m+1}^{m+n}\theta_k x_k^0\right\| \geqslant J_{m+n}(X) - J_m(X) - \varepsilon$$

and, thus, $J_{m+n}(X) \leqslant J_m(X) + J_n(X)$ since $\varepsilon > 0$ was arbitrary.

(iii) For the proof see Pisier [40, pp. 2-3], Kalton [20, p. 248], Woyczyński [46, pp. 340-341], Kalton-Peck-Roberts [22, p. 99] and Diestel-Jarchow-Tonge [6, p. 261].

(iv) By the parallelogram law we have

$$\min_{\theta_k\pm1}\left\|\sum_{k=1}^{n}\theta_k x_k\right\|^2 \leqslant \frac{\sum_{\theta_k\pm1}\|\sum_{k=1}^{n}\theta_k x_k\|^2}{2^n} = \frac{\sum_{\theta_k\pm1}\sum_{k=1}^{n}\|x_k\|^2}{2^n} \leqslant n,$$

for $x_1,\ldots,x_n \in B_X$, and so $J_n(X) \leqslant \sqrt{n}$. The equality follows from (i) in the infinite-dimensional case. If $\dim X = n$, then X is isometric to l_n^2 and $J_n(X) = J_n(l_n^2) \geqslant \sqrt{n}$, which can be seen by considering the unit vectors. The proof is complete.

Some results concerning the n-th James constants for L^p spaces are presented in the following theorem (cf. also Proposition 4):

THEOREM 2. *If* $1 \leqslant p \leqslant \infty$, *then* $J_n(L^p(\mu)) \leqslant \max(n^{1/p}, n^{1-1/p})$. *Moreover, if* $1 \leqslant p \leqslant 2$ *and* $\dim L^p(\mu) \geqslant n$, *then* $J_n(L^p(\mu)) = n^{1/p}$; *if* $2 < p \leqslant \infty$ *and* $\dim L^p(\mu) = \infty$, *then* $\sqrt{n} \leqslant J_n(L^p(\mu)) \leqslant n^{1-1/p}$.

Proof. First, let $1 \leqslant p \leqslant 2$. In this case we have Clarkson's inequality (cf. [5] or [33])

$$\|x+y\|_p^p + \|x-y\|_p^p \leqslant 2\left(\|x\|_p^p + \|y\|_p^p\right),$$

which is valid for all $x, y \in L^p(\mu)$. Then, by induction,

$$\sum_{\theta_k\pm1}\left\|\sum_{k=1}^{n}\theta_k x_k\right\|_p^p \leqslant 2^n \sum_{k=1}^{n}\|x_k\|_p^p \text{ for all } x_1,\ldots,x_n \in X.$$

In fact, by the Clarkson inequality and the induction assumption (for $n-1$) we obtain that

$$\sum_{\theta_k\pm1}\left\|\sum_{k=1}^{n}\theta_k x_k\right\|_p^p = \sum_{\theta_k\pm1}\left\|\sum_{k=1}^{n-1}\theta_k x_k + x_n\right\|_p^p + \sum_{\theta_k\pm1}\left\|\sum_{k=1}^{n-1}\theta_k x_k - x_n\right\|_p^p$$
$$\leqslant 2\sum_{\theta_k=\pm1}\left(\left\|\sum_{k=1}^{n-1}\theta_k x_k\right\|_p^p + \|x_n\|_p^p\right)$$

$$\leqslant 2\left(2^{n-1}\sum_{k=1}^{n-1}\|x_k\|_p^p + 2^{n-1}\|x_n\|_p^p\right)$$
$$= 2^n\sum_{k=1}^{n}\|x_k\|_p^p,$$

and, hence,

$$\min_{\theta_k \pm 1}\left\|\sum_{k=1}^{n}\theta_k x_k\right\|_p^p \leqslant \frac{1}{2^n}\sum_{\theta_k \pm 1}\left\|\sum_{k=1}^{n}\theta_k x_k\right\|_p^p \leqslant \sum_{k=1}^{n}\|x_k\|_p^p. \tag{5}$$

This implies that if $x_1, x_2, \ldots, x_n \in B_{L^p}$, then $\min_{\theta_k \pm 1}\|\sum_{k=1}^n \theta_k x_k\|_p \leqslant n^{1/p}$ and, hence, $J_n(L^p(\mu)) \leqslant n^{1/p}$.

Since $\dim L^p(\mu) \geqslant n$ we can find at least n pairwise disjoint subsets $A_1, \ldots, A_n$ of Ω such that $0 < \mu(A_k) < \infty$ for $k = 1, \ldots, n$. Define $x_k = \frac{\chi_{A_k}}{\mu(A_k)^{1/p}}$ for $k = 1, \ldots, n$. Then, for every choice of signs $\theta_1, \ldots, \theta_n$, we have that $\|\sum_{k=1}^n \theta_k x_k\|_p^p = n$ and, hence, $J_n(L^p(\mu)) \geqslant n^{1/p}$.

For $2 \leqslant p < \infty$ we use another Clarkson's inequality (cf. [5] or [33])

$$\|x+y\|_p^p + \|x-y\|_p^p \leqslant 2\left(\|x\|_p^{p'} + \|y\|_p^{p'}\right)^{p-1},$$

from which we obtain the estimate

$$\min\left(\|x+y\|_p^p, \|x-y\|_p^p\right) \leqslant \left(\|x\|_p^{p'} + \|y\|_p^{p'}\right)^{p-1}$$

or

$$\min\left(\|x+y\|_p^{p'}, \|x-y\|_p^{p'}\right) \leqslant \|x\|_p^{p'} + \|y\|_p^{p'}.$$

These estimates give the required estimate for n-elements

$$\min_{\theta_k=\pm 1}\left\|\sum_{k=1}^{n}\theta_k x_k\right\|_p^p \leqslant \left(\sum_{k=1}^{n}\|x_k\|_p^{p'}\right)^{p-1}. \tag{6}$$

