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JENSEN–STEFFENSEN'S AND RELATED INEQUALITIES FOR SUPERQUADRATIC FUNCTIONS

S. ABRAMOVICH, S. BANIĆ, M. MATIĆ AND J. PEČARIĆ

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Abstract. Refinements of Jensen-Steffensen's inequality, Slater-Pečarić's inequality and majorization theorems for superquadratic functions are presented.

1. Introduction

Jensen-Steffensen's inequality states that if $\varphi : I \to \mathbb{R}$ is convex, then

$$\varphi\left(\frac{1}{P_n}\sum_{i=1}^{n}\rho_i\zeta_i\right) \leqslant \frac{1}{P_n}\sum_{i=1}^{n}\rho_i\varphi\left(\zeta_i\right) \tag{1.1}$$

holds, where I is an interval in $\mathbb{R}$, $\boldsymbol{\zeta} = (\zeta_{1,\dots,}\zeta_n)$ is any monotonic n-tuple in I^n and $\boldsymbol{\rho} = (\rho_1,\dots,\rho_n)$ is a real n-tuple that satisfies

$$0 \leqslant P_j \leqslant P_n\,, \quad j = 1,\dots,n\,, \quad P_n > 0\,,$$
$$P_j = \sum_{i=1}^{j}\rho_i\,, \quad \overline{P}_j = \sum_{i=j}^{n}\rho_i\,, \quad j = 1,\dots,n\,. \tag{1.2}$$

In the sequel we assume without loss of generality that $\rho_j \neq 0,\ j = 1,\dots,n$.

In [3] the following necessary and sufficient conditions for the equality case in Jensen-Steffensen's inequality was stated:

THEOREM A. [3, Theorem 1] *Let $\varphi : I \to \mathbb{R}$ be a strictly convex function. Let $(\zeta_1,\dots,\zeta_n)$ be a monotonic n-tuple in I^n and $(\rho_1,\dots,\rho_n)$ a real n-tuple satisfying* (1.2) *and $\rho_j \neq 0$, $j = 1,\dots,n$, $n \geqslant 2$. Denote $\overline{\zeta} = \frac{1}{P_n}\sum_{i=1}^{n}\rho_i\zeta_i$. Then*

(*a*) *In the case $n = 2$ Jensen-Steffensen's inequality* (1.1) *becomes equality if and only if $\zeta_1 = \zeta_2$.*

(*b*) *In the case $n \geqslant 3$ Jensen-Steffensen's inequality* (1.1) *becomes equality if and only if one of the following two cases occurs:*

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(1) *either* $\overline{\zeta} = \zeta_1$ *or* $\overline{\zeta} = \zeta_n$

(2) *there exists* $k \in \{3, ..., n-2\}$ *such that* $\overline{\zeta} = \zeta_k$ *and*

$$\begin{cases} P_j\left(\zeta_j - \zeta_{j+1}\right) = 0, & j = 1, ..., k-1, \\ \overline{P}_j\left(\zeta_j - \zeta_{j-1}\right) = 0, & j = k+1, ..., n. \end{cases}$$

In [6] Pečarić proved a refinement of Slater's inequality established in [9]. It states (in the discrete case) that under the same conditions leading to Jensen-Steffensen's inequality if $\sum_{i=1}^{n} \rho_i \varphi'\left(\zeta_i\right) \neq 0$ and if

$$M = \frac{\sum_{i=1}^{n} \rho_i \zeta_i \varphi'\left(\zeta_i\right)}{\sum_{i=1}^{n} \rho_i \varphi'\left(\zeta_i\right)} \in I,$$

then

$$\sum_{i=1}^{n} \rho_i \varphi\left(\zeta_i\right) \leqslant P_n \varphi\left(M\right). \tag{1.3}$$

In this article we refine the above theorems and we also refine theorems on majorization that were dealt with in [5] and [7]. These refinements are achieved by using superquadratic functions which were introduced in [1] and [2].

First we quote some definitions and state a list of basic properties of superquadratic functions.

DEFINITION 1. ([1, Definition 2.1]) A function $\varphi : [0, \infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geqslant 0$ there exists a constant $C(x) \in \mathbb{R}$ such that

$$\varphi\left(y\right) - \varphi\left(x\right) - \varphi\left(\left|y - x\right|\right) \geqslant C\left(x\right)\left(y - x\right) \tag{1.4}$$

for all $y \geqslant 0$.

DEFINITION 2. A function $\varphi : [0, \infty) \to \mathbb{R}$ is said to be strictly superquadratic if (1.4) is strict for all $x \neq y$ where $xy \neq 0$.

LEMMA A. ([2, Lemma 2.3]) *Suppose that* φ *is superquadratic. Let* $x_i \geqslant 0$, $i = 1, ..., n$ *and let* $\overline{x} = \sum_{i=1}^{n} \rho_i x_i$ *where* $\rho_i \geqslant 0$ *and* $\sum_{i=1}^{n} \rho_i = 1$. *Then*

$$\sum_{i=1}^{n} \rho_i \varphi\left(x_i\right) - \varphi\left(\overline{x}\right) \geqslant \sum_{i=1}^{n} \rho_i \varphi\left(\left|x_i - \overline{x}\right|\right).$$

LEMMA B. ([1, Lemma 2.2]) *Let* φ *be a superquadratic function with* $C(x)$ *as in Definition* 1. *Then:*

(*i*) $\varphi(0) \leqslant 0$,

(*ii*) *If* $\varphi(0) = \varphi'(0) = 0$, *then* $C(x) = \varphi'(x)$ *whenever* φ *is differentiable at* $x > 0$.

(*iii*) *If* $\varphi \geqslant 0$, *then* φ *is convex and* $\varphi(0) = \varphi'(0) = 0$.

As Hölder's inequality is Jensen's inequality for the convex functions $f(x) = x^p$, $p \geqslant 1$, so the inequalities satisfied by superquadratic functions are extensions of the inequalities satisfied by the special superquadratic functions $f(x) = x^p$, $p \geqslant 2$ (see [1], [2] and [8]).

It is easy to verify that all the results for differentiable nonnegative superquadratic functions also follow when the differentiability condition is removed. As in this case, φ is anyways convex, and therefore all needed is to replace $C(x) = \varphi'(x)$ ($C(x)$ as defined in Definition 1) with $C(x)$ is any value from $\left[\varphi'_{-}(x), \varphi'_{+}(x)\right]$.

COROLLARY 1. *Let φ be a differentiable nonnegative strictly superquadratic function. Then $\varphi(x) > 0$ for $x > 0$, $\varphi'(x) - \varphi'(y) > 0$ for $x > y \geqslant 0$, and φ is strictly convex for $x \geqslant 0$.*

Proof. From Lemma B we know that if $\varphi \geqslant 0$ then $\varphi'(x) \geqslant 0$ for $x \geqslant 0$ and that $C(x) = \varphi'(x)$. Assume that there is a point $y > 0$ such that $\varphi(y) = 0$. Let $0 < x < y$. Then from

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) > \varphi'(x)(y-x)$$

we get that

$$-\varphi(x) - \varphi(|y-x|) > \varphi'(x)(y-x) \geqslant 0,$$

and as it is given that $\varphi(x) \geqslant 0$ this is a contradiction. Therefore $\varphi(x) > 0$ when $x > 0$.

