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THE REVERSE HARDY INEQUALITY WITH MEASURES

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Abstract. In this paper we characterize the validity of the inequalities

$$\|g\|_{p,(a,b),\lambda} \leqslant c \left\| u(x) \int_{(a,x)} g(y)\, d\mu \right\|_{q,(a,b),\nu}$$

and

$$\|g\|_{p,(a,b),\lambda} \leqslant c \left\| u(x) \int_{(x,b)} g(y)\, d\mu \right\|_{q,(a,b),\nu}$$

for non-negative Borel measurable functions g on the interval $(a,b) \subseteq \mathbb{R}$, where $0 < p \leqslant 1$, $0 < q \leqslant +\infty$, λ, μ and ν are non-negative Borel measures on (a,b), and u is a weight function on (a,b).

1. Introduction

In [8] G. H. Hardy proved the following celebrated inequality: Let $1 < p < +\infty$ and f a non-negative measurable function on $(0,+\infty)$. Then, if $\varepsilon < 1/p' = 1 - 1/p$,

$$\int_0^{+\infty} \left(x^{\varepsilon-1} \int_0^x f(t)\, dt \right)^p dx \leqslant c \int_0^{+\infty} \left(x^{\varepsilon} f(x) \right)^p dx \tag{1.1}$$

for some constant c independent of f. If $\varepsilon > 1/p'$, the inequality takes the form

$$\int_0^{+\infty} \left(x^{\varepsilon-1} \int_x^{+\infty} f(t)\, dt \right)^p dx \leqslant c \int_0^{+\infty} \left(x^{\varepsilon} f(x) \right)^p dx. \tag{1.2}$$

The best possible constants c in (1.1) and (1.2) are equal and this common value was determined by E. Landau in [10] as

$$c = |\varepsilon - 1/p'|^{-p}. \tag{1.3}$$

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In [3] G. A. Bliss established the inequality

$$\left(\int_0^{+\infty}\left(x^{-\frac{1}{q}-\frac{1}{p'}}\int_0^x f(t)\,dt\right)^q dx\right)^{\frac{1}{q}} \leqslant c\left(\int_0^{+\infty} f(x)^p\,dx\right)^{\frac{1}{p}}$$

for $1<p<q<+\infty$ and proved that the best possible constant is

$$c=\left(\frac{p'r^r}{q}\right)^{1/q}\left[B\left(\frac{1}{r},\frac{q-1}{r}\right)\right]^{-r/q},$$

where $r=q/p-1$ and B is the classical beta function.

During the last two decades, many authors have considered extensions of the form

$$\left(\int_a^b\left(w(x)\int_a^x f(t)\,dt\right)^q dx\right)^{\frac{1}{q}} \leqslant c\left(\int_a^b (v(x)f(x))^p\,dx\right)^{\frac{1}{p}} \tag{1.4}$$

and

$$\left(\int_a^b\left(w(x)\int_x^b f(t)\,dt\right)^q dx\right)^{\frac{1}{q}} \leqslant c\left(\int_a^b (v(x)f(x))^p\,dx\right)^{\frac{1}{p}}, \tag{1.5}$$

with $-\infty \leqslant a<b \leqslant +\infty$, w,v weights on (a,b), $0<q\leqslant+\infty$, $1\leqslant p\leqslant+\infty$. The weights w and v for which (1.4) and (1.5) hold for all non-negative f have been completely characterized. The solution of this problem (under different assumptions on p and q) is associated with the names M. Artola, J. S. Bradley, V. Kokilashvili, V. G. Maz'ja, B. Muckenhoupt, A. L. Rozin, E. Sawyer, G. Sinnamon, G. Talenti, G. Tomaselli and others. We refer to [13] and [9] for a survey of results.

In [16] E. Sawyer noted that if $0<p<1$, then the inequalities (1.4) and (1.5) hold only for trivial weights. This observations led to a study of the so-called reverse Hardy inequalities

$$\left(\int_a^b\left(w(x)\int_a^x f(t)\,dt\right)^q dx\right)^{\frac{1}{q}} \geqslant c\left(\int_a^b (v(x)f(x))^p\,dx\right)^{\frac{1}{p}} \tag{1.6}$$

and

$$\left(\int_a^b\left(w(x)\int_x^b f(t)\,dt\right)^q dx\right)^{\frac{1}{q}} \geqslant c\left(\int_a^b (v(x)f(x))^p\,dx\right)^{\frac{1}{p}} \tag{1.7}$$

in the case $0<p\leqslant 1$.

However, these are integral forms of inequalities first considered by E. T. Copson in [4, 5] for infinite series; such reverse inequalities for infinite series were also investigated by G. Bennett [2] and K.-G. Grosse-Erdmann [7]. Conditions on the weights w, v, which are either necessary or sufficient for (1.6) and (1.7) to hold when $0<q\leqslant p\leqslant 1$ were established by P. R. Beesack and H. P. Heinig [1]. Discrete analogues of (1.6) and (1.7) were proved in [7], where it is also remarked that the techniques used in the proofs may be applicable to the continuous versions of the inequalities, namely to

(1.6) and (1.7). No estimates of the constants c are mentioned in [7]. In the case that $0 < p, q < 1$ the characterization of inequalities (1.6) and (1.7) was given in [14].

In this paper we make a comprehensive study of general inequalities of the form

$$\|gw\|_{p,(a,b),\mu} \leqslant c \left\| u(x) \int_{(a,x)} g(y)\, d\mu \right\|_{q,(a,b),\nu} \tag{1.8}$$

and

$$\|gw\|_{p,(a,b),\mu} \leqslant c \left\| u(x) \int_{(x,b)} g(y)\, d\mu \right\|_{q,(a,b),\nu} \tag{1.9}$$

involving non-negative Borel measures μ and ν, with complete proofs and estimates for c, provided that $0 < p \leqslant 1$ and $0 < q \leqslant +\infty$. In addition to the extra generality and the filling of gaps in previous works on these inequalities, this approach unifies the continuous and discrete problems, so that the integral and series inequalities follow as particular cases. As in [7], our method is based on a discretization of function norms. The general inequalities (involving three non-negative Borel measures λ, μ and ν) mentioned in the Abstract of this paper are reduced either to (1.8) or to (1.9).

The paper is organized as follows. We start with notation and preliminary results in Section 2. General discretization formulae of weighted function norms are given in Section 3 while necessary and sufficient conditions for the validity of the inequality (1.8) or (1.9) can be found in Section 5 or in Section 4, respectively. Finally, in Section 6 we show that the results from Sections 4 and 5 can be used to characterize the validity of inequalities mentioned at the Abstract of this paper.

2. Notation and preliminaries

Throughout the paper we assume that $I := (a,b) \subseteq \mathbb{R}$. Let μ be a non-negative Borel measure on I. We denote by $B^+(I)$ the set of all non-negative Borel measurable functions on I. If E is a nonempty Borel measurable subset of I and f is a Borel measurable function on E, then we put

$$\|f\|_{p,E,\mu} := \left(\int_E |f(y)|^p\, d\mu \right)^{\frac{1}{p}}, \qquad 0 < p < +\infty,$$
$$\|f\|_{\infty,E,\mu} := \sup\{\alpha : \mu\{y \in E : |f(y)| \geqslant \alpha\} > 0\};$$

the symbol χ_E stands for the characteristic function of the set E. In the notation $\|f\|_{p,E,\mu}$, $0 < p \leqslant +\infty$, we omit the symbol μ if μ is the Lebesgue measure on I.

By $A \lesssim B$ we mean that $A \leqslant CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B *are equivalent*.

We put

$$p' := \begin{cases} \frac{p}{1-p} & \text{if} \quad 0 < p < 1, \\ +\infty & \text{if} \quad p = 1, \\ \frac{p}{p-1} & \text{if} \quad 1 < p < +\infty, \\ 1 & \text{if} \quad p = +\infty, \end{cases}$$

and $1/(\pm\infty)=0$, $0/0=0$, $0\cdot(\pm\infty)=0$ and $\overline{\mathbb{Z}}=\mathbb{Z}\cup\{-\infty,+\infty\}$.