In fact, by the first and second estimates above and the induction we obtain that

$$\min_{\theta_k=\pm 1}\left\|\sum_{k=1}^{n}\theta_k x_k\right\|_p^p = \min_{\theta_k=\pm 1}\min\left(\left\|\sum_{k=1}^{n-1}\theta_k x_k + x_n\right\|_p^p, \left\|\sum_{k=1}^{n-1}\theta_k x_k - x_n\right\|_p^p\right)$$
$$\leqslant \min_{\theta_k=\pm 1}\left(\left\|\sum_{k=1}^{n-1}\theta_k x_k\right\|_p^{p'} + \|x_n\|_p^{p'}\right)^{p-1}$$
$$= \min_{\theta_k=\pm 1}\left[\min\left(\left\|\sum_{k=1}^{n-2}\theta_k x_k + x_{n-1}\right\|_p^{p'} + \|x_n\|_p^{p'},\right.\right.$$
$$\left.\left.\left\|\sum_{k=1}^{n-2}\theta_k x_k - x_{n-1}\right\|_p^{p'} + \|x_n\|_p^{p'}\right)\right]^{p-1}$$

$$\leqslant \left[\min_{\theta_k=\pm 1} \left\| \sum_{k=1}^{n-2} \theta_k x_k \right\|_p^{p'} + \|x_{n-1}\|_p^{p'} + \|x_n\|_p^{p'} \right]^{p-1} \leqslant \dots$$
$$\leqslant (\sum_{k=1}^{n} \|x_k\|_p^{p'})^{p-1}.$$

Hence, if $x_1, x_2, \dots, x_n \in B_{L^p}$, then $\min_{\theta_k \pm 1} \| \sum_{k=1}^n \theta_k x_k\|_p \leqslant n^{1-1/p}$ and we conclude that $J_n(L^p(\mu)) \leqslant n^{1-1/p}$.

The estimate from below $J_n(L^p(\mu)) \geqslant \sqrt{n}$ follows from the property (i). The proof is complete.

PROBLEM 1. Find the exact formula for $J_n(L^p(\mu))$ when $p > 2$ and $n \geqslant 3$.

It is well-known (see [6]) that a Banach space X is B-convex if and only if its dual space $X^\star$ is B-convex. In this connection and in view of the estimates (v) between the James constants $J(X)$ and $J(X^\star)$ we can ask the following question:

PROBLEM 2. Find some relations between $J_n(X)$ and $J_m(X^\star)$ for $m, n \in \mathbb{N}$ and $n \geqslant 3$.

We can also consider the n*-th strong James constants* of a Banach space X defined by

$$J_n^s(X) := \sup \left\{ \min_{\theta_k=\pm 1} \left\| \sum_{k=1}^{n} \theta_k x_k \right\| : \ x_1, \dots, x_n \in S_X \right\}.$$

Then, obviously $J_n^s(X) \leqslant J_n(X)$ and $J_3^s(X) < J_3(X)$ for $X = l_2^\infty$ (that is, when $X = \mathbb{R}^2$ with the norm of $x = (x_1, x_2)$ equal to $\|x\| = \max\{|x_1|, |x_2|\}$). In fact,

$$J_3^s(l_2^\infty) = 1 \quad \text{and} \quad J_3(l_2^\infty) \geqslant 2.$$

The first equality can be proved by considering two extreme cases $x_1 = (1,a), x_2 = (1,b), x_3 = (1,c)$ and $x_1 = (1,a), x_2 = (1,b), x_3 = (c,1)$ and the second estimate we obtain by taking $x_1 = (1,1), x_2 = (-1,1), x_3 = (0,\varepsilon)$ with $0 < \varepsilon < 1$ since then

$$\begin{aligned} J_3(l_2^\infty) &\geqslant \min\{\|x_1 + x_2 + x_3\|, \|x_1 + x_2 - x_3\|, \|x_1 - x_2 + x_3\|, \|x_1 - x_2 - x_3\|\} \\ &= \min\{2+\varepsilon, 2-\varepsilon, 2\} = 2 - \varepsilon, \end{aligned}$$

and our claim follows by letting $\varepsilon \to 0^+$.

We don't know any example of a Banach space X such that $\dim X \geqslant 3$ and $J_3^s(X) < J_3(X)$ but we guess that only $J_2^s(X) = J_2(X)$ for $\dim X \geqslant 2$. We easily see that $J_n^s(l^1) = J_n^s(l_m^1) = n$ for $m \geqslant n$ and in the paper [34] it was proved that for the Cesàro sequence spaces $ces_p, 1 < p \leqslant \infty$ we have the equalities $J_n^s(ces_p) = n$ for all natural $n \geqslant 2$, which means that they are not B-convex.

Our main result in this Section is to estimate $J_n(X)$ by $J_n^s(X)$ constants.

THEOREM 3. *For any Banach space X and $n \geqslant 2$ we have the estimate*

$$J_n(X) \leqslant \frac{1}{2}\left[J_n^s(X) - 1 + \sqrt{(2n - J_n^s(X) - 1)^2 + 4n}\right]. \tag{7}$$

Proof. Denote for simplicity $J_n^s(X) = a$. Let $c = c(a)$ be a number from $[0,1]$ which we will determine later on in a suitable way. For fixed $x_1, \dots, x_n \in B_X$ consider two cases:

I. $\min_{k=1,2,\dots,n} \|x_k\| \leqslant c$.

Then, for every choice of signs θ_k, we have that $\| \sum_{k=1}^n \theta_k x_k\| \leqslant c + n - 1$.

II. $\min_{k=1,2,\dots,n} \|x_k\| > c$.

There is a choice of signs θ_k such that $\left\|\sum_{k=1}^n \theta_k \frac{x_k}{\|x_k\|}\right\| \leqslant J_n^s(X) = a$. Since

$$\begin{aligned}\left\|\sum_{k=1}^n \theta_k \left(\frac{x_k}{\|x_k\|} - x_k\right)\right\| &\leqslant \sum_{k=1}^n \|x_k\| \left(\frac{1}{\|x_k\|} - 1\right) \\ &\leqslant \sum_{k=1}^n \left(\frac{1}{c} - 1\right) = n\left(\frac{1}{c} - 1\right)\end{aligned}$$

it follows that for this choice of signs

$$\begin{aligned}\left\|\sum_{k=1}^n \theta_k x_k\right\| &\leqslant \left\|\sum_{k=1}^n \theta_k \frac{x_k}{\|x_k\|}\right\| + \left\|\sum_{k=1}^n \theta_k \left(\frac{x_k}{\|x_k\|} - x_k\right)\right\| \\ &\leqslant \left\|\sum_{k=1}^n \theta_k \frac{x_k}{\|x_k\|}\right\| + n\left(\frac{1}{c} - 1\right) \leqslant a + n\left(\frac{1}{c} - 1\right).\end{aligned}$$