For all $x, y > 0$, such that $x \neq y$, by Definition 1., we have

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) > \varphi'(x)(y-x),$$
$$\varphi(x) - \varphi(y) - \varphi(|y-x|) > \varphi'(y)(x-y).$$

Summing these two inequalities, as $\varphi(x) > 0$ for $x > 0$, we get

$$\left[\varphi'(x) - \varphi'(y)\right](x-y) > 2\varphi(|y-x|) > 0.$$

Hence, for $x > y > 0$ we get that $\varphi'(x) - \varphi'(y) > 0$. As a result φ is also strictly convex for $x \geqslant 0$.

The following Lemma 1 is used in the sequel. This lemma is an immediate consequence of the definition of superquadratic functions.

LEMMA 1. *Let φ be a superquadratic function with $C(x)$ as in Definition 1. Then for all probability measures μ and all non-negative $\mu-$measurable functions f and g*

$$\int \varphi(g(s))\, d\mu(s) - \int \varphi(f(s))\, d\mu(s) \geqslant \int \varphi(|g(s) - f(s)|)\, d\mu(s) + \int C(f(s))\,(g(s) - f(s))\,d\mu(s) \tag{1.5}$$

holds. Moreover, if

$$\int C\left(f\left(s\right)\right)\left(g(s)-f\left(s\right)\right)d\mu(s) \geqslant 0, \tag{1.6}$$

then

$$\int \varphi\left(g(s)\right)d\mu(s) - \int \varphi\left(f\left(s\right)\right)d\mu(s) \geqslant \int \varphi(|g(s)-f\left(s\right)|)d\mu(s).$$

Similarly, if φ *is subquadratic (which means that* $-\varphi$ *is superquadratic), then*

$$\begin{aligned}&\int \varphi\left(g(s)\right)d\mu(s) - \int \varphi\left(f\left(s\right)\right)d\mu(s)\\ &\qquad\leqslant \int \varphi\left(|g(s)-f\left(s\right)|\right)d\mu(s) + \int C\left(f\left(s\right)\right)\left(g(s)-f\left(s\right)\right)d\mu(s),\end{aligned}$$

and if also

$$\int C\left(f\left(s\right)\right)\left(g(s)-f\left(s\right)\right)d\mu(s) \leqslant 0,$$

then

$$\int \varphi\left(g(s)\right)d\mu(s) - \int \varphi\left(f\left(s\right)\right)d\mu(s) \leqslant \int \varphi(|g(s)-f\left(s\right)|)d\mu(s).$$

THEOREM B. [1, Theorem 2.4] *Suppose that* φ *is superquadratic and C is as in Definition* 1. *If* μ *is a probability measure, f is a non-negative* μ *-measurable function such that* $\int C\left(f\left(s\right)\right)d\mu(s) \neq 0$, *then*

$$\begin{aligned}&\varphi(m) + \int \varphi\left(|f\left(s\right)-m|\right)d\mu(s)\\ &\qquad\leqslant \int \varphi\left(f\left(s\right)\right)d\mu(s) \leqslant \varphi\left(M\right) - \int \varphi\left(|f\left(s\right)-M|\right)d\mu(s),\end{aligned}$$

where

$$m = \int f\left(s\right)d\mu(s) \qquad \text{and} \qquad M = \frac{\int f\left(s\right)C\left(f\left(s\right)\right)d\mu(s)}{\int C\left(f\left(s\right)\right)d\mu(s)}.$$

A discrete version of this theorem is also used in the sequel. It can be obtained by choosing a discrete measure μ on $\{1,\ldots,n\}$, defined by $\mu(i) = \rho_i / \sum_{j=1}^{n}\rho_j$, $\rho_i \geqslant 0$, and the function f defined by $f\left(i\right) = \zeta_i$.

REMARK 1. All our results hold also if instead of dealing with a superquadratic function on $[0,\infty)$, we deal with superquadratic functions on $[0,L]$, where x, y, x_i, $g(s)$, $f\left(s\right)$, M, that appear in $\varphi\left(x\right)$, $\varphi\left(y\right)$, $\varphi\left(x_i\right)$, $\varphi\left(g(s)\right)$, $\varphi\left(f\left(s\right)\right)$, $\varphi\left(M\right)$, above and in the sequel are included in $[0,L]$ for which φ is superquadratic.

An example for such $\varphi\left(x\right)$ is

$$\varphi\left(x\right) = \begin{cases} xe^{-\frac{1}{x}}, & x > 0; \\ 0, & x = 0. \end{cases}$$

This function is convex on $[0,\infty)$, but superquadratic only on $[0,L]$, where $0.8955 \leqslant L \leqslant 1$, (see [4]).

2. Jensen-Steffensen's Inequality

The first theorem that we prove here is a refinement of Jensen-Steffensen's inequality for nonnegative superquadratic functions. The proof uses the same technique as in [3]. Steffensen's coefficients in the case $n = 2$, leads always to $\rho_i > 0$, and this case was already dealt with in [1]. Therefore we deal in this chapter and in the next chapter with $n \geqslant 3$.

THEOREM 1. *Let $\varphi : [0,\infty) \to \mathbb{R}$ be differentiable superquadratic and nonnegative, let $\boldsymbol{\zeta}$ be a nonnegative monotonic n-tuple in $\mathbb{R}^n$, and $\boldsymbol{\rho}$ a real n-tuple satisfying Steffensen's coefficients, that is*

$$0 \leqslant P_j \leqslant P_n , \quad j = 1,...,n , \quad P_n > 0,$$

$$P_j = \sum_{i=1}^{j} \rho_i , \quad \overline{P}_j = \sum_{i=j}^{n} \rho_i , \quad j = 1,...,n. \tag{2.1}$$

Let $\overline{\zeta}$ be defined by

$$\overline{\zeta} = \frac{1}{P_n} \sum_{i=1}^{n} \rho_i \zeta_i. \tag{2.2}$$

Then

$$\begin{aligned}
\sum_{i=1}^{n} \rho_i \varphi(\zeta_i) - P_n \varphi\left(\overline{\zeta}\right) &\geqslant \sum_{j=1}^{k-1} P_j \varphi(\zeta_{j+1} - \zeta_j) + P_k \varphi\left(\overline{\zeta} - \zeta_k\right) \\
&\quad + \overline{P}_{k+1} \varphi\left(\zeta_{k+1} - \overline{\zeta}\right) + \sum_{j=k+2}^{n} \overline{P}_j \varphi(\zeta_j - \zeta_{j-1}) \\
&\geqslant \left(\sum_{i=1}^{k} P_i + \sum_{i=k+1}^{n} \overline{P_i}\right) \varphi\left(\frac{\sum_{i=1}^{n} \rho_i \left(\left|\zeta_i - \overline{\zeta}\right|\right)}{\sum_{i=1}^{k} P_i + \sum_{i=k+1}^{n} \overline{P_i}}\right) \\
&\geqslant ((n-1) P_n) \varphi\left(\frac{\sum_{i=1}^{n} \rho_i \left(\left|\zeta_i - \overline{\zeta}\right|\right)}{(n-1) P_n}\right),
\end{aligned} \tag{2.3}$$

where $k \in \{1,...,n-1\}$ satisfies

$$\zeta_k \leqslant \overline{\zeta} \leqslant \zeta_{k+1}. \tag{2.4}$$

In case φ is, additionally, strictly superquadratic,

$$\sum_{i=1}^{n} \rho_i \varphi(\zeta_i) - P_n \varphi\left(\overline{\zeta}\right) > ((n-1) P_n) \varphi\left(\frac{\sum_{i=1}^{n} \rho_i \left(\left|\zeta_i - \overline{\zeta}\right|\right)}{(n-1) P_n}\right)$$

holds for $\boldsymbol{\zeta} > \mathbf{0}$ unless one of the following two cases occurs:

(1) *either* $\overline{\zeta} = \zeta_1$ *or* $\overline{\zeta} = \zeta_n$,

(2) *there exists* $k \in \{3, ..., n-2\}$ *such that* $\overline{\zeta} = \zeta_k$ *and*

$$\begin{cases} P_j\left(\zeta_j - \zeta_{j+1}\right) = 0, & j = 1, ..., k-1, \\ \overline{P}_j\left(\zeta_j - \zeta_{j-1}\right) = 0, & j = k+1, ..., n. \end{cases}$$

In these two cases
$$\sum_{i=1}^{n} \rho_i \varphi\left(\zeta_i\right) - P_n \varphi\left(\overline{\zeta}\right) = 0.$$

Proof. The following identities are used in the proof in a similar way as they were used in [3]:

$$\sum_{i=1}^{n} \rho_i \varphi\left(\zeta_i\right) - P_n \varphi\left(\overline{\zeta}\right) = \sum_{j=1}^{k-1} P_j\left(\varphi\left(\zeta_j\right) - \varphi\left(\zeta_{j+1}\right)\right) + P_k\left(\varphi\left(\zeta_k\right) - \varphi\left(\overline{\zeta}\right)\right)$$
$$+ \overline{P}_{k+1}\left(\varphi\left(\zeta_{k+1}\right) - \varphi\left(\overline{\zeta}\right)\right) + \sum_{j=k+2}^{n} \overline{P}_j\left(\varphi\left(\zeta_j\right) - \varphi\left(\zeta_{j-1}\right)\right). \tag{2.5}$$

In the case $k = 1$ we assume $\sum_{j=1}^{k-1}$ to be 0 , while in the case $k = n-1$ we assume $\sum_{k+2}^{n}$ to be 0.

In particular

$$0 = \sum_{i=1}^{n} \rho_i \zeta_i - P_n \overline{\zeta} = \sum_{j=1}^{k-1} P_j\left(\zeta_j - \zeta_{j+1}\right) + P_k\left(\zeta_k - \overline{\zeta}\right)$$
$$+ \overline{P}_{k+1}\left(\zeta_{k+1} - \overline{\zeta}\right) + \sum_{j=k+2}^{n} \overline{P}_j\left(\zeta_j - \zeta_{j-1}\right) \tag{2.6}$$

and

$$P_n\left(\overline{\zeta} - \zeta_1\right) = \sum_{j=2}^{n} \overline{P}_j\left(\zeta_j - \zeta_{j-1}\right), \quad P_n\left(\zeta_n - \overline{\zeta}\right) = \sum_{j=1}^{n-1} P_j\left(\zeta_{j+1} - \zeta_j\right). \tag{2.7}$$

Without loss of generality we assume that $\boldsymbol{\zeta}$ is increasing, that is, $\zeta_j \leqslant \zeta_{j+1}$, $j = 1, ..., n-1$. It is obvious from (2.7) that in this case $\zeta_1 \leqslant \overline{\zeta} \leqslant \zeta_n$.

As φ is differentiable superquadratic and nonnegative it follows that

$$\varphi(x) - \varphi\left(\overline{\zeta}\right) \geqslant \varphi'\left(\overline{\zeta}\right)\left(x - \overline{\zeta}\right) + \varphi\left(\left|x - \overline{\zeta}\right|\right) \tag{2.8}$$

holds.

From Lemma B we know that φ is also convex increasing. A convex increasing superquadratic function φ satisfies the following inequalities for all $y, z \in [0, \infty)$:

$$\begin{aligned} \varphi(y) - \varphi(z) &\geqslant \varphi'(z)(y - z) + \varphi(|y - z|) \\ &\geqslant \varphi'\left(\overline{\zeta}\right)(y - z) + \varphi(|y - z|), \quad y \leqslant z \leqslant \overline{\zeta}, \\ \varphi(z) - \varphi(y) &\geqslant \varphi'(y)(z - y) + \varphi(|z - y|) \\ &\geqslant \varphi'\left(\overline{\zeta}\right)(z - y) + \varphi(|z - y|), \quad \overline{\zeta} \leqslant y \leqslant z. \end{aligned} \tag{2.9}$$

From (2.8), (2.9), (2.4), (2.5), and (2.1), we get that

$$\begin{aligned}
\sum_{i=1}^{n}\rho_i\varphi\left(\zeta_i\right)-P_n\varphi\left(\overline{\zeta}\right) \geqslant & \left[\sum_{j=1}^{k-1}P_j\varphi'\left(\zeta_{j+1}\right)\left(\zeta_j-\zeta_{j+1}\right)+P_k\varphi'\left(\overline{\zeta}\right)\left(\zeta_k-\overline{\zeta}\right)\right.\\
& \left.+\overline{P}_{k+1}\varphi'\left(\overline{\zeta}\right)\left(\zeta_{k+1}-\overline{\zeta}\right)+\sum_{j=k+2}^{n}\overline{P}_j\varphi'\left(\zeta_{j-1}\right)\left(\zeta_j-\zeta_{j-1}\right)\right]\\
& +\left[\sum_{j=1}^{k-1}P_j\varphi\left(\zeta_{j+1}-\zeta_j\right)+P_k\varphi\left(\overline{\zeta}-\zeta_k\right)+\overline{P}_{k+1}\varphi\left(\zeta_{k+1}-\overline{\zeta}\right)+\sum_{j=k+2}^{n}\overline{P}_j\varphi\left(\zeta_j-\zeta_{j-1}\right)\right]\\
\geqslant & \left[\sum_{j=1}^{k-1}P_j\varphi'\left(\overline{\zeta}\right)\left(\zeta_j-\zeta_{j+1}\right)+P_k\varphi'\left(\overline{\zeta}\right)\left(\zeta_k-\overline{\zeta}\right)\right.\\
& \left.+\overline{P}_{k+1}\varphi'\left(\overline{\zeta}\right)\left(\zeta_{k+1}-\overline{\zeta}\right)+\sum_{j=k+2}^{n}\overline{P}_j\varphi'\left(\overline{\zeta}\right)\left(\zeta_j-\zeta_{j-1}\right)\right]\\
& +\left[\sum_{j=1}^{k-1}P_j\varphi\left(\zeta_{j+1}-\zeta_j\right)+P_k\varphi\left(\overline{\zeta}-\zeta_k\right)+\overline{P}_{k+1}\varphi\left(\zeta_{k+1}-\overline{\zeta}\right)\sum_{j=k+2}^{n}\overline{P}_j\varphi\left(\zeta_j-\zeta_{j-1}\right)\right]. \qquad (2.10)
\end{aligned}$$

Using (2.6) we get from (2.10) that

$$\begin{aligned}
\sum_{i=1}^{n}\rho_i\varphi\left(\zeta_i\right)-P_n\varphi\left(\overline{\zeta}\right) \geqslant & \sum_{j=1}^{k-1}P_j\varphi\left(\zeta_{j+1}-\zeta_j\right)+P_k\varphi\left(\overline{\zeta}-\zeta_k\right)\\
& +\overline{P}_{k+1}\varphi\left(\zeta_{k+1}-\overline{\zeta}\right)+\sum_{j=k+2}^{n}\overline{P}_j\varphi\left(\zeta_j-\zeta_{j-1}\right).
\end{aligned}$$

As φ is convex and $P_j \geqslant 0$, $\overline{P}_j \geqslant 0$, $j=1,...,n$ and $P_k+\overline{P}_{k+1}=P_n>0$, we get that

$$\begin{aligned}
\sum_{i=1}^{n}\rho_i\varphi\left(\zeta_i\right)-P_n\varphi\left(\overline{\zeta}\right) \geqslant & \left(\sum_{j=1}^{k}P_j+\sum_{j=k+1}^{n}\overline{P}_j\right)\times\\
& \times\varphi\left(\frac{\sum_{j=1}^{k-1}P_j\left(\zeta_{j+1}-\zeta_j\right)+P_k\left(\overline{\zeta}-\zeta_k\right)+\overline{P}_{k+1}\left(\zeta_{k+1}-\overline{\zeta}\right)+\sum_{j=k+2}^{n}\overline{P}_j\left(\zeta_j-\zeta_{j-1}\right)}{\sum_{j=1}^{k}P_j+\sum_{j=k+1}^{n}\overline{P}_j}\right)\\
= & \left(\sum_{j=1}^{k}P_j+\sum_{j=k+1}^{n}\overline{P}_j\right)\varphi\left(\frac{\sum_{i=1}^{n}\rho_i\left(\left|\zeta_i-\overline{\zeta}\right|\right)}{\sum_{j=1}^{k}P_j+\sum_{j=k+1}^{n}\overline{P}_j}\right). \qquad (2.11)
\end{aligned}$$

The identity in (2.11) follows from (2.7).