DEFINITION 2.1. Let $N,M\in\overline{\mathbb{Z}}$, $N<M$. A positive non-increasing sequence $\{\tau_k\}_{k=N}^{M}$ is called *almost geometrically decreasing* if there are $\alpha\in(1,+\infty)$ and $L\in\mathbb{N}$ such that

$$\tau_k\leqslant\frac{1}{\alpha}\tau_{k-L}\quad\text{for all}\quad k\in\{N+L,\dots,M\}.$$

A positive non-decreasing sequence $\{\sigma_k\}_{k=N}^{M}$ is called *almost geometrically increasing* if there are $\alpha\in(1,+\infty)$ and $L\in\mathbb{N}$ such that

$$\sigma_k\geqslant\alpha\sigma_{k-L}\quad\text{for all}\quad k\in\{N+L,\dots,M\}.$$

REMARK 2.2. Definition 2.1 implies that if $0<q<+\infty$, then following three statements are equivalent:

(i) $\{\tau_k\}_{k=N}^{M}$ is an almost geometrically decreasing sequence;
(ii) $\{\tau_k^q\}_{k=N}^{M}$ is an almost geometrically decreasing sequence;
(iii) $\{\tau_k^{-q}\}_{k=N}^{M}$ is an almost geometrically increasing sequence.

Let $\emptyset\neq\mathcal{Z}\subseteq\overline{\mathbb{Z}}$, $0<q\leqslant+\infty$ and let $\{w_k\}=\{w_k\}_{k\in\mathcal{Z}}$ be a sequence of positive numbers. We denote by $\ell^q(\{w_k\},\mathcal{Z})$ the following discrete analogue of a weighted Lebesgue space: if $0<q<+\infty$, then

$$\ell^q(\{w_k\},\mathcal{Z})=\Big\{\{a_k\}_{k\in\mathcal{Z}}:\quad\|a_k\|_{\ell^q(\{w_k\},\mathcal{Z})}:=\Big(\sum_{k\in\mathcal{Z}}|a_kw_k|^q\Big)^{\frac{1}{q}}<+\infty\Big\}$$

and

$$\ell^\infty(\{w_k\},\mathcal{Z})=\Big\{\{a_k\}_{k\in\mathcal{Z}}:\quad\|a_k\|_{\ell^\infty(\{w_k\},\mathcal{Z})}:=\sup_{k\in\mathcal{Z}}|a_kw_k|<+\infty\Big\}.$$

If $w_k=1$ for all $k\in\mathcal{Z}$, we write simply $\ell^q(\mathcal{Z})$ instead of $\ell^q(\{w_k\},\mathcal{Z})$.

Given two (quasi-)Banach spaces X and Y, we write $X\hookrightarrow Y$ if $X\subset Y$ and if the natural embedding of X in Y is continuous.

We quote some known results. Proofs can be found in [11] and [12].

LEMMA 2.3. *Let $N,M\in\overline{\mathbb{Z}}$, $N\leqslant M$. Then, for any positive sequence $\{\tau_k\}_{k=N}^{M}$ and all $m\in\overline{\mathbb{Z}}$ satisfying $N<m<M$,*

$$\sum_{k=m}^{M}\tau_k\lesssim\tau_m \tag{2.1}$$

or

$$\sum_{k=N}^{m}\tau_k\lesssim\tau_m \tag{2.2}$$

if and only if the sequence $\{\tau_k\}_{k=N}^{M}$ is almost geometrically decreasing or increasing, respectively.

LEMMA 2.4. *Let* $q \in (0,+\infty]$, $N,M \in \overline{\mathbb{Z}}$, $N \leqslant M$, $\mathcal{Z} = \{N,N+1,\dots,M-1,M\}$ *and let* $\{\tau_k\}_{k=N}^{M}$ *be an almost geometrically decreasing sequence. Then*

$$\left\| \tau_k \sum_{m=N}^{k} a_m \right\|_{\ell^q(\mathcal{Z})} \approx \|\tau_k a_k\|_{\ell^q(\mathcal{Z})} \tag{2.3}$$

and

$$\|\tau_k \sup_{N\leqslant m\leqslant k} a_m\|_{\ell^q(\mathcal{Z})} \approx \|\tau_k a_k\|_{\ell^q(\mathcal{Z})} \tag{2.4}$$

for all non-negative sequences $\{a_k\}_{k=N}^{M}$.

LEMMA 2.5. *Let* $q \in (0,+\infty]$, $N \leqslant M$, $N,M \in \overline{\mathbb{Z}}$, $\mathcal{Z} = \{N,N+1,\dots,M-1,M\}$ *and let* $\{\sigma_k\}_{k=N}^{M}$ *be an almost geometrically increasing sequence. Then*

$$\left\| \sigma_k \sum_{m=k}^{M} a_m \right\|_{\ell^q(\mathcal{Z})} \approx \|\sigma_k a_k\|_{\ell^q(\mathcal{Z})} \tag{2.5}$$

and

$$\|\sigma_k \sup_{k\leqslant m\leqslant M} a_m\|_{\ell^q(\mathcal{Z})} \approx \|\sigma_k a_k\|_{\ell^q(\mathcal{Z})} \tag{2.6}$$

for all non-negative sequences $\{a_k\}_{k=N}^{M}$.

The following two lemmas are discrete versions of the classical Landau resonance theorems. Proofs can be found, for example, in [6].

LEMMA 2.6. *Let* $0 < p \leqslant q \leqslant +\infty$, $\emptyset \neq \mathcal{Z} \subseteq \overline{\mathbb{Z}}$ *and let* $\{v_k\}_{k\in\mathcal{Z}}$ *and* $\{w_k\}_{k\in\mathcal{Z}}$ *be two sequences of positive numbers. Assume that*

$$\ell^p(\{v_k\},\mathcal{Z}) \hookrightarrow \ell^q(\{w_k\},\mathcal{Z}). \tag{2.7}$$

Then

$$\|\{w_k v_k^{-1}\}\|_{\ell^\infty(\mathcal{Z})} \leqslant C, \tag{2.8}$$

where C *stands for the norm of the embedding* (2.7).

LEMMA 2.7. *Let* $0 < q < p \leqslant +\infty$, $\emptyset \neq \mathcal{Z} \subseteq \overline{\mathbb{Z}}$ *and let* $\{v_k\}_{k\in\mathcal{Z}}$ *and* $\{w_k\}_{k\in\mathcal{Z}}$ *be two sequences of positive numbers. Assume that* (2.7) *holds. Then*

$$\|\{w_k v_k^{-1}\}\|_{\ell^r(\mathcal{Z})} \leqslant C, \tag{2.9}$$

where $1/r := 1/q - 1/p$ *and* C *stands for the norm of the embedding* (2.7).

3. Discretization of function norms

In this section we define a discretizing sequence for a non-negative, non-decreasing, finite and right-continuous function φ on $(a,b) \subseteq \mathbb{R}$. We use this sequence to discretize function norms, more precisely, we find discrete norms equivalent to the original ones.

If φ is a non-negative and monotone function on (a,b), then by $\varphi(a)$ and $\varphi(b)$ we mean the values $\varphi(a+) := \lim_{t\to a+} \varphi(t)$ and $\varphi(b-) := \lim_{t\to b-} \varphi(t)$, respectively.

LEMMA 3.1. *Let φ be a non-negative, non-decreasing, finite and right-continuous function on (a,b). There is a strictly increasing sequence $\{x_k\}_{k=N}^{M+1}$, $-\infty \leqslant N \leqslant M \leqslant +\infty$, with elements from the closure of the interval (a,b), such that:*

(i) if $N > -\infty$, then $\varphi(x_N) > 0$ and $\varphi(x) = 0$ for every $x \in (a,x_N)$; if $M < +\infty$, then $x_{M+1} = b$;

(ii) $\varphi(x_{k+1}-) \leqslant 2\varphi(x_k)$ if $N \leqslant k \leqslant M$;

(iii) $2\varphi(x_k-) \leqslant \varphi(x_{k+1})$ if $N < k < M$.