Putting these two cases together we obtain that

$$\left\|\sum_{k=1}^n \theta_k x_k\right\| \leqslant \max\left\{c + n - 1, a + n\left(\frac{1}{c} - 1\right)\right\}$$

and taking the supremum over all $x_1, \dots, x_n \in B_X$ we get that

$$J_n(X) \leqslant \max\left\{c + n - 1, a + n\left(\frac{1}{c} - 1\right)\right\}.$$

Denote the right hand side by $f(c)$ and let us look for the minimum of this function on $[0,1]$. The minimum is attained at $c = c_0$ when $c + n - 1 = a + \frac{n}{c} - n$. Considering the function

$$g(c) = c^2 - (a + 1 - 2n)c - n,\ c \in [0,1]$$

we have that $g(c_0) = 0, g(0) = -n < 0$ and $g(1) = n - a \geqslant 0$. Thus $c_0 \in (0,1]$; more precisely, if $a < n$, then $0 < c_0 < 1$ and if $a = n$, then $c_0 = 1$. Also $c_0 = \left[a + 1 - 2n + \sqrt{(2n - a - 1)^2 + 4n}\right]/2$ and, hence,

$$J_n(X) \leqslant c_0 + n - 1 = \left(a - 1 + \sqrt{(2n - a - 1)^2 + 4n}\right)/2,$$

and the proof is complete.

Immediately from Theorem 3 (since $a < n$ implies $c_0 < 1$) we see that in the definition of uniformly non- l_1^n space we can have elements from the unit sphere or from the unit ball (see also Kamińska-Turett [23, Lemma 2]).

COROLLARY 1. *For any Banach space X and $n \geqslant 2$ it yields that $J_n(X) < n$ if and only if $J_n^s(X) < n$.*

We pose the following conjecture:

CONJECTURE. *If $J_n(X) < n$, then $J_n^s(X) < J_n(X)$ for $n \geqslant 3$ and $\dim X \geqslant 3$.*

3. Type and the n-th Khintchine constants of Banach spaces

For given $n \in \mathbf{N}$, $0 < p, q \leqslant \infty$ and a Banach space X, we define *the n-th Khintchine constants* $K_{p,q}^n(X)$ to be the smallest of all numbers $C \geqslant 1$ such that

$$\left(\int_0^1 \left\|\sum_{k=1}^n r_k(t)x_k\right\|^q dt\right)^{\frac{1}{q}} \leqslant C\left(\sum_{k=1}^n \|x_k\|^p\right)^{\frac{1}{p}}$$

for every choice $x_1, \ldots, x_n \in X$, where $\{r_k\}_{k=1}^n$ are the Rademacher functions. If $X = \mathbb{R}$ with the absolute value as the norm, then we will write $K_{p,q}^n(\mathbb{R})$. Moreover, for $p = q$ we denote these constants by $t_{p,n}(X)$ as the numbers connected with the type p of the space X and if $p = q = 2$ we denote them shortly by $t_n(X)$ as the numbers connected with the type 2 of the space X. Note that $t_2(X) = \sqrt{C_{NJ}(X)}$.

REMARK 1. For $0 < q < \infty$ we have equality

$$\int_0^1 \left\|\sum_{k=1}^n r_k(t)x_k\right\|^q dt = \frac{1}{2^n}\sum_{\theta_k=\pm 1}\left\|\sum_{k=1}^n \theta_k x_k\right\|^q. \tag{8}$$

A Banach space X is of *type* p, $1 \leqslant p \leqslant 2$, if $T_{p,q}(X) := \sup_{n\in\mathbb{N}} K_{p,q}^n(X) < \infty$. Pisier [39] proved in 1973 that a Banach space X has non-trivial type, i.e., is of type $p > 1$ if and only if it is B-convex.

The Khintchine constants $K_{p,q}^n(X)$ for some choices of p, q were studied by several authors, e.g. Pisier [39], [40] in 1973, Enflo-Lindenstrauss-Pisier [8] in 1975 and Figiel-Lindenstrauss-Milman [10] in 1977 considered $t_n(X)$, Maurey-Pisier [33] in 1976, Woyczyński [46] in 1978, and Milman-Schechtman [36] in 1986 investigated $t_{p,n}(X)$ for $1 \leqslant p \leqslant 2$. Let us collect some properties of these constants in the next proposition:

PROPOSITION 2. *The following properties for the n-th Khintchine constants hold*

(i) $1 \leqslant K_{p,q}^n(\mathbb{R}) \leqslant K_{p,q}^n(X) \leqslant n^{(1-1/p)_+}$.

(ii) $K_{p,q}^n(X)$ *are increasing in* n, p, q, $K_{1,1}^n(X) = 1$ *and* $K_{p,q}^n(X) \leqslant n^{\frac{1}{r}-\frac{1}{p}} K_{r,q}^n(X)$ *for all* $0 < q \leqslant \infty, 0 < r \leqslant p \leqslant \infty$.

(iii) If $1 \leqslant q < \infty$*, then* $K^n_{p,q}(X)$ *are subadditive in* n*, that is,* $K^{m+n}_{p,q}(X) \leqslant K^m_{p,q}(X) + K^n_{p,q}(X)$ *for all* $m, n \in \mathbb{N}$*.*

(iv) If $0 < p \leqslant q < \infty$*, then* $K^n_{p,q}(X)$ *are submultiplicative in* n*, that is,* $K^{mn}_{p,q}(X) \leqslant K^m_{p,q}(X) K^n_{p,q}(X)$ *for all* $m, n \in \mathbb{N}$*.*

(v) If X *is a Hilbert space, then* $K^n_{p,q}(X) = 1$ *for* $0 < p, q \leqslant 2$*;* $K^n_{2,2}(X) = 1$ *for* $n \geqslant 2$ *if and only if* X *is a Hilbert space.*