Since φ is convex with $\varphi(0)=0$, it yields that $\frac{\varphi(x)}{x}$ is increasing and $x\varphi\left(\frac{1}{x}\right)$ is decreasing on $(0,\infty)$. Therefore, from

$$\sum_{j=1}^{k} P_j+\sum_{j=k+1}^{n}\overline{P}_j\leqslant(n-1)P_n$$

we have that

$$\left(\sum_{j=1}^{k} P_j+\sum_{j=k+1}^{n}\overline{P}_j\right)\varphi\left(\frac{\sum_{i=1}^{n}\rho_i\left(\left|\zeta_i-\overline{\zeta}\right|\right)}{\sum_{j=1}^{k}P_j+\sum_{j=k+1}^{n}\overline{P}_j}\right)\geqslant((n-1)P_n)\,\varphi\left(\frac{\sum_{i=1}^{n}\rho_i\left(\left|\zeta_i-\overline{\zeta}\right|\right)}{(n-1)P_n}\right)$$

holds and hence we get (2.3).

The assertion on the equality case is proved using the very detailed proof in Chapter 2 in [3] that leads to Theorem A.

The equality case follows since for differentiable strictly superquadratic φ we have strict inequality in (1.4) when $y\neq x, yx\neq 0$ for $C(x)=\varphi'(x)$. In this case we get from (2.5) as $P_k+\overline{P}_k=P_n>0$ that the first inequality in (2.10) is strict for $\boldsymbol{\zeta}>\mathbf{0}$ unless (1) or (2) (as appeared in the statement of the theorem) holds.

The fact that the second inequality in (2.10) is strict unless (1) or (2) holds is obtained from Corollary 1 that states that $\varphi'(y)-\varphi'(x)>0,\ y>x>0$ and that $\varphi'(x)>0$ and from (2.8) and (2.9). These two strict inequalities in (2.10) lead to strict inequality in the first inequality in (2.3).

Hence, from the strict inequality (2.3) we get unless (1) or (2) holds that for differentiable nonnegative strictly superquadratic function $\varphi(x)$

$$\sum_{i=1}^{n}\rho_i\varphi(\zeta_i)-P_n\varphi\left(\overline{\zeta}\right)>((n-1)P_n)\,\varphi\left(\frac{\sum_{i=1}^{n}\rho_i\left(\left|\zeta_i-\overline{\zeta}\right|\right)}{(n-1)P_n}\right)$$

holds.

In our case $\varphi(0)=0$, therefore, when (1) or (2) holds, we get that

$$\sum_{j=1}^{k-1}P_j\varphi(\zeta_{j+1}-\zeta_j)+P_k\varphi\left(\overline{\zeta}-\zeta_k\right)+\overline{P}_{k+1}\varphi\left(\zeta_{k+1}-\overline{\zeta}\right)+\sum_{j=k+2}^{n}\overline{P}_j\varphi(\zeta_j-\zeta_{j-1})=0.$$

This means that in case (1) or (2)

$$\sum_{i=1}^{n}\rho_i\varphi(\zeta_i)-P_n\varphi\left(\overline{\zeta}\right)=0.$$

Hence the proof of the Theorem 1 is complete.

REMARK 2. In the case $\rho_i\geqslant 0,\ i=1,...,n,$ we have

$$\sum_{j=1}^{k}P_j+\sum_{j=k+1}^{n}\overline{P}_j\leqslant\max\{k,n-k\}P_n. \tag{2.12}$$

Since the function $x\varphi\left(\frac{1}{x}\right)$ is decreasing (as mentioned in the proof of Theorem 1), we get the following refinement of the last inequality in (2.3):

$$\left(\sum_{j=1}^{k} P_j + \sum_{j=k+1}^{n} \overline{P}_j\right) \varphi\left(\frac{\sum_{i=1}^{n} \rho_i \left|\zeta_i - \overline{\zeta}\right|}{\sum_{j=1}^{k} P_j + \sum_{j=k+1}^{n} \overline{P}_j}\right)$$
$$\geqslant \left(\max\{k, n-k\} P_n\right) \varphi\left(\frac{\sum_{i=1}^{n} \rho_i \left|\zeta_i - \overline{\zeta}\right|}{\max\{k, n-k\} P_n}\right).$$

REMARK 3. Under the conditions of Theorem 1, in the case $\rho_i \geqslant 0$, $i = 1, ..., n$, from Theorem B (its discrete version) and the convexity of φ we have

$$\sum_{i=1}^{n} \rho_i \varphi(\zeta_i) - P_n \varphi\left(\overline{\zeta}\right) \geqslant \sum_{i=1}^{n} \rho_i \varphi\left(\left|\zeta_i - \overline{\zeta}\right|\right) \geqslant P_n \varphi\left(\frac{\sum_{i=1}^{n} \rho_i \left|\zeta_i - \overline{\zeta}\right|}{P_n}\right).$$

Since the function $x\varphi\left(\frac{1}{x}\right)$ is decreasing and

$$\sum_{i=1}^{n} \rho_i = P_n \leqslant \sum_{j=1}^{k} P_j + \sum_{j=k+1}^{n} \overline{P}_j \tag{2.13}$$

holds, we get that

$$P_n \varphi\left(\frac{\sum_{i=1}^{n} \rho_i \left|\zeta_i - \overline{\zeta}\right|}{P_n}\right) \geqslant \left(\sum_{j=1}^{k} P_k + \sum_{j=k+1}^{n} \overline{P}_j\right) \varphi\left(\frac{\sum_{i=1}^{n} \rho_i \left(\left|\zeta_i - \overline{\zeta}\right|\right)}{\sum_{j=1}^{k} P_j + \sum_{j=k+1}^{n} \overline{P}_j}\right).$$

Therefore, in this case Theorem 1 is weaker than the left hand side inequality in Theorem B (its discrete version).

3. Slater-Pečarić's Inequality

We prove in the following theorem a Slater-Pečarić type of inequality for Steffensen's coefficients $\boldsymbol{\rho} = (\rho_1, ..., \rho_n)$ and for nonnegative superquadratic functions. The proof uses the same technique as used in [6].