Proof. Define the sets A_k by

$$A_k = \left\{t \in (a,b) : 2^k \leqslant \varphi(t) < 2^{k+1}\right\}, \quad k \in \mathbb{Z}. \tag{3.1}$$

Let $\{A_{m_k}\}_{k=N}^{M}$ be the maximal subsequence of $\{A_k\}_{k\in\mathbb{Z}}$ which contains only nonempty sets and let $x_k = \inf A_{m_k}$. The assumptions on φ and (3.1) imply that the sequence $\{x_k\}_{k=N}^{M}$ is strictly increasing. Moreover, if $N > -\infty$, then $\varphi(x_N) > 0$ and $\varphi(x) = 0$ for every $x \in (a,x_N)$. If $M < +\infty$, then $x_M < b$ and we put $x_{M+1} = b$.

By the right continuity of φ,

$$2^{m_k} \leqslant \varphi(x_k) < 2^{m_k+1} \quad \text{if} \quad N \leqslant k \leqslant M. \tag{3.2}$$

If $x_k \leqslant t < x_{k+1}$, then $t \in A_{m_k}$. Together with (3.2), this implies that

$$\varphi(t) < 2^{m_k+1} \leqslant 2\varphi(x_k)$$

and (ii) follows. Similarly, if $x_{k-1} \leqslant t < x_k$, then

$$2\varphi(t) < 2 \cdot 2^{m_{k-1}+1} \leqslant 2 \cdot 2^{m_k} = 2^{m_k+1} \leqslant 2^{m_{k+1}} \leqslant \varphi(x_{k+1})$$

and (iii) follows. □

DEFINITION 3.2. Let φ be a non-negative, non-decreasing, finite and right-continuous function on (a,b). A strictly increasing sequence $\{x_k\}_{k=N}^{M+1}$, $-\infty \leqslant N < M \leqslant +\infty$, is said to be *a discretizing sequence of the function* φ if it satisfies the conditions $(i)-(iii)$ of Lemma 3.1.

REMARK 3.3. We shall use the following *convention*: if $N = -\infty$, then we put $x_N = \lim_{k\to-\infty} x_k$. It is clear that if $N = -\infty$ and $x_N > a$, then $\varphi(x) = 0$ for all $x \in (a,x_N)$ (cf. condition (i) of Lemma 3.1).

THEOREM 3.4. *Let ν be a non-negative Borel measure on $I = (a,b)$ such that the function $\varphi(t) = \nu(a,t]$ is finite on I. If $\{x_k\}_{k=N}^{M+1}$ is a discretizing sequence of the function φ, then*

$$\int_{(a,b)} h(t)\, d\nu(t) \approx \sum_{k=N}^{M} h(x_k)\nu(a,x_k] \tag{3.3}$$

for all non-negative and non-increasing functions h on I.

Proof. It is easy to see that function φ is right-continuous on (a,b). Since

$$\lim_{t\to a+}\varphi(t)=\lim_{n\to+\infty}\varphi(a+1/n)=\lim_{n\to+\infty}\nu(a,a+1/n]$$
$$=\nu\left(\bigcap_n (a,a+1/n]\right)=\nu(\emptyset)=0,$$

the discretizing sequence $\{x_k\}_{k=N}^{M+1}$ of the function φ satisfies $N=-\infty$ or $x_n>a$. Moreover, by the construction of the sequence $\{x_k\}_{k=N}^{M+1}$, (cf. conditions (ii) and (iii) of Lemma 3.1),

$$\nu(a,x_{k+1})\leqslant 2\nu(a,x_k] \qquad \text{if} \quad N\leqslant k\leqslant M, \tag{3.4}$$

and

$$\nu(a,x_{k-1})\leqslant \frac{1}{2}\nu(a,x_k] \qquad \text{if} \quad N<k-1<M. \tag{3.5}$$

Using (3.5), we obtain

$$\nu(a,x_k]=\nu(a,x_{k-1})+\nu[x_{k-1},x_k]\leqslant \frac{1}{2}\nu(a,x_k]+\nu[x_{k-1},x_k].$$

Consequently,

$$\nu(a,x_k]\leqslant 2\nu[x_{k-1},x_k] \quad \text{if} \quad N<k-1<M. \tag{3.6}$$

Applying the equality $\nu(a,x_N)=0$ and (3.4), we arrive at

$$\int_{(a,b)}h(t)\,d\nu(t)=\sum_{k=N}^{M}\int_{[x_k,x_{k+1})}h(t)\,d\nu(t)\leqslant\sum_{k=N}^{M}h(x_k)\nu(a,x_{k+1})\leqslant 2\sum_{k=N}^{M}h(x_k)\nu(a,x_k]$$

for all non-negative and non-increasing functions h on I. To prove the reverse estimate, we distinguish two cases.

First assume that $M-N\leqslant 1$. Consequently,

$$\int_{(a,b)}h(t)\,d\nu(t)\geqslant\frac{1}{2}\sum_{k=N}^{M}\int_{(a,x_k]}h(t)\,d\nu(t)\geqslant\frac{1}{2}\sum_{k=N}^{M}h(x_k)\nu(a,x_k],$$

which is the desired reverse estimate.

Suppose now that $N+1<M$. Using the estimate

$$\chi_{(a,x_N]}(t)+\chi_{(a,x_{N+1}]}(t)+\sum_{k=N+2}^{M}\chi_{[x_{k-1},x_k]}(t)\leqslant 2 \quad \text{for all} \quad t\in I$$

and the monotonicity of h, we obtain

$$\int_{(a,b)}h(t)\,d\nu(t)\geqslant\frac{1}{2}\left(\int_{(a,x_N]}h(t)\,d\nu(t)+\int_{(a,x_{N+1}]}h(t)\,d\nu(t)+\sum_{k=N+2}^{M}\int_{[x_{k-1},x_k]}h(t)\,d\nu(t)\right)$$
$$\geqslant\frac{1}{2}\left(h(x_N)\nu(a,x_N]+h(x_{N+1})\nu(a,x_{N+1}]+\sum_{k=N+2}^{M}h(x_k)\nu[x_{k-1},x_k]\right).$$

Now we apply (3.6) to get

$$\int_{(a,b)} h(t)\,d\nu(t) \geqslant \frac{1}{4}\sum_{k=N}^{M} h(x_k)\nu(a,x_k],$$

and the result follows. $\square$

We shall need an analogue of Theorem 3.4, where $L^1(\nu)$-norm is replaced by a weighted $L^\infty(\nu)$-norm. But there is a substantial difference between these two cases. While the function $\varphi(t) := \nu(a,t]$, $t \in I$, corresponding to the former case is right-continuous on I, the function

$$\varphi(t) := \|u\|_{\infty,(a,t],\nu}, \quad t \in I, \quad \text{with} \quad u \in B^+(I), \tag{3.7}$$

cannot be right-continuous on I. (To see it, let $I = (0,2)$, $u = \chi_{(0,1]} + 2\chi_{(1,2)}$ and let ν be the Lebesgue measure on I. Then $\varphi(1) = 1$ but $\varphi(1+) = 2$). Therefore, in the following theorem we consider the function φ defined by

$$\varphi(t) = \|u\|_{\infty,(a,t+],\nu} := \lim_{s\to t+} \|u\|_{\infty,(a,s],\nu}, \quad t \in I, \tag{3.8}$$

instead of φ given by (3.7). Note also that the assumptions on h are more restrictive there.

THEOREM 3.5. *Let ν be a non-negative Borel measure on $I = (a,b)$ and let $u \in B^+(I)$ be such that the function $\|u\|_{\infty,(a,t],\nu} < +\infty$ for all $t \in I$. If $\{x_k\}_{k=N}^{M+1}$ is a discretizing sequence of the function $\varphi(t) = \|u\|_{\infty,(a,t+],\nu}$, $t \in I$, then*

$$\|hu\|_{\infty,(a,b),\nu} \approx \sup_{N\leqslant k\leqslant M} h(x_k)\|u\|_{\infty,(a,x_k+],\nu} \tag{3.9}$$

for all non-negative, non-increasing and right-continuous functions h on I.