(vi) If $1 \leqslant r < \infty$ *and* $0 < p \leqslant r \leqslant q < \infty$*, then* $K^n_{p,q}(L^r(\mu)) = K^n_{p,q}(\mathbb{R})$*. In particular, if* $1 \leqslant r \leqslant \infty$ *and* $0 < p \leqslant r \leqslant q \leqslant 2$*, then* $K^n_{p,q}(L^r(\mu)) = 1$*.*

Proof. (i) There exists $x \in X$ with $\|x\| = 1$. Take $x_k = a_k x$, $k = 1, \ldots, n$, with arbitrary $a_k \in \mathbb{R}$. Then $K^n_{p,q}(X) \geqslant K^n_{p,q}(\mathbb{R})$. Moreover, for every $x_1, x_2, \ldots, x_n \in X$,

$$\left(\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|^q dt \right)^{\frac{1}{q}} \leqslant \sum_{k=1}^n \|x_k\| \leqslant n^{(1-1/p)_+} \left(\sum_{k=1}^n \|x_k\|^p \right)^{\frac{1}{p}}.$$

(ii) This follows from the properties of the l^p_n and $L^p[0,1]$ spaces.

(iii) This statement for $t_n(X)$ can be found in Pisier [40, p. 7] and for $t_{p,n}(X)$ in Woyczyński [46, p. 344]. We can easily see that the subadditivity of $K^n_{p,q}(X)$ holds if $q \geqslant 1$. This follows directly from the following two estimates

$$\left(\int_0^1 \left\| \sum_{k=1}^{m+n} r_k(t) x_k \right\|^q dt \right)^{1/q} \leqslant \left(\int_0^1 \left\| \sum_{k=1}^{m} r_k(t) x_k \right\|^q dt \right)^{1/q} + \left(\int_0^1 \left\| \sum_{k=m+1}^{m+n} r_k(t) x_k \right\|^q dt \right)^{1/q}$$

and

$$\left(\int_0^1 \left\| \sum_{k=m+1}^{m+n} r_k(t) x_k \right\|^q dt \right)^{1/q} \leqslant K^n_{p,q}(X) \left(\sum_{k=m+1}^{m+n} \|x_k\|^p \right)^{1/p}.$$

(iv) This statement for $t_n(X)$ was proved by Pisier [39, pp. 991-992], [40, pp. 7-8], Enflo-Lindenstrauss-Pisier [8, p. 200] and Figiel-Lindenstrauss-Milman [10, p. 82] (see also Beauzamy [2, p. 313], Diestel-Jarchow-Tonge [6, p. 265], Wojtaszczyk [45, p. 142]). For $t_{p,n}(X)$ it was proved by Maurey-Pisier [33, p. 71], Woyczyński [46, p. 345] and Milman-Schechtman [36, p, 86] (see also Benyamini-Lindenstrauss [3, p. 443]).

We modify the proof in [36] and [3], where the statement (iv) was proved for $p = q$, and prove it for $p \leqslant q$. Let $m, n \in \mathbb{N}$ and $x_1, \ldots, x_{mn} \in X$. For each $k = 1, \ldots, n$ and $t \in [0,1]$ define

$$y_k(t) := \sum_{j=(k-1)m+1}^{km} r_j(t) x_j.$$

Then

$$\int_0^1 \|y_k(t)\|^q dt \leqslant K^m_{p,q}(X)^q \left(\sum_{j=(k-1)m+1}^{km} \|x_j\|^p \right)^{q/p}.$$

The products $\{r_k(s)r_j(t)\}$ have the same joint distribution as $\{r_j(t)\}$. Hence, by the Minkowski inequality (since $q/p \geqslant 1$),

$$\begin{aligned}
\int_0^1 \left\| \sum_{j=1}^{mn} r_j(t)x_j \right\|^q dt &= \int_0^1 \int_0^1 \left\| \sum_{k=1}^{n} r_k(s)y_k(t) \right\|^q dsdt \\
&\leqslant K_{p,q}^n(X)^q \int_0^1 \left(\sum_{k=1}^{n} \|y_k(t)\|^p \right)^{q/p} dt \\
&\leqslant K_{p,q}^n(X)^q \left[\sum_{k=1}^{n} \left(\int_0^1 \|y_k(t)\|^q dt \right)^{p/q} \right]^{q/p} \\
&\leqslant K_{p,q}^n(X)^q \left\{ \sum_{k=1}^{n} \left[K_{p,q}^m(X)^q \left(\sum_{j=(k-1)m+1}^{km} \|x_j\|^p \right)^{q/p} \right]^{p/q} \right\}^{q/p} \\
&= K_{p,q}^n(X)^q K_{p,q}^m(X)^q \left(\sum_{k=1}^{n} \sum_{j=(k-1)m+1}^{km} \|x_j\|^p \right)^{q/p} \\
&= K_{p,q}^n(X)^q K_{p,q}^m(X)^q \left(\sum_{j=1}^{mn} \|x_j\|^p \right)^{q/p},
\end{aligned}$$

i.e., $K_{p,q}^{mn}(X) \leqslant K_{p,q}^m(X) K_{p,q}^n(X)$.

(v) If X is a Hilbert space, then $K_{2,2}^n(X) = 1$ and the rest of the proof follows from the properties (i) and (ii).

(vi) In fact, by using the Minkowski inequality twice we obtain that

$$\begin{aligned}
\int_0^1 \left\| \sum_{k=1}^{n} r_k(t)x_k \right\|_r^q dt &= \int_0^1 \left(\int_\Omega \left| \sum_{k=1}^{n} r_k(t)x_k(s) \right|^r d\mu(s) \right)^{q/r} dt \\
&\leqslant \left[\int_\Omega \left(\int_0^1 \left| \sum_{k=1}^{n} r_k(t)x_k(s) \right|^q dt \right)^{r/q} d\mu(s) \right]^{q/r} \\
&\leqslant K_{p,q}^n \left[\int_\Omega \left(\sum_{k=1}^{n} |x_k(t)|^p \right)^{r/p} d\mu(s) \right]^{q/r} \\
&\leqslant K_{p,q}^n \left[\sum_{k=1}^{n} \left(\int_\Omega |x_k(t)|^r \, d\mu(s) \right)^{p/r} \right]^{q/p} \\
&= K_{p,q}^n \left(\sum_{k=1}^{n} \|x_k\|_r^p \right)^{q/p}.
\end{aligned}$$

Hence $K_{p,q}^n(L^r(\mu)) \leqslant K_{p,q}^n$ and the reversed inequality follows from (i). The proof is complete.