THEOREM 2. *Let $\varphi : [0, \infty) \to \mathbb{R}$ be a differentiable nonnegative superquadratic function. Let $\boldsymbol{\rho} = (\rho_1, ..., \rho_n)$ be a real n-tuple satisfying (2.1) and $\boldsymbol{\zeta} = (\zeta_1, ..., \zeta_n)$ be a nonnegative increasing n-tuple. If $\sum_{i=1}^{n} \rho_i \varphi'(\zeta_i) \neq 0$ we define*

$$M = \frac{\sum_{i=1}^{n} \rho_i \zeta_i \varphi'(\zeta_i)}{\sum_{i=1}^{n} \rho_i \varphi'(\zeta_i)},$$

then,

Case A. for s *satisfying* $\zeta_s \leqslant M \leqslant \zeta_{s+1},\ s+1 \leqslant n$*,*

$$\begin{aligned}\sum_{i=1}^{n} \rho_i \varphi(\zeta_i) &\leqslant P_n \varphi(M) - \left(\sum_{j=1}^{s-1} P_j \varphi(\zeta_{j+1} - \zeta_j) + \right. \\ &\qquad \left. + P_s \varphi(M - \zeta_s) + \overline{P}_{s+1} \rho(\zeta_{s+1} - M) + \sum_{j=s+2}^{n} \overline{P}_j \varphi(\zeta_j - \zeta_{j-1}) \right) \\ &\leqslant P_n \varphi(M) - \left(\sum_{j=1}^{s} P_j + \sum_{j=s+1}^{n} \overline{P}_j \right) \varphi \left(\frac{\sum_{i=1}^{n} \rho_i |\zeta_i - M|}{\sum_{j=1}^{s} P_j + \sum_{j=s+1}^{n} \overline{P}_j} \right) \\ &\leqslant P_n \varphi(M) - ((n-1) P_n) \varphi \left(\frac{\sum_{i=1}^{n} \rho_i |\zeta_i - M|}{(n-1) P_n} \right). \end{aligned} \tag{3.1}$$

If φ *is also strictly superquadratic, then the inequality*

$$\sum_{i=1}^{n} \rho_i \varphi(\zeta_i) < P_n \varphi(M) - ((n-1) P_n) \varphi \left(\frac{\sum_{i=1}^{n} \rho_i |\zeta_i - M|}{(n-1) P_n} \right)$$

holds for $\boldsymbol{\zeta} > \mathbf{0}$ *unless one of the following two cases occurs:*
$(1')$ $M = \zeta_1$*,*
$(2')$ *there exists* $s \in \{3, ..., n\}$ *such that* $M = \zeta_s$ *and*

$$\begin{cases} P_j (\zeta_j - \zeta_{j+1}) = 0, & j = 1, ..., s-1, \\ \overline{P}_j (\zeta_j - \zeta_{j-1}) = 0, & j = s+1, ..., n. \end{cases}$$

In these two cases

$$\sum_{i=1}^{n} \rho_i \varphi(\zeta_i) = P_n \varphi(M).$$

Case B. for $M > \zeta_n$*,*

$$\sum_{i=1}^{n} \rho_i \varphi(\zeta_i) \leqslant P_n \varphi(M) - (nP_n) \varphi \left(\frac{\sum_{i=1}^{n} \rho_i |\zeta_i - M|}{nP_n} \right). \tag{3.1*}$$

When φ *is also strictly superquadratic,* $(3.1*)$ *is strict.*

Proof. First, from Lemma B we get that as φ is superquadratic nonnegative, it is increasing and convex too. It was proved in [3] that when $\boldsymbol{\rho}$ is satisfying (2.1) and $\boldsymbol{\zeta}$ is increasing then

$$\zeta_1 \leqslant \frac{\sum_{i=1}^{n} \rho_i \zeta_i}{\sum_{i=1}^{n} \rho_i} = \overline{\zeta} \leqslant \zeta_n$$

holds.

Similarly, as $\varphi(\zeta)$ is increasing and convex and as $\boldsymbol{\zeta}$ is nonnegative increasing we get that $\sum_{i=1}^{n}\rho_i\varphi'(\zeta_i) > 0$, $\sum_{i=1}^{n}\rho_i\zeta_i\varphi'(\zeta_i) \geqslant 0$, and that

$$\zeta_1 \leqslant \frac{\sum_{i=1}^{n}\rho_i\zeta_i\varphi'(\zeta_i)}{\sum_{i=1}^{n}\rho_i\varphi'(\zeta_i)} = M$$

holds too.

Case A. For $\zeta_1 \leqslant M \leqslant \zeta_n$, we want to get a lower bound, to $P_n\varphi(M) - \sum_{i=1}^{n}\rho_i\varphi(\zeta_i)$. For this we use the following identity for $k \in \{1, ..., n-1\}$:

$$\begin{aligned} P_n\varphi(M) - \sum_{i=1}^{n}\rho_i\varphi(\zeta_i) &= \sum_{j=1}^{k-1}P_j(\varphi(\zeta_{j+1}) - \varphi(\zeta_j)) + P_k(\varphi(M) - \varphi(\zeta_k)) \\ &+ \overline{P}_{k+1}(\varphi(M) - \varphi(\zeta_{k+1})) + \sum_{j=k+2}^{n}\overline{P}_j(\varphi(\zeta_{j-1}) - \varphi(\zeta_j)). \end{aligned} \tag{3.2}$$

As $P_j \geqslant 0, \overline{P}_j \geqslant 0, j = 1, ..., n,$ and φ is superquadratic we get from Lemma B that

$$\begin{aligned} &\sum_{j=1}^{k-1}P_j(\varphi(\zeta_{j+1}) - \varphi(\zeta_j)) + P_k(\varphi(M) - \varphi(\zeta_k)) \\ &\quad + \overline{P}_{k+1}(\varphi(M) - \varphi(\zeta_{k+1})) + \sum_{j=k+2}^{n}\overline{P}_j(\varphi(\zeta_{j-1}) - \varphi(\zeta_j)) \\ &\geqslant \sum_{j=1}^{k-1}P_j\varphi'(\zeta_j)(\zeta_{j+1} - \zeta_j) + P_k\varphi'(\zeta_k)(M - \zeta_k) + \overline{P}_{k+1}\varphi'(\zeta_{k+1})(M - \zeta_{k+1}) \\ &+ \sum_{j=k+2}^{n}\overline{P}_j\varphi'(\zeta_j)(\zeta_{j-1} - \zeta_j) + \sum_{j=1}^{k-1}P_j\varphi(|\zeta_{j+1} - \zeta_j|) + P_k\varphi(|M - \zeta_k|) \\ &+ \overline{P}_{k+1}\varphi(|\zeta_{k+1} - M|) + \sum_{j=k+2}^{n}\overline{P}_j\varphi(|\zeta_j - \zeta_{j-1}|). \end{aligned} \tag{3.3}$$