Proof. Since, cf. Lemma 3.1 and Remark 3.3,

$$\|u\|_{\infty,(a,x_N),\nu} = 0 \quad \text{when} \quad x_N > a,$$

and

$$\|u\|_{\infty,(a,x_{k+1}),\nu} \leqslant 2\|u\|_{\infty,(a,x_k+],\nu} \quad \text{if} \quad N \leqslant k \leqslant M,$$

we obtain

$$\begin{aligned}\|uh\|_{\infty,(a,b),\nu} &\leqslant \sup_{N\leqslant k\leqslant M} \|uh\|_{\infty,[x_k,x_{k+1}),\nu}\\ &\leqslant \sup_{N\leqslant k\leqslant M} h(x_k)\|u\|_{\infty,[x_k,x_{k+1}),\nu}\\ &\leqslant 2\sup_{N\leqslant k\leqslant M} h(x_k)\|u\|_{\infty,(a,x_k+],\nu}.\end{aligned}$$

The reverse estimate is obvious since the properties of h imply that

$$\begin{aligned}\|uh\|_{\infty,(a,b),\nu} &\geqslant \sup_{N\leqslant k\leqslant M} \|uh\|_{\infty,(a,x_k+],\nu}\\ &\geqslant \sup_{N\leqslant k\leqslant M} h(x_k)\|u\|_{\infty,(a,x_k+],\nu}. \qquad \square\end{aligned}$$

Let φ be a non-negative, non-decreasing, finite and right-continuous function on (a,b). Using a discretizing sequence $\{x_k\}_{k=N}^{M+1}$ of φ, we define the sequence of intervals $\{J_k\}_{k=N}^{M}$ as follows:

$$J_i = (x_i, x_{i+1}], \quad \text{if} \quad N \leqslant i < M, \quad \text{and} \quad J_M = (x_M, b) \quad \text{if} \quad M < +\infty. \tag{3.10}$$

COROLLARY 3.6. *Let $0 < q < +\infty$. Suppose that μ and ν are non-negative Borel measures on $I = (a,b)$. Let ν be such that the function $\varphi(t) = \nu(a,t]$ is finite on (a,b). If $\{x_k\}_{k=N}^{M+1}$ is a discretizing sequence of φ, then*

$$\left\| \int_{(x,b)} g\, d\mu \right\|_{q,I,\nu} \approx \left(\sum_{k=N}^{M} \left(\int_{J_k} g\, d\mu \right)^q \nu(a,x_k] \right)^{\frac{1}{q}} \tag{3.11}$$

and

$$\big\| \|g\|_{\infty,(x,b),\mu} \big\|_{q,I,\nu} \approx \left(\sum_{k=N}^{M} \|g\|_{\infty,J_k,\mu}^q \, \nu(a,x_k] \right)^{\frac{1}{q}} \tag{3.12}$$

for all $g \in B^+(I)$, where $\{J_k\}_{k=N}^{M}$ is defined by (3.10).

Proof. We prove (3.11) only (the proof of (3.12) is analogous). By Theorem 3.4,

$$\left\| \int_{(x,b)} g\, d\mu \right\|_{q,I,\nu} \approx \left(\sum_{k=N}^{M} \left(\int_{(x_k,b)} g\, d\mu \right)^q \nu(a,x_k] \right)^{\frac{1}{q}}$$
$$= \left(\sum_{k=N}^{M} \left(\sum_{i=k}^{M} \int_{J_k} g\, d\mu \right)^q \nu(a,x_k] \right)^{\frac{1}{q}}.$$

The condition (iii) of Lemma 3.1 implies that $\{\nu(a,x_k]\}_{k=N}^{M}$ is an almost geometrically increasing sequence. (We can take $\alpha = L = 2$ in Definition 2.1. Indeed, by the monotonicity of φ and the condition (iii) of Lemma 3.1, $2\varphi(x_{k-1}) \leqslant 2\varphi(x_k-) \leqslant \varphi(x_{k+1})$ if $N < k < M$, and, on putting $k-1 = m-2$, we arrive at $2\varphi(x_{m-2}) \leqslant \varphi(x_m)$ if $N+2 \leqslant m \leqslant M$.) Thus $\{\nu(a,x_k]^{\frac{1}{q}}\}_{k=N}^{M}$ is also an almost geometrically increasing sequence and (3.11) follows on applying Lemma 2.5. □

COROLLARY 3.7. *Suppose that μ and ν are non-negative Borel measures on $I = (a,b)$. Let $u \in B^+(I)$ be such that the function $\varphi(t) = \|u\|_{\infty,(a,t],\nu}$ is finite on (a,b). If $\{x_k\}_{k=N}^{M+1}$ is a discretizing sequence of the function $\varphi(t) = \|u\|_{\infty,(a,t+],\nu}$, $t \in I$, then*

$$\left\| u(x) \int_{(x,b)} g\, d\mu \right\|_{\infty,I,\nu} \approx \sup_{N \leqslant k \leqslant M} \left(\int_{J_k} g\, d\mu \right) \|u\|_{\infty,(a,x_k+],\nu} \tag{3.13}$$

and

$$\big\| u(x) \|g\|_{\infty,(x,b),\mu} \big\|_{\infty,I,\nu} \approx \sup_{N \leqslant k \leqslant M} \|g\|_{\infty,J_k,\mu} \|u\|_{\infty,(a,x_k],\nu} \tag{3.14}$$

for all $g \in B^+(I)$, where $\{J_k\}_{k=N}^{M}$ is defined by (3.10).

Proof. This follows from Theorem 3.5 and Lemma 2.5. □

4. The reverse Hardy inequality

In this section we characterize the validity of the inequality

$$\|gw\|_{p,(a,b),\mu} \leqslant c \left\| u(x) \int_{(x,b)} g(y)\,d\mu \right\|_{q,(a,b),\nu}, \quad g \in B^+(I). \tag{4.1}$$

Our first result concerns the case when $0 < q \leqslant p \leqslant 1$.

THEOREM 4.1. *Assume that* $0 < q \leqslant p \leqslant 1$. *Let* μ *and* ν *be non-negative Borel measures on* $I = (a,b) \subseteq \mathbb{R}$. *Let* $w \in B^+(I)$ *and let* $u \in B^+(I)$ *satisfy* $\|u\|_{q,(a,t],\nu} < +\infty$ *for all* $t \in I$. *Then the inequality* (4.1) *holds for all* $g \in B^+(I)$ *if and only if*

$$A_1 := \sup_{x \in (a,b)} \|w\|_{p',(a,x],\mu} \|u\|_{q,(a,x),\nu}^{-1} < +\infty.$$

The best possible constant c *in* (4.1) *satisfies* $c \approx A_1$.

Proof. Let $0 < q \leqslant 1$. By Corollary 3.6,

$$\left\| u(x) \int_{(x,b)} g(y)\,d\mu \right\|_{q,(a,b),\nu} \approx \left(\sum_{k=N}^{M} \left(\int_{J_k} g\,d\mu \right)^q \|u\|_{q,(a,x_k],\nu}^q \right)^{\frac{1}{q}} \tag{4.2}$$

for all $g \in B^+(I)$, where $\{x_k\}_{k=N}^{M+1}$ is a discretizing sequence of the function $\varphi(t) = \|u\|_{q,(a,t],\nu}^q$, $t \in (a,b)$, and $\{J_k\}_{k=N}^{M}$ is defined by (3.10). By Lemma 3.1 (cf. also Remark 3.3),

$$\text{if} \quad x_N > a, \quad \text{then} \quad \|u\|_{q,(a,x_N),\nu} = 0; \tag{4.3}$$
$$\text{if} \quad M < +\infty, \quad \text{then} \quad x_{M+1} = b;$$

$$\|u\|_{q,(a,x_{k+1}),\nu}^q \leqslant 2\|u\|_{q,(a,x_k],\nu}^q \quad \text{if} \quad N \leqslant k \leqslant M; \tag{4.4}$$

$$2\|u\|_{q,(a,x_k),\nu}^q \leqslant \|u\|_{q,(a,x_{k+1}],\nu}^q \quad \text{if} \quad N < k < M. \tag{4.5}$$

Assume that $A_1 < +\infty$. This condition and (4.3) imply that

$$\|w\|_{p',(a,x_N],\mu} = 0 \quad \text{if} \quad x_N > a. \tag{4.6}$$

Consequently, $w = 0$ μ-a.e. in $(a,x_N]$ if $x_N > a$, which yields $\|w\|_{p,(a,x_N],\mu} = 0$ if $x_N > a$. Therefore,

$$\|gw\|_{p,(a,b),\mu} = \left(\sum_{k=N}^{M} \|gw\|_{p,J_k,\mu}^p \right)^{\frac{1}{p}} \quad \text{for any} \quad g \in B^+(I). \tag{4.7}$$