COROLLARY 2. *For any Banach space* $X, 0 < p \leqslant q < \infty$ *and* $n \geqslant 2$ *we have*

$$K_{p,q}^{n+1}(X) \leqslant K_{p,q}^{2}(X) K_{p,q}^{n}(X) \text{ and } K_{p,q}^{n}(X) \leqslant \left[K_{p,q}^{2}(X)\right]^{n-1}. \tag{9}$$

In fact, using properties (ii) and (iv) in Proposition 2 we obtain that

$$K_{p,q}^{n+1}(X) \leqslant K_{p,q}^{2n}(X) \leqslant K_{p,q}^{2}(X) K_{p,q}^{n}(X)$$

and then

$$K_{p,q}^{n}(X) \leqslant K_{p,q}^{2}(X) K_{p,q}^{n-1}(X) \leqslant \left[K_{p,q}^{2}(X)\right]^{2} K_{p,q}^{n-2}(X) \leqslant \ldots \leqslant \left[K_{p,q}^{2}(X)\right]^{n-1}.$$

PROBLEM 3. Is $K_{p,q}^{n}(X)$ a submultiplicative function of n for $p > q > 0$?

Let us note that Pisier [41] proved the equality $K_{2,2}^{n}(l_m^p) = [\min(m,n)]^{1/p-1/2}$ for $1 \leqslant p \leqslant 2$ and by using this equality he proved that for any $n \geqslant 2$ and any $C \in [1, \sqrt{n}]$ there exists a Banach space X such that $K_{2,2}^{n}(X) = C$.

4. An estimate of the n-th Khintchine constants by the n-th James constants

Our main result is an estimate of the n-th Khintchine constants by the n-th James constants. For the proof we need the following crucial lemma (corresponding to Lemma 1), which has been proved in Kutzarova-Nikolova-Zachariades [29, Lemma 6].

LEMMA 2. *Let* X *be a normed space and* $n \geqslant 2$. *Then, for every* $x_1, x_2, \ldots, x_n \in X$, *with* $\|x_n\| \leqslant \|x_k\|$ *for* $k = 1, 2, \ldots, n-1$, *there exist* $\theta_1, \ldots, \theta_n \in \{-1, 1\}$ *such that*

$$\left\|\sum_{k=1}^{n} \theta_k x_k\right\| \leqslant \sum_{k=1}^{n-1} \|x_k\| + [J_n^s(X) - n + 1]\, \|x_n\|.$$

Proof (see also [29]). If $x_n = 0$ the statement in the lemma is clear. Let $x_n \neq 0$. Then there exists a choice of signs $\theta_1, \ldots, \theta_n$ such that $\| \sum_{k=1}^{n} \theta_k \frac{x_k}{\|x_k\|} \| \leqslant J_n^s(X)$ and, hence, we have that

$$\begin{aligned}
\left\|\sum_{k=1}^{n} \theta_k x_k\right\| &= \left\|\sum_{k=1}^{n} \theta_k \left(1 - \frac{\|x_n\|}{\|x_k\|}\right) x_k + \sum_{k=1}^{n} \theta_k \frac{\|x_n\|}{\|x_k\|} x_k\right\| \\
&\leqslant \sum_{k=1}^{n} \left(1 - \frac{\|x_n\|}{\|x_k\|}\right) \|x_k\| + \|x_n\| \left\|\sum_{k=1}^{n} \theta_k \frac{x_k}{\|x_k\|}\right\| \\
&\leqslant \sum_{k=1}^{n-1} \|x_k\| + [J_n^s(X) - n + 1]\, \|x_n\|.
\end{aligned}$$

Our main result reads:

THEOREM 4. *Let X be a Banach space X, $1 \leqslant p,q \leqslant \infty$ and $n \geqslant 2$. Then*

$$\frac{J_n(X)}{n^{\frac{1}{p}}} \leqslant K_{p,q}^n(X) \leqslant \frac{1}{2^{\frac{n-1}{q}}}\left[2^{\frac{(n-1)p'}{q}}(n-1)+c_n^{\frac{p'}{q}}\right]^{\frac{1}{p'}}, \tag{10}$$

where $c_n = a_n^q + 2^{n-1} - 1$ and $a_n = [J_n^s(X) - n + 1]_+$. For $p = 1$ the right hand side in (10) if it is as usual interpreted is equal to 1.

Proof. It is clear that

$$\begin{aligned}\min_{\theta_k=\pm1}\left\|\sum_{k=1}^n \theta_k x_k\right\| &\leqslant \left(\frac{1}{2^n}\sum_{\theta_k=\pm1}\left\|\sum \theta_k x_k\right\|^q\right)^{\frac{1}{q}} \\ &\leqslant K_{p,q}^n(X)\left(\sum_{k=1}^n \|x_k\|^p\right)^{\frac{1}{p}} \leqslant K_{p,q}^n(X)\, n^{\frac{1}{p}},\end{aligned}$$

which gives the left hand side inequality of (10). It is also clear that $J_n(X) \leqslant K_{\infty,q}^n(X)$.

To prove the upper estimate in (10) we may suppose, without loss of generality, that $x_1,\dots,x_n \in X$ are not all zero, $\|x_n\| \leqslant \|x_k\|$ for every $k = 2,3,\dots,n-1$ and $\sum_{k=1}^{n-1}\|x_k\| = 1$.