We choose now $k = s$ to be such that $\zeta_s \leqslant M \leqslant \zeta_{s+1}$. Once we show that under our conditions

$$\begin{aligned} &\sum_{j=1}^{s-1}P_j\varphi'(\zeta_j)(\zeta_{j+1} - \zeta_j) + P_s\varphi'(\zeta_s)(M - \zeta_s) \\ &\quad + \overline{P}_{s+1}\varphi'(\zeta_{s+1})(M - \zeta_{s+1}) + \sum_{j=s+2}^{n}\overline{P}_j\varphi'(\zeta_j)(\zeta_{j-1} - \zeta_j) \geqslant 0\,, \end{aligned} \tag{3.4}$$

then from (3.2), (3.3), (3.4) and from the convexity of φ we get that

$$
\begin{aligned}
&P_n\varphi(M)-\sum_{i=1}^{n}\rho_i\varphi(\zeta_i)\\
&\quad\geqslant\sum_{j=1}^{s-1}P_j\varphi(|\zeta_{j+1}-\zeta_j|)+P_s\varphi|M-\zeta_s|+\overline{P}_{s+1}\varphi(|\zeta_{s+1}-M|)+\sum_{j=s+2}^{n}\overline{P}_j\varphi(|\zeta_j-\zeta_{j-1}|)\\
&\quad\geqslant\left(\sum_{j=1}^{s}P_j+\sum_{j=s+1}^{n}\overline{P}_j\right)\times\\
&\times\varphi\left(\frac{\sum_{j=1}^{s-1}P_j(\zeta_{j+1}-\zeta_j)+P_s(M-\zeta_s)+\overline{P}_{s+1}(\zeta_{s+1}-M)+\sum_{j=s+2}^{n}\overline{P}_j(\zeta_j-\zeta_{j-1})}{\sum_{j=1}^{s}P_j+\sum_{j=s+1}^{n}\overline{P}_j}\right)\\
&\quad=\left(\sum_{j=1}^{s}P_j+\sum_{j=s+1}^{n}\overline{P}_j\right)\varphi\left(\frac{\sum_{i=1}^{n}\rho_i|\zeta_i-M|}{\sum_{j=1}^{s}P_j+\sum_{j=s+1}^{n}\overline{P}_j}\right).
\end{aligned}
$$

Since $\rho_1=P_1$, $\rho_n=\overline{P}_n$, and $\rho_j=P_j-P_{j-1}=\overline{P}_j-\overline{P}_{j+1}$, $j=2,\ldots,n-1$, the last equality follows from the identities:

$$
\begin{aligned}
\sum_{j=1}^{s}\rho_j(M-\zeta_j)&=\sum_{j=1}^{s-1}P_j(\zeta_{j+1}-\zeta_j)+P_s(M-\zeta_s)\ ,\\
\sum_{j=s+1}^{n}\rho_j(\zeta_j-M)&=\overline{P}_{s+1}(\zeta_{s+1}-M)+\sum_{j=s+2}^{n}\overline{P}_j(\zeta_j-\zeta_{j-1})\ .
\end{aligned}
$$

In the same way as in the proof of the Theorem 1, we conclude that

$$
\left(\sum_{j=1}^{s}P_j+\sum_{j=s+1}^{n}\overline{P}_j\right)\varphi\left(\frac{\sum_{j=1}^{n}\rho_i(|\zeta_i-M|)}{\sum_{j=1}^{s}P_j+\sum_{j=s+1}^{n}\overline{P}_j}\right)\geqslant((n-1)P_n)\,\varphi\left(\frac{\sum_{i=1}^{n}\rho_i(|\zeta_i-M|)}{(n-1)P_n}\right)
$$

holds and hence we get (3.1).

In order to complete the proof of (3.1) we only need to prove (3.4). For this we use the following two identities:

$$
\begin{aligned}
\sum_{i=1}^{n}\rho_i\varphi'(\zeta_i)=&\sum_{j=1}^{s-1}P_j\left(\varphi'(\zeta_j)-\varphi'(\zeta_{j+1})\right)+P_s\varphi'(\zeta_s)\\
&+\overline{P}_{s+1}\varphi'(\zeta_{s+1})+\sum_{j=s+2}^{n}\overline{P}_j\left(\varphi'(\zeta_j)-\varphi'(\zeta_{j-1})\right)
\end{aligned}\tag{3.5}
$$

and

$$\sum_{i=1}^{n}\rho_i\varphi'(\zeta_i)\zeta_i=\sum_{j=1}^{s-1}P_j\left(\varphi'(\zeta_j)\zeta_j-\varphi'(\zeta_{j+1})\zeta_{j+1}\right)+P_s\varphi'(\zeta_s)\zeta_s$$
$$+\overline{P}_{s+1}\varphi'(\zeta_{s+1})\zeta_{s+1}+\sum_{j=s+2}^{n}\overline{P}_j\left(\varphi'(\zeta_j)\zeta_j-\varphi'(\zeta_{j-1})\zeta_{j-1}\right). \tag{3.6}$$

In the case $s=1$ we assume $\sum_{j=1}^{s-1}$ to be 0, and in case $s=n-1$ we assume $\sum_{s+2}^{n}$ to be 0.

Since
$$M\sum_{i=1}^{n}\rho_i\varphi'(\zeta_i)=\sum_{i=1}^{n}\rho_i\varphi'(\zeta_i)\zeta_i,$$
by substituting (3.5) and (3.6) into the left side of (3.4) we get

$$\sum_{j=1}^{s-1}P_j\varphi'(\zeta_j)(\zeta_{j+1}-\zeta_j)+P_s\varphi'(\zeta_s)(M-\zeta_s)$$
$$+\overline{P}_{s+1}\varphi'(\zeta_{s+1})(M-\zeta_{s+1})+\sum_{j=s+2}^{n}\overline{P}_j\varphi'(\zeta_j)(\zeta_{j-1}-\zeta_j)$$
$$=\sum_{j=1}^{s-1}P_j(M-\zeta_{j+1})\left(\varphi'(\zeta_{j+1})-\varphi'(\zeta_j)\right)$$
$$+\sum_{j=s+2}^{n}\overline{P}_j(\zeta_{j-1}-M)\left(\varphi'(\zeta_j)-\varphi'(\zeta_{j-1})\right)\geqslant 0. \tag{3.7}$$

The last inequality holds because φ is convex and $\zeta_s\leqslant M\leqslant\zeta_{s+1}$. Therefore (3.4) is satisfied which is the reason for (3.1) to hold for nonnegative superquadratic functions $\varphi(\zeta)$ which according to Lemma B are also increasing and convex.

The proof of the assertions on the equality case in the theorem follow similarly to the proofs of the equality cases in Theorem 1.

Case B for $M>\zeta_n$ is proved similarly to Case A.

Hence the proof of Theorem 2 is complete.

Similarly, as in Remark 2 and Remark 3, we can observe the following:

REMARK 4. Since in the case $\rho_i\geqslant 0,\ i=1,...,n,$ (2.12) holds and the function $x\varphi\left(\frac{1}{x}\right)$ is decreasing, we get the following refinement of the last inequality in (3.1):

$$P_n\varphi(M)-\left(\sum_{j=1}^{s}P_j+\sum_{j=s+1}^{n}\overline{P}_j\right)\varphi\left(\frac{\sum_{i=1}^{n}\rho_i|\zeta_i-M|}{\sum_{j=1}^{s}P_j+\sum_{j=s+1}^{n}\overline{P}_j}\right)$$
$$\leqslant P_n\varphi(M)-(\max\{s,n-s\}P_n)\varphi\left(\frac{\sum_{i=1}^{n}\rho_i|\zeta_i-M|}{\max\{s,n-s\}P_n}\right).$$

REMARK 5. Under the conditions of Theorem 2 together with the restriction that $\rho_i\geqslant 0,\ i=1,...,n,$ from Theorem B (its discrete version) and the convexity of φ we have

$$P_n\varphi(M)-\sum_{i=1}^{n}\rho_i\varphi(\zeta_i)\geqslant\sum_{i=1}^{n}\rho_i\varphi(|\zeta_i-M|)\geqslant P_n\varphi\left(\frac{\sum_{i=1}^{n}\rho_i|\zeta_i-M|}{P_n}\right).$$

Since $x\varphi\left(\frac{1}{x}\right)$ is decreasing, from (2.13) we get

$$P_n\varphi\left(\frac{\sum_{i=1}^{n}\rho_i|\zeta_i-M|}{P_n}\right)\geqslant\left(\sum_{j=1}^{s}P_j+\sum_{j=s+1}^{n}\overline{P}_j\right)\varphi\left(\frac{\sum_{i=1}^{n}\rho_i(|\zeta_i-M|)}{\sum_{j=1}^{s}P_j+\sum_{j=s+1}^{n}\overline{P}_j}\right),$$

so we conclude that, in this case, Theorem 2 is weaker than the right hand side inequality proven in Theorem B (its discrete version).