By Hölder's inequality (with the exponents $1/p$ and p'/p),

$$\|gw\|_{p,J_k,\mu}^p \leqslant \|g\|_{1,J_k,\mu}^p \|w\|_{p',J_k,\mu}^p, \quad N \leqslant k \leqslant M. \tag{4.8}$$

Identity (4.7) and inequality (4.8) give

$$\|gw\|_{p,(a,b),\mu} \leqslant \left(\sum_{k=N}^{M} \|g\|_{1,J_k,\mu}^{p} \|w\|_{p',J_k,\mu}^{p}\right)^{\frac{1}{p}}$$
$$\leqslant \left(\sup_{N\leqslant k\leqslant M} \|w\|_{p',J_k,\mu} \|u\|_{q,(a,x_k],\nu}^{-1}\right) \left(\sum_{k=N}^{M} \|g\|_{1,J_k,\mu}^{p} \|u\|_{q,(a,x_k],\nu}^{p}\right)^{\frac{1}{p}}.$$

Moreover, using the inequality $0 < q/p \leqslant 1$ and (4.2), we arrive at

$$\|gw\|_{p,(a,b),\mu} \leqslant \left(\sup_{N\leqslant k\leqslant M} \|w\|_{p',J_k,\mu} \|u\|_{q,(a,x_k],\nu}^{-1}\right) \left(\sum_{k=N}^{M} \|g\|_{1,J_k,\mu}^{q} \|u\|_{q,(a,x_k],\nu}^{q}\right)^{\frac{1}{q}}$$
$$\approx \left(\sup_{N\leqslant k\leqslant M} \|w\|_{p',J_k,\mu} \|u\|_{q,(a,x_k],\nu}^{-1}\right) \left\| u(x) \int_{(x,b)} g\, d\mu \right\|_{q,(a,b),\nu}. \tag{4.9}$$

Applying (4.4), we get

$$\sup_{N\leqslant k\leqslant M} \|w\|_{p',J_k,\mu} \|u\|_{q,(a,x_k],\nu}^{-1} \leqslant 2^{\frac{1}{q}} \sup_{N\leqslant k\leqslant M} \|w\|_{p',(a,x_{k+1}]\cap I,\mu} \|u\|_{q,(a,x_{k+1}),\nu}^{-1}$$
$$\leqslant 2^{\frac{1}{q}} A_1. \tag{4.10}$$

The inequality (4.1) (with $c \lesssim A_1$) follows from (4.9) and (4.10).

We now prove necessity. The validity of the inequality (4.1) on $B^+(I)$ and (4.2) imply that

$$\left(\sum_{k=N}^{M} \|gw\|_{p,J_k,\mu}^{p}\right)^{\frac{1}{p}} \lesssim c \left(\sum_{k=N}^{M} \left(\int_{J_k} g\, d\mu\right)^{q} \|u\|_{q,(a,x_k],\nu}^{q}\right)^{\frac{1}{q}} \tag{4.11}$$

for all $g \in B^+(I)$.

Let $g_k \in B^+(I)$, $N \leqslant k \leqslant M$, be functions that saturate Hölder's inequality (4.8), that is, functions satisfying

$$\sup g_k \subset J_k, \quad \|g_k\|_{1,J_k,\mu} = 1 \quad \text{and} \quad \|g_k w\|_{p,J_k,\mu}^{p} \geqslant \frac{1}{2} \|w\|_{p',J_k,\mu}^{p}. \tag{4.12}$$

Then we define the test function g by

$$g = \sum_{k=N}^{M} a_k\, g_k, \tag{4.13}$$

where $\{a_k\}$ is a sequence of non-negative numbers. Consequently, (4.11) yields

$$\left(\sum_{k=N}^{M} a_k^{p} \|w\|_{p',J_k,\mu}^{p}\right)^{\frac{1}{p}} \lesssim c \left(\sum_{k=N}^{M} a_k^{q} \|u\|_{q,(a,x_k],\nu}^{q}\right)^{\frac{1}{q}}, \tag{4.14}$$

and, by Lemma 2.6,

$$\sup_{N\leqslant k\leqslant M} \|w\|_{p',J_k,\mu} \|u\|_{q,(a,x_k],\nu}^{-1} \lesssim c. \tag{4.15}$$

Assuming that $x_N > a$, testing (4.1) with $g = \chi_{(a,x_N]}$ and using (4.3), we arrive at $\|w\|_{p,(a,x_N],\mu} = 0$. This implies that $w = 0$ μ-a.e. in $(a, x_N]$. Consequently, (4.6) holds. Therefore, on using (3.10),

$$A_1 = \sup_{N\leqslant k\leqslant M} \sup_{x\in J_k} \|w\|_{p',(a,x],\mu} \|u\|^{-1}_{q,(a,x),\nu}$$

and hence

$$A_1 \leqslant \sup_{N\leqslant k\leqslant M} \|w\|_{p',(a,x_{k+1}]\cap I,\mu} \|u\|^{-1}_{q,(a,x_k],\nu}.$$

Applying (4.6) and (3.10) again, we arrive at

$$A_1 \leqslant \sup_{N\leqslant k\leqslant M} \left(\sum_{i=N}^{k} \|w\|^{p'}_{p',J_i,\mu} \right)^{\frac{1}{p'}} \|u\|^{-1}_{q,(a,x_k],\nu} \quad \text{if} \quad 0 < p < 1$$

and

$$A_1 \leqslant \sup_{N\leqslant k\leqslant M} \left(\sup_{N\leqslant i\leqslant k} \|w\|_{p',J_i,\mu} \right) \|u\|^{-1}_{q,(a,x_k],\nu} \quad \text{if} \quad p = 1.$$

Now, the fact that $\{\|u\|^{-1}_{q,(a,x_k],\nu}\}_{k=N}^{M}$ is almost geometrically decreasing (cf. (4.5)) and Lemma 2.4 imply that

$$A_1 \lesssim \sup_{N\leqslant k\leqslant M} \|w\|_{p',J_k,\mu} \|u\|^{-1}_{q,(a,x_k],\nu},$$

which, together with (4.15), yields $A_1 \lesssim c$. □

REMARK 4.2. Let A_1 be the number defined in Theorem 4.1. If $p = 1$, then

$$A_1 = \left\| w(x) \|u\|^{-1}_{q,(a,x),\nu} \right\|_{\infty,(a,b),\mu}.$$

Indeed, exchanging essential suprema, we obtain

$$\begin{aligned} A_1 &= \left\| \|w\|_{\infty,(a,x],\mu} \|u\|^{-1}_{q,(a,x),\nu} \right\|_{\infty,(a,b)} \\ &= \left\| \left\| w(s) \|u\|^{-1}_{q,(a,x),\nu} \right\|_{\infty,(a,x],\mu} \right\|_{\infty,(a,b)} \\ &= \left\| \left\| w(s)\chi_{(a,x]}(s) \|u\|^{-1}_{q,(a,x),\nu} \right\|_{\infty,(a,b),\mu} \right\|_{\infty,(a,b)} \\ &= \left\| \left\| w(s) \|u\|^{-1}_{q,(a,x),\nu} \right\|_{\infty,[s,b)} \right\|_{\infty,(a,b),\mu} \\ &= \left\| w(s) \|u\|^{-1}_{q,(a,s),\nu} \right\|_{\infty,(a,b),\mu}. \end{aligned}$$

In the rest of the paper we shall need the Lebesgue-Stieltjes integral. To this end, we recall some basic facts.

Let φ be non-decreasing and finite function on the interval $I := (a,b) \subseteq \mathbb{R}$. We assign to φ the function λ defined on subintervals of I by

$$\lambda([\alpha,\beta]) = \varphi(\beta+) - \varphi(\alpha-), \tag{4.16}$$

$$\lambda([\alpha,\beta)) = \varphi(\beta-) - \varphi(\alpha-), \tag{4.17}$$
$$\lambda((\alpha,\beta]) = \varphi(\beta+) - \varphi(\alpha+), \tag{4.18}$$
$$\lambda((\alpha,\beta)) = \varphi(\beta-) - \varphi(\alpha+). \tag{4.19}$$

The function λ is a non-negative, additive and regular function of intervals. Thus (cf. [15]), it admits a unique extension to a non-negative Borel measure λ on I. The Lebesgue-Stieltjes integral $\int_I f\, d\varphi$ is defined as $\int_I f\, d\lambda$.