Let $1 < p < \infty$ and $1 \leqslant q < \infty$. By using Lemma 2 and Minkowski's inequality, we obtain that

$$\begin{aligned}&\frac{1}{2^{\frac{1}{q}}}\left[\sum_{\theta_k\pm1}\left\|\sum \theta_k x_k\right\|^q\right]^{\frac{1}{q}} \\ &\leqslant \left[\left(\sum_{k=1}^{n-1}\|x_k\| + a_n\|x_n\|\right)^q + (2^{n-1}-1)\left(\sum_{k=1}^{n-1}\|x_k\| + \|x_n\|\right)^q\right]^{\frac{1}{q}} \\ &\leqslant \left[2^{n-1}\left(\sum_{k=1}^{n-1}\|x_k\|\right)^q\right]^{\frac{1}{q}} + \left[a_n^q\|x_n\|^q + (2^{n-1}-1)\|x_n\|^q\right]^{\frac{1}{q}} \\ &= 2^{\frac{n-1}{q}} + c_n^{\frac{1}{q}}\|x_n\|.\end{aligned}$$

Hence

$$K_{p,q}^n(X) \leqslant \frac{2^{\frac{n-1}{q}} + c_n^{\frac{1}{q}}\|x_n\|}{2^{\frac{n-1}{q}}\left(\sum_{k=1}^n \|x_k\|^p\right)^{\frac{1}{p}}}.$$

Moreover, according to Hölder's inequality,

$$1 = \sum_{k=1}^{n-1}\|x_k\| \leqslant (n-1)^{\frac{1}{p'}}\left(\sum_{k=1}^{n-1}\|x_k\|^p\right)^{\frac{1}{p}},$$

where $\frac{1}{p}+\frac{1}{p'}=1$, and, thus, $\sum_{k=1}^{n-1}\|x_k\|^p \geqslant (n-1)^{1-p}$. We conclude that

$$K_{p,q}^n(X) \leqslant \frac{2^{\frac{n-1}{q}}+c_n^{\frac{1}{q}}\|x_n\|}{2^{\frac{n-1}{q}}\left(\|x_n\|^p+(n-1)^{1-p}\right)^{\frac{1}{p}}}.$$

We consider the function $f(t)=\frac{2^{\frac{n-1}{q}}+c_n^{\frac{1}{q}}t}{2^{\frac{n-1}{q}}(t^p+(n-1)^{1-p})^{\frac{1}{p}}}$ for $t \geqslant 0$. It is easy to see that $f(t) \leqslant f(t_0)$, where $t_0=\frac{c_n^{\frac{1}{q(p-1)}}}{(n-1)2^{\frac{n-1}{q(p-1)}}}$. Thus, we obtain that

$$K_{p,q}^n(X) \leqslant \frac{1}{2^{\frac{n-1}{q}}}\left[2^{\frac{(n-1)p'}{q}}(n-1)+c_n^{\frac{p'}{q}}\right]^{\frac{1}{p'}}.$$

Therefore the right hand side inequality of (10) holds,

If $p=1$, then the estimate $K_{1,q}^n(X) \leqslant 1$ is clear and the right hand side is equal to

$$2^{(1-n)/q}\max\{2^{(n-1)/q},\ldots,2^{(n-1)/q},c_n^{1/q}\}=1.$$

If $p=\infty$, then $p'=1$ and $1=\sum_{k=1}^{n-1}\|x_k\| \leqslant (n-1)\max_{k=1,\ldots,n-1}\|x_k\|$ from which it follows that

$$\max_{k=1,\ldots,n}\|x_k\| = \max_{k=1,\ldots,n-1}\|x_k\| \geqslant \frac{1}{n-1}$$

and, thus,

$$K_{\infty,q}^n(X) \leqslant \frac{2^{\frac{n-1}{q}}+c_n^{\frac{1}{q}}\|x_n\|}{2^{\frac{n-1}{q}}\max_{k=1,\ldots,n}\|x_k\|} \leqslant \frac{2^{\frac{n-1}{q}}+c_n^{\frac{1}{q}}\frac{1}{n-1}}{2^{\frac{n-1}{q}}\frac{1}{n-1}}.$$

This gives the required estimate $K_{\infty,q}^n(X) \leqslant n-1+\frac{c_n^{\frac{1}{q}}}{2^{\frac{n-1}{q}}}$.

If $q=\infty$, then

$$\max_{\theta_k=\pm 1}\left\|\sum_{k=1}^{n}\theta_k x_k\right\| \leqslant \sum_{k=1}^{n-1}\|x_k\|+\|x_n\|=1+t,$$

and

$$K_{p,\infty}^n(X) \leqslant \frac{1+t}{(\sum_{k=1}^n\|x_k\|^p)^{1/p}} \leqslant \frac{1+t}{[t^p+(n-1)]^{1/p}} := g(t).$$

The function g has maximum at $t_0=\frac{1}{n-1}$ and, hence, $g(t) \leqslant g(t_0)=n^{1/p'}$, which is again the right hand side of (10) and the proof is complete.

We will now point out some direct consequences of Theorem 4.

COROLLARY 3. *If $n \geqslant 2$, then*

$$\frac{J_n(X)}{\sqrt{n}} \leqslant t_n(X)=K_{2,2}^n(X) \leqslant 2^{\frac{1-n}{2}}\left\{2^{n-1}n-1+[J_n^s(X)-n+1]_+^2\right\}.$$

In particular, $\frac{J(X)^2}{2} \leqslant C_{NJ}(X) = K_{2,2}^2(X)^2 \leqslant \frac{J(X)^2}{2} + 2 - J(X)$.

Pisier [39] proved that a Banach space X is uniformly non-l_n^1 if and only if $K_{2,2}^n(X) < \sqrt{n}$. By using the the estimates (10) we can state similar result for the n-th Khintchine constants $K_{p,q}^n(X)$.

COROLLARY 4. *An infinite dimensional Banach space X is uniformly non-l_n^1 if and only if $K_{p,q}^n(X) < n^{1/p'}$ for $1 < p \leqslant \infty, 1 \leqslant q < \infty$.*

COROLLARY 5. *Let X be a Banach space. For $n \geqslant 2$ and $1 < p \leqslant \infty, 1 \leqslant q < \infty$ fixed we have that $J_n(X) = n$ if and only if $K_{p,q}^n(X) = n^{1/p'}$.*

The assertion of the following proposition is known for $p = 1$ and means that the Banach space X is B-convex (cf. [14] and [6]). Note that a Banach space X is B-convex if and only if $\lim_{n\to\infty} \frac{J_n(X)}{n} = 0$. We extend this result to $1 \leqslant p < 2$.

PROPOSITION 3. *Let X be an infinite dimensional Banach space and $1 \leqslant p < 2$. The following conditions are equivalent*

(i) X *is of type strictly bigger than* p.

(ii) $\lim_{n\to\infty} \frac{J_n(X)}{n^{1/p}} = 0$.

(iii) $\inf_{n\geqslant 2} \frac{J_n(X)}{n^{1/p}} < 1$.

(iv) l^p *is not finitely representable in* X.