Summerizing Theorem 1 and Theorem 2 we get the following theorem:

THEOREM 3. *Under the conditions of Theorem 1 and Theorem 2, when $M\leqslant\zeta_n$,*

$$P_n\varphi\left(\overline{\zeta}\right)+(n-1)P_n\varphi\left(\frac{\sum_{i=1}^{n}\rho_i\left(\left|\zeta_i-\overline{\zeta}\right|\right)}{(n-1)P_n}\right)\leqslant\sum_{i=1}^{n}\rho_i\varphi(\zeta_i)$$
$$\leqslant P_n\varphi(M)-(n-1)P_n\varphi\left(\frac{\sum_{i=1}^{n}\rho_i(|\zeta_i-M|)}{(n-1)P_n}\right)\qquad(3.8)$$

holds.

If $\boldsymbol{\zeta}>\mathbf{0}$ and φ is also strictly superquadratic, strict inequality holds to the left of $\sum_{i=1}^{n}\rho_i\varphi(\zeta_i)$ in (3.8) unless one of the following two cases occurs:

(1) *either $\overline{\zeta}=\zeta_1$ or $\overline{\zeta}=\zeta_n$,*

(2) *there exists $k\in\{3,...,n-2\}$ such that $\overline{\zeta}=\zeta_k$ and*

$$\begin{cases}P_j(\zeta_j-\zeta_{j+1})=0, & j=1,...,k-1,\\ \overline{P}_j(\zeta_j-\zeta_{j-1})=0, & j=k+1,...,n.\end{cases}$$

When (1) or (2) occurs,

$$P_n\varphi\left(\overline{\zeta}\right)=\sum_{i=1}^{n}\rho_i\varphi(\zeta_i).$$

Strict inequality holds to the right of $\sum_{i=1}^{n}\rho_i\varphi(\zeta_i)$ in (3.8) unless one of the following two cases occurs:

(1′) $M=\zeta_1$,

(2′) *there exists $s\in\{3,...,n\}$ such that $M=\zeta_s$ and*

$$\begin{cases}P_j(\zeta_j-\zeta_{j+1})=0, & j=1,...,s-1,\\ \overline{P}_j(\zeta_j-\zeta_{j-1})=0, & j=s+1,...,n.\end{cases}$$

When (1′) or (2′) occurs,

$$\sum_{i=1}^{n}\rho_i\varphi(\zeta_i)=P_n\varphi(M).$$

4. Majorization theorems

In the sequel we use the following version of Lemma 1.

LEMMA 2. *Let* $w > 0$, f *and* g *be nonnegative integrable functions on* $[a,b]$. *Suppose that* $\varphi(x)$ *is a differentiable superquadratic function on* $[0,\infty)$ *satisfying* $\varphi(0) = \varphi'(0) = 0$. *Then, for* $a \leqslant \xi \leqslant b$

$$\int_a^\xi (\varphi(g(s)) - \varphi(f(s)))\, w(s)ds \geqslant \int_a^\xi \varphi(|g(s) - f(s)|)\, w(s)ds + \int_a^\xi \frac{d\varphi(f(s))}{df}(g(s) - f(s))\, w(s)ds \tag{4.1}$$

holds. Moreover, if

$$\int_a^\xi \frac{d\varphi(f(s))}{ds}(g(s) - f(s))\, w(s)ds \geqslant 0 \tag{4.2}$$

then,

$$\int_a^\xi (\varphi(g(s)) - \varphi(f(s)))\, w(s)ds \geqslant \int_a^\xi \varphi(|g(s) - f(s)|)\, w(s)ds. \tag{4.3}$$

Similarly, if φ *is differentiable and subquadratic on* $[0,\infty)$, *and* $C(x) = \varphi'(x)$, *then*

$$\int_a^\xi (\varphi(g(s)) - \varphi(f(s)))\, w(s)ds \leqslant \int_a^\xi \frac{d\varphi(f(s))}{df}(g(s) - f(s))\, w(s)ds + \int_a^\xi \varphi(|g(s) - f(s)|)\, w(s)ds. \tag{4.4}$$

If also

$$\int_a^\xi \frac{d\varphi(f(s))}{df}(g(s) - f(s))\, w(s)ds \leqslant 0, \tag{4.5}$$

then

$$\int_a^\xi (\varphi(g(s)) - \varphi(f(s)))\, w(s)ds \leqslant \int_a^\xi \varphi(|g(s) - f(s)|)\, w(s)ds. \tag{4.6}$$

Proof. According to Lemma B, if $\varphi(x)$ is superquadratic and differentiable satisfying $\varphi(0) = \varphi'(0) = 0$, then $C(x) = \varphi'(x)$. Hence Lemma 2 follows from Lemma 1.

Now we state conditions for which (4.2) holds leading to (4.3) for superquadratic functions. Analogously we state conditions for which (4.5) holds leading to (4.6) for subquadratic functions.

The results follow by using the identity

$$\begin{aligned} R(\xi) &= \int_a^\xi \frac{d\varphi(f(s))}{df}(g(s) - f(s))\, w(s)ds \\ &= \frac{d\varphi}{df}(f(\xi))\, H(\xi) - \int_a^\xi H(s)d\varphi'(f(s)), \end{aligned} \tag{4.7}$$

where $H(\xi)$ is defined by

$$H(\xi) = \int_a^{\xi} (g(s) - f(s))\, w(s)ds. \tag{4.8}$$

In [5] Maligranda, Pečarić and Persson, and Pečarić and Abramovich in [7], established conditions for which $R(\xi) \geqslant 0$ and conditions for $R(\xi) \leqslant 0$. There the sign of $R(\xi)$ was used to prove inequalities related to convex or concave functions. Here the conditions leading to the sign of $R(\xi)$ are used to prove inequalities related to superquadratic functions and to subquadratic functions.

LEMMA D. ([5, Theorem 2] and [7, Lemma 2]) *Let $w > 0$, f and g be integrable functions on $[a,b]$. Suppose that $\varphi(x)$ is differentiable on an interval $I \supseteq f([a,b])$. Let $R(\xi)$ and $H(\xi)$ be as in (4.7) and (4.8). Then we get that*

$$R(b) \geqslant 0, \quad (R(b) \leqslant 0)$$

when

$$H(b) = 0$$

in each of the cases (a) - (d) where

(a) *φ is convex (concave) on $I \supseteq f([a,b])$, $H(t) \geqslant 0$, $a \leqslant t \leqslant b$, $f(t)$ decreases on $a \leqslant t \leqslant b$.*

(b) *φ is convex (concave) on $I \supseteq f([a,b])$, $H(t) \leqslant 0$, $a \leqslant t \leqslant b$, $f(t)$ increases on $a \leqslant t \leqslant b$.*

(c) *φ is concave (convex) on $I \supseteq f([a,b])$, $H(t) \geqslant 0$, $a \leqslant t \leqslant b$, $f(t)$ increases on $a \leqslant t \leqslant b$.*

(d) *φ is concave (convex) on $I \supseteq f([a,b])$, $H(t) \leqslant 0$, $a \leqslant t \leqslant b$, $f(t)$ decreases on $a \leqslant t \leqslant b$.*

If we replace the condition $H(b) = 0$ by monotonicity conditions on φ, then we get that $R(\xi) \geqslant 0$, $(R(\xi) \leqslant 0)$, for every ξ, $a \leqslant \xi \leqslant b$:

(a') *if φ increases (decreases) on $I \supseteq f([a,b])$ in addition to condition (a) or condition (c);*

(b') *if φ decreases (increases) on $I \supseteq f([a,b])$ in addition to condition (b) or condition (d).*

From Lemma 2 and Lemma D we get the following immediate results which refine inequalities given in [5] and [7].