In this section the role of the function φ will be played by a function h which will be *non-decreasing* and *right-continuous* on I. Consequently, the associated Borel measure λ will be determined by (cf. (4.18))

$$\lambda((\alpha,\beta]) = h(\beta) - h(\alpha) \quad \text{for any} \quad (\alpha,\beta] \subset I \tag{4.20}$$

(since the Borel subsets of I can be generated by subintervals $(\alpha,\beta] \subset I$).

Consider now the inequality (4.1) in the case when $0 < p \leqslant 1$, $p < q \leqslant +\infty$ and define r by

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{q}. \tag{4.21}$$

In such a case we shall write a condition characterizing the validity of inequality (4.1) in a compact form involving $\int_{(a,b)} f\, dh$, where $f(t) = \|w\|^r_{p',(a,t],\mu}$ and $h(t) = -\|u\|^{-r}_{q,(a,t+],\nu}$, $t \in (a,b)$. (Hence, the Lebesgue-Stieltjes integral $\int_{(a,b)} f\, dh$ is defined by the non-decreasing and right-continuous function h on (a,b)). However, it can happen that $\|u\|_{q,(a,t+],\nu} = 0$ for all $t \in (a,c)$ with a convenient $c \in (a,b)$ (provided that we omit the trivial case when $u = 0$ ν-a.e. on (a,b)). Then we have to explain what is the meaning of the Lebesgue-Stieltjes integral since in such a case the function $h = -\infty$ on (a,c). To this end, we adopt the following convention.

CONVENTION 4.3. Let $I = (a,b) \subseteq \mathbb{R}$, $f : I \to [0,+\infty]$ and $h : I \to [-\infty, 0]$. Assume that h is non-decreasing and right-continuous on I. If $h : I \to (-\infty, 0]$, then the symbol $\int_I f\, dh$ means the usual Lebesgue-Stieltjes integral. However, if $h = -\infty$ on some subinterval (a,c) with $c \in I$, then we define $\int_I f\, dh$ only if $f = 0$ on $(a,c]$ and we put

$$\int_I f\, dh = \int_{(c,b)} f\, dh.$$

In the proof of the next theorem we shall use frequently the Lebesgue-Stieltjes integral $\int_J d\varphi$, where φ is a non-decreasing, finite and right-continuous function on $I = (a,b)$ and J is a subinterval of I of the form (α,β), $[\alpha,\beta)$ or $(\alpha,\beta]$. The formulae (4.19), (4.17) and (4.18) imply that

$$\int_{(\alpha,\beta)} d\varphi = \varphi(\beta-) - \varphi(\alpha), \tag{4.22}$$

$$\int_{[\alpha,\beta)} d\varphi = \varphi(\beta-) - \varphi(\alpha-), \tag{4.23}$$

$$\int_{(\alpha,\beta]} d\varphi = \varphi(\beta) - \varphi(\alpha). \tag{4.24}$$

THEOREM 4.4. *Assume that* $0 < p \leqslant 1$, $p < q \leqslant +\infty$ *and* r *is given by* (4.21). *Let* μ *and* ν *be non-negative Borel measures on* $I = (a,b) \subseteq \mathbb{R}$. *Let* $w \in B^+(I)$ *and let* $u \in B^+(I)$ *satisfy* $\|u\|_{q,(a,t],\nu} < +\infty$ *for all* $t \in I$ *and* $u \neq 0$ ν*-a.e. on* I. *Then the inequality* (4.1) *holds for all* $g \in B^+(I)$ *if and only if*

$$A_2 := \left(\int_{(a,b)} \|w\|^r_{p',(a,t],\mu} \, d\left(-\|u\|^{-r}_{q,(a,t+],\nu}\right) \right)^{\frac{1}{r}} + \frac{\|w\|_{p',(a,b),\mu}}{\|u\|_{q,(a,b),\nu}} < +\infty.$$

The best possible constant c *in* (4.1) *satisfies* $c \approx A_2$.

REMARK 4.5. Let $q < +\infty$ in Theorem 4.4. Then

$$\|u\|_{q,(a,t+],\nu} = \|u\|_{q,(a,t],\nu} \quad \text{for all} \quad t \in I,$$

which implies that

$$A_2 = \left(\int_{(a,b)} \|w\|^r_{p',(a,t],\mu} \, d\left(-\|u\|^{-r}_{q,(a,t],\nu}\right) \right)^{\frac{1}{r}} + \frac{\|w\|_{p',(a,b),\mu}}{\|u\|_{q,(a,b),\nu}}.$$

Proof of Theorem 4.4. Let $0 < p \leqslant 1$ and $p < q \leqslant +\infty$.

(i) Suppose first that $q < +\infty$. Let $\{x_k\}_{k=N}^{M+1}$ be the discretizing sequence of the function $\varphi(t) = \|u\|^q_{q,(a,t],\nu}$, $t \in (a,b)$. Then (4.3)–(4.5) are satisfied. Moreover, by Corollary 3.6, (4.2) holds, where $\{J_k\}_{k=N}^{M}$ is given by (3.10).

Assume that $A_2 < +\infty$. This condition, (4.3) and Convention 4.3 imply that (4.6) holds and, as in the proof of Theorem 4.1, we arrive at (4.7). Thus, using (4.8), the discrete version of Hölder's inequality (with the exponents q/p and r/p) and (4.2), we obtain

$$\begin{aligned} \|gw\|_{p,(a,b),\mu} &\leqslant \left(\sum_{k=N}^{M} \|g\|^p_{1,J_k,\mu} \|w\|^p_{p',J_k,\mu} \right)^{\frac{1}{p}} \\ &\leqslant \left(\sum_{k=N}^{M} \|g\|^q_{1,J_k,\mu} \|u\|^q_{q,(a,x_k],\nu} \right)^{\frac{1}{q}} \left(\sum_{k=N}^{M} \|w\|^r_{p',J_k,\mu} \|u\|^{-r}_{q,(a,x_k],\nu} \right)^{\frac{1}{r}} \\ &\approx \left\| u(x) \int_{(x,b)} g \, d\mu \right\|_{q,(a,b),\nu} \left(\sum_{k=N}^{M} \|w\|^r_{p',J_k,\mu} \|u\|^{-r}_{q,(a,x_k],\nu} \right)^{\frac{1}{r}}. \end{aligned} \tag{4.25}$$

By (4.5),

$$2\|u\|^q_{q,(a,x_{k+1}),\nu} \leqslant \|u\|^q_{q,(a,x_{k+2}],\nu} \leqslant \|u\|^q_{q,(a,x_{k+3}),\nu} \quad \text{if} \quad N < k+1 < M.$$

Therefore,

$$\|u\|^{-r}_{q,(a,x_{k+3}),\nu} \leqslant 2^{-\frac{r}{q}} \|u\|^{-r}_{q,(a,x_{k+1}),\nu},$$

which yields

$$\|u\|^{-r}_{q,(a,x_{k+1}),\nu} - \|u\|^{-r}_{q,(a,x_{k+3}),\nu} \geqslant (1 - 2^{-\frac{r}{q}}) \|u\|^{-r}_{q,(a,x_{k+1}),\nu} \quad \text{if} \quad N \leqslant k \leqslant M-2.$$

Assume that $N \leqslant M-2$. On using (4.4) and the last estimate, we arrive at

$$\begin{aligned}\sum_{k=N}^{M} \|w\|^r_{p',J_k,\mu}\|u\|^{-r}_{q,(a,x_k],\nu} &\lesssim \sum_{k=N}^{M} \|w\|^r_{p',J_k,\mu}\|u\|^{-r}_{q,(a,x_{k+1}),\nu} \\ &\lesssim \sum_{k=N}^{M-2} \|w\|^r_{p',J_k,\mu}\left(\|u\|^{-r}_{q,(a,x_{k+1}),\nu} - \|u\|^{-r}_{q,(a,x_{k+3}),\nu}\right) \\ &\quad + \|w\|^r_{p',J_{M-1},\mu}\left(\|u\|^{-r}_{q,(a,x_M),\nu} - \|u\|^{-r}_{q,(a,b),\nu}\right) \\ &\quad + \|w\|^r_{p',J_{M-1},\mu}\|u\|^{-r}_{q,(a,b),\nu} + \|w\|^r_{p',J_M,\mu}\|u\|^{-r}_{q,(a,b),\nu}.\end{aligned} \tag{4.26}$$