Proof. $(i) \Rightarrow (ii)$. Let X be of type r for some $p < r \leqslant 2$. Then, according to the first estimate in (10), we obtain that $\sup_{n\geqslant 2} \frac{J_n(X)}{n^{\frac{1}{r}}} < \infty$. Hence, $\lim_{n\to\infty} \frac{J_n(X)}{n^{\frac{1}{p}}} = \lim_{n\to\infty} [\frac{J_n(X)}{n^{\frac{1}{r}}} n^{\frac{1}{r}-\frac{1}{p}}] = 0.$

$(ii) \Rightarrow (iii)$. This implication is obvious.

$(iii) \Rightarrow (iv)$. Since $J_n(l^p) = n^{1/p}$ we conclude that l^p is not finitely representable in X. In fact, if it is so, then by Proposition 1 we will have that $n^{1/p} = J_n(l^p) \leqslant J_n(X)$ for every n, which contradicts the assumption (iii).

$(iv) \Rightarrow (i)$. We use the Maurey-Pisier theorem (cf. [33]): *If X is an infinite dimensional Banach space, then $l^{p(X)}$ is finitely representable in X and even more: l^r is finitely representable in X for every $r \in [p(X), 2]$, where*

$$p(X) := \sup\{p \geqslant 1 : X \textit{ is of type } p\}.$$

From this fact we conclude that $p < p(X)$ and, thus, X is of type strictly bigger than p. The proof is complete.

COROLLARY 6. *If X be an infinite dimensional Banach space, $1 < p \leqslant 2$ and $\sup_{n\geqslant 2} \frac{J_n(X)}{n^{1/p}} < \infty$, then X is of type r for every $1 < r < p$.*

Proof. The result follows from Proposition 3 since for $1 < r < p$ we have that $\lim_{n\to\infty} \frac{J_n(X)}{n^{1/r}} = \lim_{n\to\infty} \frac{J_n(X)}{n^{1/p}} n^{1/p-1/r} = 0$.

THEOREM 5. *If X is an infinite dimensional Banach space, then*

$$\lim_{n\to\infty} \frac{\ln n}{\ln J_n(X)} = \sup_{n\geqslant 2} \frac{\ln n}{\ln J_n(X)} = p(X). \tag{11}$$

Proof. Define $l(X) := \sup_{n\geqslant 2} \frac{\ln n}{\ln J_n(X)}$ and note also that from the estimates $\sqrt{n} \leqslant J_n(X) \leqslant n$ we obtain that $1 \leqslant \frac{\ln n}{\ln J_n(X)} \leqslant 2$ and, hence, $1 \leqslant l(X) \leqslant 2$.

We suppose that $p(X) < l(X)$. Let $p(X) < r < l(X)$. Then there exists $m \geqslant 2$ such that $r < \frac{\ln m}{\ln J_m(X)}$. We conclude that $\frac{J_m(X)}{m^{\frac{1}{r}}} < 1$ and, thus, according to Proposition 3, X is of type bigger than r which is a contradiction. Hence $l(X) \leqslant p(X)$. Now we suppose that $\liminf_{n\to\infty} \frac{\ln n}{\ln J_n(X)} < p(X)$. Let $\liminf_{n\to\infty} \frac{\ln n}{\ln J_n(X)} < s < p(X)$. Then there exists a subsequence $\{n_k\}$ such that $\frac{\ln n_k}{\ln J_{n_k}(X)} < s$, i.e. $1 < \frac{\ln J_{n_k}(X)}{n_k^{1/s}}$ which means that $\frac{\ln J_n(X)}{n^{1/s}}$ does not converge to 0. Thus, again by using Proposition 3, we find that X is not of type bigger than s which is a contradiction. Hence $p(X) \leqslant \liminf_{n\to\infty} \frac{\ln n}{\ln J_n(X)}$ and so

$$p(X) \leqslant \liminf_{n\to\infty} \frac{\ln n}{\ln J_n(X)} \leqslant \limsup_{n\to\infty} \frac{\ln n}{\ln J_n(X)} \leqslant \sup_{n\geqslant 2} \frac{\ln n}{\ln J_n(X)} = l(X) \leqslant p(X),$$

which means that

$$\lim_{n\to\infty} \frac{\ln n}{\ln J_n(X)} = \sup_{n\geqslant 2} \frac{\ln n}{\ln J_n(X)} = p(X)$$

and the proof is complete.

We should mention here that the first equality in (11) follows also from the well-known property of submultiplicative sequences. In fact, if $\{a_n\}$ is a submultiplicative sequence, then $\lim_{n\to\infty} \frac{\ln a_n}{\ln n}$ exists and is equal to $\inf_{n\geqslant 2} \frac{\ln a_n}{\ln n}$.

Note that Woyczyński [46, p. 347] proved the following related result: if X is an infinite dimensional Banach space and $0 < p < \infty$, then $p(X) = \lim_{n\to\infty} \frac{\ln n}{\ln[n^{1/p} K_{p,p}^n(X)]}$, and Milman-Schechtman [36, p. 87] for $p(X) \leqslant p \leqslant 2$ that

$$p(X) = \lim_{n\to\infty} \frac{\ln n}{\ln[n^{1/p} K_{p,2}^n(X)]}$$

or, equivalently,

$$\lim_{n\to\infty} \frac{\ln K_{p,2}^n(X)}{\ln n} = \frac{1}{p(X)} - \frac{1}{p}.$$

PROPOSITION 4. *If $p \geqslant 2$, then*

$$J_n(L^p(\mu)) \leqslant \min\left[n^{1-1/p}, \left(\int_0^1 \left| \sum_{k=1}^n r_k(t) \right|^p dt \right)^{1/p} \right].$$

The first estimate was proved already in Theorem 2. The other estimate follows from Theorem 4 and the Figiel-Iwaniec-Pełczyński estimate [9]:

$$J_n(L^p(\mu)) \leqslant n^{1/p} K^n_{p,p}(L^p(\mu)) = K^n_{p,p}(\mathbb{R}) \leqslant \left(\int_0^1 \left|\sum_{k=1}^n r_k(t)\right|^p dt\right)^{1/p}.$$

Note that for $n=2$ and $n=3$ the constant $n^{1-1/p}$ is smaller than $(\int_0^1 |\sum_{k=1}^n r_k(t)|^p dt)^{1/p}$ and for large n we have reverse inequality. Moreover, $\lim_{n\to\infty} \frac{J_n(L^p)}{n} = \frac{1}{2}$.