THEOREM 4. *Let $w > 0$ f, g be nonnegative integrable functions on $[a,b]$. Suppose that $\varphi(x)$ is a differentiable superquadratic (subquadratic) function on $[0,\infty)$, satisfying $C(x) = \varphi'(x)$ ($C(x)$ is as in Definition 1). Then inequality (4.3), (inequality (4.6)) holds for $\xi = b$ in cases (a) - (d) in Lemma D, and for every ξ, $a < \xi \leqslant b$ in cases (a') - (b') there.*

In other words, for instance, under the conditions of Lemma D (a) and under the condition of Lemma D (b) it was proved in [5] and [7] that for φ convex on $I \supseteq f([a,b])$ the inequality

$$\int_a^b (\varphi(g(s)) - \varphi(f(s)))\, w(s)ds \geqslant 0 \tag{4.9}$$

holds.

However, if φ is also superquadratic on $[0,\infty)$, we get whenever

$$\int_a^b \varphi\left(|g(s)-f(s)|\right)w(s)ds \geqslant 0 \tag{4.10}$$

the "tighter inequality"

$$\int_a^b \left(\varphi\left(g(s)\right)-\varphi(f(s))\right)w(s)ds \geqslant \int_a^b \varphi\left(|g(s)-f(s)|\right)w(s)ds. \tag{4.11}$$

Also, if φ is superquaratic and convex on $[0,\infty)$, by using also Jensen's inequality we get

$$\begin{aligned}\int_a^b \left(\varphi\left(g(s)\right)-\varphi(f(s))\right)w(s)ds &\geqslant \int_a^b \varphi\left(|g(s)-f(s)|\right)w(s)ds\\ &\geqslant \left(\int_a^b w(s)ds\right)\varphi\left(\frac{\int_a^b |g(s)-f(s)|\,w(s)\,ds}{\int_a^b w(s)ds}\right).\end{aligned} \tag{4.12}$$

This gives a better inequality then (4.9) whenever (4.10) holds, especially when φ is nonnegative superquadratic.

If φ is convex and subquadratic we get under the conditions of Lemma D (c) that (4.6) for $\zeta=b$ holds.

On the other hand, if φ is superquadratic and also concave, we get that (4.11) holds too.

In the following we establish some examples:

Let $w>0$ f, g be nonnegative integrable functions on $[a,b]$. If f/g is decreasing, then it is easy to verify that

$$H(\xi)=\int_a^\xi \left(g(s)\cdot Z-f(s)\right)w(s)ds \leqslant 0\,, \qquad a\leqslant \xi \leqslant b,$$

where

$$Z=\frac{\int_a^b f(t)w(t)dt}{\int_a^b g(t)w(t)dt}\,,$$

and if f/g is increasing then

$$H(\xi)=\int_a^\xi \left(g(s)\cdot Z-f(s)\right)w(s)ds \geqslant 0\,, \qquad a\leqslant \xi \leqslant b\,.$$

Therefore Theorem 4 holds in cases (a), (c) and (a') of Lemma D, when f/g is increasing, and in cases (b), (d) and (b') of Lemma D, Theorem 4 holds when f/g is decreasing.

In particular for a nonnegative concave function $f(x)$, $f(x)/x$ decreases (of course there are nonconcave functions for which $f(x)/x$ decreases), and for nonnegative convex function $f(x)$ satisfying $f(0)=0$, $f(x)/x$ increases.

From these observations, we get the following examples of Theorem 4:

EXAMPLE 1. Let $w > 0$, f be nonnegative, integrable and increasing on $[a,b] \subseteq [0,\infty)$ and let $f(x)/x$ be decreasing on $[a,b]$. Using Theorem 4, if φ is differentiable superquadratic on $[0,\infty)$ and convex on $[f(a), f(b)]$, then

$$\int_a^b (\varphi(Z \cdot s) - \varphi(f(s)))\, w(s) ds \geqslant \int_a^b \varphi(|Zs - f(s)|)\, w(s)\, ds \tag{4.13}$$

holds, where

$$Z = \frac{\int_a^b f(s) w(s) ds}{\int_a^b s w(s) ds}.$$

Moreover, if φ is superquadratic and convex on $[0,\infty)$, then

$$\begin{aligned} \int_a^b (\varphi(Z \cdot s) - \varphi(f(s)))\, w(s) ds &\geqslant \int_a^b \varphi(|Zs - f(s)|)\, w(s) ds \\ &\geqslant \left(\int_a^b w(s) ds\right) \varphi\left(\frac{\int_a^b (|Zs - f(s)|)\, w(s) ds}{\int_a^b w(s) ds}\right). \end{aligned} \tag{4.14}$$

For instance, the function $\varphi(x) = 2x^2 \log x - 3x^2$, $x > 0$, $\varphi(0) = 0$ is superquadratic on $x \geqslant 0$ and convex on $x \geqslant 1$. Therefore, under the above assumptions on f, if $[f(a), f(b)] \subset [1,\infty)$, we get for our φ that (4.13) holds. If instead we choose φ to be $\varphi(x) = x^p$, $p \geqslant 2$ then (4.14) holds.

EXAMPLE 2. Let $w > 0$, g be nonnegative and integrable on $[a,b] \subseteq [0,\infty)$. Let $g(x)/x$ be decreasing on $[a,b]$ and $\int_a^b (g(x) - x)\, w(x)\, dx = 0$.
If φ is concave on $[a,b]$ and φ is differentiable superquadratic on $[0,\infty)$, then from Theorem 4 we get (4.11).

The following illustrates this case:
The function $\varphi(x) = 2x^2 \log x - 3x^2$, $x > 0$, $\varphi(0) = 0$ is superquadratic on $[0,\infty)$ and concave on $[0,1]$. Let

$$g(x) = \begin{cases} e^{2+n}, & 0 \leqslant x \leqslant \dfrac{1}{2e^{2+n}}; \\ 0, & \dfrac{1}{2e^{2+n}} \leqslant x \leqslant 1. \end{cases}$$

Then for a constant n which is large enough

$$\int_0^1 \varphi(|g(x) - x|)\, dx > \frac{e^n(1+4n)}{2}.$$

Hence as $\int_0^1 (g(x) - x)\, dx = 0$ and $\int_0^\xi (g(x) - x)\, dx \geqslant 0$, $0 \leqslant \xi \leqslant 1$ we get for the chosen $\varphi(x)$, $g(x)$ and $f(x) = x$ that

$$\int_0^1 (\varphi(g(x)) - \varphi(x))\, dx \geqslant \int_0^1 \varphi(|g(x) - x|)\, dx > \frac{e^n(1+4n)}{2} > 0.$$

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S. Abramovich
Department of Mathematics
University of Haifa
Haifa, 31905
Israel
e-mail: abramos@math.haifa.ac.il

S. Banić
Faculty of civil engineering and architecture
University of Split
Matice hrvatske 15
21000 Split
Croatia
e-mail: Senka.Banic@gradst.hr

M. Matić
Department of Mathematics
Faculty of Natural Sciences, Mathematics and Education
University of Split
Teslina 12
21000 Split
Croatia
e-mail: mmatic@fesb.hr

J. Pečarić
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6
10000 Zagreb
Croatia
e-mail: pecaric@hazu.hr