Now, by (4.23) with $\varphi(t) = -\|u\|^{-r}_{q,(a,t],\nu}$, $t \in I$, and $[\alpha,\beta) = [x_{k+1},x_{k+3})$, $N \leqslant k \leqslant M-2$, or $[\alpha,\beta) = [x_M,b)$, we obtain that

$$\begin{aligned}&\sum_{k=N}^{M} \|w\|^r_{p',J_k,\mu}\|u\|^{-r}_{q,(a,x_k],\nu} \\ &\quad\leqslant \sum_{k=N}^{M-2} \|w\|^r_{p',J_k,\mu}\int_{[x_{k+1},x_{k+3})} d\left(-\|u\|^{-r}_{q,(a,t],\nu}\right) \\ &\qquad + \|w\|^r_{p',J_{M-1},\mu}\int_{[x_M,b)} d\left(-\|u\|^{-r}_{q,(a,t],\nu}\right) + 2\|w\|^r_{p',(a,b),\mu}\|u\|^{-r}_{q,(a,b),\nu} \\ &\quad\leqslant \sum_{k=N}^{M-2}\int_{[x_{k+1},x_{k+3})} \|w\|^r_{p',(a,t],\mu}\, d\left(-\|u\|^{-r}_{q,(a,t],\nu}\right) \\ &\qquad + \int_{[x_M,b)} \|w\|^r_{p',(a,t],\mu}\, d\left(-\|u\|^{-r}_{q,(a,t],\nu}\right) + 2\|w\|^r_{p',(a,b),\mu}\|u\|^{-r}_{q,(a,b),\nu} \\ &\quad\leqslant 2\int_{(a,b)} \|w\|^r_{p',(a,t],\mu}\, d\left(-\|u\|^{-r}_{q,(a,t],\nu}\right) + 2\|w\|^r_{p',(a,b),\mu}\|u\|^{-r}_{q,(a,b),\nu} \\ &\quad\lesssim A_2^r\end{aligned}$$

(note that we have used (4.6) and Convention 4.3), that is,

$$\sum_{k=N}^{M} \|w\|^r_{p',J_k,\mu}\|u\|^{-r}_{q,(a,x_k],\nu} \lesssim A_2^r. \tag{4.27}$$

If $N > M-2$, then (4.27) can be proved more simply (and we leave it to the reader). The inequality (4.1) (with $c \leqslant A_2$) follows from (4.25) and (4.27).

For necessity we apply the same argument as in the proof of Theorem 4.1 to get (4.14). Next, by Lemma 2.7,

$$\left(\sum_{k=N}^{M} \|w\|^r_{p',J_k,\mu}\|u\|^{-r}_{q,(a,x_k],\nu}\right)^{\frac{1}{r}} \lesssim c. \tag{4.28}$$

As in the necessity part of the proof of the Theorem 4.1, we can show that (4.6) holds. Together with (3.10), (4.24) and (4.22), this yields

$$\begin{aligned} A_2^r &\approx \sum_{k=N}^{M} \int_{J_k} \|w\|_{p',(a,t],\mu}^r \, d\left(-\|u\|_{q,(a,t],\nu}^{-r}\right) + \|w\|_{p',(a,b),\mu}^r \|u\|_{q,(a,b),\nu}^{-r} \\ &\leqslant \sum_{k=N}^{M-1} \|w\|_{p',(a,x_{k+1}],\mu}^r \int_{J_k} d\left(-\|u\|_{q,(a,t],\nu}^{-r}\right) \\ &\quad + \|w\|_{p',(a,b),\mu}^r \int_{(x_M,b)} d\left(-\|u\|_{q,(a,t],\nu}^{-r}\right) + \|w\|_{p',(a,b),\mu}^r \|u\|_{q,(a,b),\nu}^{-r} \\ &\lesssim \sum_{k=N}^{M-1} \|w\|_{p',(a,x_{k+1}],\mu}^r \|u\|_{q,(a,x_k],\nu}^{-r} + \|w\|_{p',(a,b),\mu}^r \|u\|_{q,(a,x_M],\nu}^{-r}. \end{aligned} \tag{4.29}$$

Thus, using (4.6) and (3.10) again, we arrive at

$$A_2^r \lesssim \sum_{k=N}^{M} \left(\sum_{i=N}^{k} \|w\|_{p',J_i,\mu}^{p'} \right)^{\frac{r}{p'}} \|u\|_{q,(a,x_k],\nu}^{-r} \quad \text{if} \quad 0 < p < 1$$

and

$$A_2^r \lesssim \sum_{k=N}^{M} \left(\sup_{N \leqslant i \leqslant k} \|w\|_{p',J_i,\mu} \right)^{r} \|u\|_{q,(a,x_k],\nu}^{-r} \quad \text{if} \quad p = 1.$$

Now, the fact that $\{\|u\|_{q,(a,x_k],\nu}^{-r}\}_{k=N}^{M}$ is almost geometrically decreasing (cf. (4.5)) and Lemma 2.4 imply that

$$A_2^r \lesssim \sum_{k=N}^{M} \|w\|_{p',J_k,\mu}^r \|u\|_{q,(a,x_k],\nu}^{-r}, \tag{4.30}$$

which, together with (4.28), yields $A_2 \lesssim c$.

(ii) Suppose now that $q = +\infty$. Let $\{x_k\}_{k=N}^{M+1}$ be a discretizing sequence of the function $\varphi(t) = \|u\|_{\infty,(a,t+],\nu}$, $t \in (a,b)$. By Lemma 3.1 (cf. also Remark 3.3),

$$\begin{aligned} &\text{if} \quad x_N > a, \quad \text{then} \quad \|u\|_{\infty,(a,x_N),\nu} = 0; \\ &\text{if} \quad M < +\infty, \quad \text{then} \quad x_{M+1} = b; \end{aligned} \tag{4.31}$$

$$\|u\|_{\infty,(a,x_{k+1}),\nu} \leqslant 2\|u\|_{\infty,(a,x_k+],\nu} \quad \text{if} \quad N \leqslant k \leqslant M; \tag{4.32}$$

$$2\|u\|_{\infty,(a,x_k),\nu} \leqslant \|u\|_{\infty,(a,x_{k+1}+],\nu} \quad \text{if} \quad N < k < M. \tag{4.33}$$

Moreover, by Corollary 3.7,

$$\left\| u(x) \int_{(x,b)} g(y)\, d\mu \right\|_{\infty,(a,b),\nu} \approx \sup_{N \leqslant k \leqslant M} \left(\int_{J_k} g\, d\mu \right) \|u\|_{\infty,(a,x_k+],\nu} \tag{4.34}$$

for all $g \in B^+(I)$, where $\{J_k\}_{k=N}^{M}$ is given by (3.10).

Assume that $A_2 < +\infty$. This condition, (4.31) and Convention 4.3 imply that (4.6) holds, and, as in the proof of Theorem 4.1, we arrive at (4.7). Thus, using (4.8) and (4.34), we obtain

$$\begin{aligned}\|gw\|_{p,(a,b),\mu} &\leqslant \left(\sum_{k=N}^{M} \|g\|_{1,J_k,\mu}^p \|w\|_{p',J_k,\mu}^p\right)^{\frac{1}{p}} \\ &\leqslant \left(\sup_{N\leqslant k\leqslant M} \|g\|_{1,J_k,\mu}\|u\|_{\infty,(a,x_k+],\nu}\right)\left(\sum_{k=N}^{M} \|w\|_{p',J_k,\mu}^p \|u\|_{\infty,(a,x_k+],\nu}^{-p}\right)^{\frac{1}{p}} \\ &\approx \left\|u(x)\int_{(x,b)} g\,d\mu\right\|_{\infty,(a,b),\nu}\left(\sum_{k=N}^{M} \|w\|_{p',J_k,\mu}^p \|u\|_{\infty,(a,x_k+],\nu}^{-p}\right)^{\frac{1}{p}}.\end{aligned} \tag{4.35}$$

Analogously as in the case (i), we arrive at

$$\left(\sum_{k=N}^{M} \|w\|_{p',J_k,\mu}^p \|u\|_{\infty,(a,x_k+],\nu}^{-p}\right)^{\frac{1}{p}} \lesssim A_2. \tag{4.36}$$

Therefore, (4.1) (with $c \lesssim A_2$) follows from (4.35) and (4.36).