5. Banach-Mazur distance and stability under norm perturbations

For isomorphic Banach spaces X and Y, the *Banach-Mazur distance* between X and Y, denoted by $d(X,Y)$, is defined to be the infimum of $\|T\|\|T^{-1}\|$ taken over all bicontinuous linear operators T from X onto Y (cf. [39]).

We follow considerations in the paper by Kato-Maligranda-Takahashi [25], where the results for $n=2$ were proved.

THEOREM 6. *If X and Y are isomorphic Banach spaces, then for any $n \geqslant 2$*

$$\frac{J_n(X)}{d(X,Y)} \leqslant J_n(Y) \leqslant J_n(X) d(X,Y) \text{ and } \frac{K^n_{p,q}(X)}{d(X,Y)} \leqslant K^n_{p,q}(Y) \leqslant K^n_{p,q}(X) d(X,Y).$$

In particular, if the spaces X and Y are isometric, then for any $n \geqslant 2$ $J_n(X) = J_n(Y)$ and $K^n_{p,q}(X) = K^n_{p,q}(Y)$.

Proof. Let $x_1,\ldots,x_n \in B_X$ be arbitrary. For each $\epsilon > 0$ there exists an isomorphism T from X onto Y such that $\|T\|\|T^{-1}\| \leqslant (1+\epsilon) d(X,Y)$. Put

$$y_k = \frac{Tx_k}{\|T\|},\ k = 1,\ldots,n.$$

Then $y_k \in B_Y, k=1,\ldots,n$ since $\|y_k\| = \frac{\|Tx_k\|}{\|T\|} \leqslant \|x_k\| \leqslant 1$ and $x_k = \|T\| T^{-1}(y_k)$. We obtain

$$\begin{aligned}
\min_{\theta_k=\pm 1} \left\|\sum_{k=1}^n \theta_k x_k\right\|_X &= \|T\| \min_{\theta_k=\pm 1} \left\|T^{-1}(\sum_{k=1}^n \theta_k y_k)\right\|_X \\
&\leqslant \|T\|\|T^{-1}\| \min_{\theta_k=\pm 1} \left\|\sum_{k=1}^n \theta_k y_k\right\|_Y \\
&\leqslant \|T\|\|T^{-1}\| J_n(Y) \leqslant (1+\epsilon) d(X,Y) J_n(Y),
\end{aligned}$$

and, since $x_1,\ldots,x_n \in B_X$ were arbitrary, $J_n(X) \leqslant (1+\epsilon) d(X,Y) J_n(Y)$, which gives the first estimate. The second estimate follows by just interchanging X and Y. The other two estimates for $K^n_{p,q}(\cdot)$ constants can be proved similarly.

COROLLARY 7. *If X and Y are isomorphic Banach spaces and X is B-convex, then also Y is B-convex.*

COROLLARY 8. *Let* $X = (X, \|\cdot\|)$ *be a non-trivial Banach space and let* $X_1 = (X, \|\cdot\|_1)$, *where* $\|\cdot\|_1$ *is an equivalent norm on X satisfying, for some* $a, b > 0$ *and all* $x \in X, a\|x\| \leqslant \|x\|_1 \leqslant b\|x\|$. *Then*

$$\frac{a}{b}J_n(X) \leqslant J_n(X_1) \leqslant \frac{b}{a}J_n(X) \text{ and } \frac{a}{b}K^n_{p,q}(X) \leqslant K^n_{p,q}(X_1) \leqslant \frac{b}{a}K^n_{p,q}(X).$$

The proof follows immediately from Theorem 6 and the fact that $d(X, X_1) \leqslant b/a$.

We illustrate the above corollary by the following example:

EXAMPLE 1. For $1 \leqslant p \leqslant 2$ and $\lambda \geqslant 1$ let $X_{\lambda,p}$ be the space $L^p[0,1]$ with the norm $\|x\|_{\lambda,p} = \max\{\|x\|_p, \lambda\|x\|_1\}$. Then

$$J_n(X_{\lambda,p}) = \min\{n, \lambda n^{1/p}\}, \text{ and } K^n_{p,p}(X_{\lambda,p}) = \min\{n^{1-1/p}, \lambda\}.$$

The inequalities from above follow from the estimates $\|x\|_p \leqslant \|x\|_{\lambda,p} \leqslant \lambda\|x\|_p$ for all $x \in L^p$ and Corollary 8. The equalities we are getting by taking functions $x_{k,n} = a\chi_{[\frac{k-1}{n},\frac{k}{n})}$ for $k = 1, 2, \ldots, n, n = 2, 3, \ldots$ with $a = \min(n^{1/p}, n/\lambda)$, since

$$\|x_{k,n}\|_{\lambda,p} = \max\{\frac{a}{n^{1/p}}, \lambda\frac{a}{n}\} = \frac{a}{\min(n^{1/p}, n/\lambda)} = 1$$

and

$$\sum_{\theta_k=\pm1}\Big\|\sum_{k=1}^{n}\theta_k x_{k,n}\Big\|_{\lambda,p} = \max(a, \lambda a) = \lambda a.$$

Thus $J_n(X_{\lambda,p}) \geqslant \lambda a$ and $K^n_{p,p}(X_{\lambda,p}) \geqslant \frac{a\lambda}{n^{1/p}} = \min\{\lambda, n^{1-1/p}\}$, which means we have estimates from below, and consequenly equalities.

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Lech Maligranda
Department of Mathematics
Luleå University of Technology
SE 971 87 Luleå
SWEDEN
e-mail: lech@sm.luth.se

Ludmila Nikolova
Department of Mathematics
Sofia University
blv. J. Bouchier 5
Sofia 1164
BULGARIA
e-mail: ludmilan@fmi.uni-sofia.bg

Lars-Erik Persson
Department of Mathematics
Luleå University of Technology
SE 971 87 Luleå
SWEDEN
e-mail: larserik@sm.luth.se

Theodossios Zachariades
Department of Mathematics
University of Athens
Panepistimiopolis, 15784 Athens
GREECE
e-mail: tzaharia@math.uoa.gr