Now, we prove necessity part. The validity of the inequality (4.1) on $B^+(I)$ and (4.34) imply that

$$\left(\sum_{k=N}^{M} \|gw\|_{p,J_k,\mu}^p\right)^{\frac{1}{p}} \lesssim c \sup_{N\leqslant k\leqslant M}\left(\int_{J_k} g\,d\mu\right)\|u\|_{\infty,(a,x_k+],\nu} \tag{4.37}$$

for all $g \in B^+(I)$. Let $g_k \in B^+(I)$, $N \leqslant k \leqslant M$, be functions satisfying (4.12) and define the test function g by (4.13). Consequently, (4.37) yields

$$\left(\sum_{k=N}^{M} a_k^p \|w\|_{p',J_k,\mu}^p\right)^{\frac{1}{p}} \lesssim \sup_{N\leqslant k\leqslant M} a_k \|u\|_{\infty,(a,x_k+],\nu},$$

and, by Lemma 2.7,

$$\left(\sum_{k=N}^{M} \|w\|_{p',J_k,\mu}^p \|u\|_{\infty,(a,x_k+],\nu}^{-p}\right)^{\frac{1}{p}} \lesssim c. \tag{4.38}$$

The same idea as that used in part (i) shows that (cf. (4.29)–(4.30))

$$A_2^p \lesssim \sum_{k=N}^{M} \|w\|_{p',J_k,\mu}^p \|u\|_{\infty,(a,x_k+],\nu}^{-p}, \tag{4.39}$$

which, together with (4.38), yields $A_2 \lesssim c$. □

In the case when $0 < p \leqslant 1$, $p \leqslant q \leqslant +\infty$ we have seen that the validity of the inequality (4.1) on $B^+(I)$ is characterized by the condition $A_2 < +\infty$. The following theorem shows that there is another quantity equivalent to A_2.

THEOREM 4.6. *Suppose that all the assumptions of Theorem* 4.4 *are satisfied.*
(i) *Let*

$$\|w\|_{p',(a,t+],\mu} \lesssim \|w\|_{p',(a,t],\mu} \quad \textit{for all} \quad t \in (a,b). \tag{4.40}$$

Then $A_2 \approx A_3$, *where*

$$A_3 := \left(\int_{(a,b)} \|u\|_{q,(a,t),\nu}^{-r} \, d\left(\|w\|_{p',(a,t+],\mu}^{r} \right) \right)^{\frac{1}{r}} + \frac{\lim_{t\to a+} \|w\|_{p',(a,t),\mu}}{\lim_{t\to a+} \|u\|_{q,(a,t),\nu}}.$$

(ii) *Let*

$$\|w\|_{p',(a,t],\mu} \lesssim \|w\|_{p',(a,t),\mu} \quad \textit{for all} \quad t \in (a,b). \tag{4.41}$$

Then $A_2 \approx A_4$, *where*

$$A_4 := \left(\int_{(a,b)} \|u\|_{q,(a,t+],\nu}^{-r} \, d\left(\|w\|_{p',(a,t+],\mu}^{r} \right) \right)^{\frac{1}{r}} + \frac{\lim_{t\to a+} \|w\|_{p',(a,t),\mu}}{\lim_{t\to a+} \|u\|_{q,(a,t),\nu}}.$$

REMARK 4.7. Theorems 4.4 and 4.6 imply that if the weight w satisfies (4.40), then the conditions $A_2 < +\infty$ and $A_3 < +\infty$ are equivalent. Similarly, if w satisfies (4.41), then the conditions $A_2 < +\infty$ and $A_4 < +\infty$ are equivalent.

Note also that if $p' < +\infty$ and $\|w\|_{p',(a,t],\mu} < +\infty$ for all $t \in (a,b)$, then $\|w\|_{p',(a,t+],\mu} = \|w\|_{p',(a,t],\mu}$ for all $t \in (a,b)$ and therefore w satisfies (4.40). In this case the condition $A_3 < +\infty$ reduces to

$$\left(\int_{(a,b)} \|u\|_{q,(a,t),\nu}^{-r} \, d\left(\|w\|_{p',(a,t],\mu}^{r} \right) \right)^{\frac{1}{r}} < +\infty \tag{4.42}$$

since $\lim_{t\to a+} \|w\|_{p',(a,t),\mu} = 0$.

Suppose now that $p' = +\infty$, the weight w satisfies (4.40), and $A_3 < +\infty$. Moreover, let $\lim_{t\to a+} \|u\|_{q,(a,t),\nu} = 0$. Then the second term in A_3 has to be finite which can happen only if $\lim_{t\to a+} \|w\|_{\infty,(a,t),\mu} = 0$ (cf. our convention that $0/0 = 0$).

Let μ be a non-negative Borel measure on (a,b) which has no atoms. Then it is clear that the condition (4.41) holds.

Suppose now that $p' = +\infty$, the weight w satisfies (4.41), and $A_4 < +\infty$. Moreover, let $\lim_{t\to a+} \|u\|_{q,(a,t),\nu} = 0$. Then the second term in A_4 has to be finite which can happen only if $\lim_{t\to a+} \|w\|_{\infty,(a,t),\mu} = 0$ (cf. our convention that $0/0 = 0$).

Proof of Theorem 4.6. It is clear that it is sufficient to verify the implications:

$$A_i < +\infty \Rightarrow A_2 \lesssim A_i, \quad i = 3,4, \tag{4.43}$$

$$A_2 < +\infty \Rightarrow A_i \lesssim A_2, \quad i = 3,4. \tag{4.44}$$

Let $\{x_k\}_{k=N}^{M+1}$ be the discretizing sequence from the proof of Theorem 4.4.

$(i$-$1)$ Assume first that $A_3 < +\infty$. If $x_N > a$, then, by (4.3) (if $q < +\infty$) or by (4.31) (if $q = +\infty$), $\|u\|_{q,(a,x_N),\nu} = 0$. This means that $\|u\|_{q,(a,t),\nu}^{-r} = +\infty$ for all $t \in (a, x_N]$. Together with the fact that the first term in A_3 is finite, this implies that

$$\|w\|_{p',(a,t],\mu} \quad \text{is constant in} \quad (a, x_N]. \tag{4.45}$$

Hence, since also the second term in A_3 is finite,

$$\|w\|_{p',(a,t],\mu} = 0 \quad \text{in} \quad (a, x_N]. \tag{4.46}$$

Therefore, (4.6) is satisfied. Consequently, (4.30) holds (cf. (4.29)–(4.30)) if $q < +\infty$, or (4.39) holds if $q = +\infty$. Thus, we obtain

$$A_2^r \lesssim \sum_{k=N}^{M-1} \|w\|^r_{p',(a,x_{k+1}],\mu} \|u\|^{-r}_{q,(a,x_k+],\nu} + \|w\|^r_{p',(a,b),\mu} \|u\|^{-r}_{q,(a,x_M+],\nu}. \tag{4.47}$$

The condition $A_3 < +\infty$ also implies that $\|w\|^r_{p',(a,t+],\mu} < +\infty$ for all $t \in (a,b)$.

Assume first that $\lim_{t\to a+} \|w\|_{p',(a,t),\mu} = 0$. Together with (4.24) and (4.22), this yields

$$A_2^r \lesssim \sum_{k=N}^{M-1} \int_{(a,x_{k+1}]} d\left(\|w\|^r_{p',(a,t+],\mu}\right) \|u\|^{-r}_{q,(a,x_k+],\nu} + \int_{(a,b)} d\left(\|w\|^r_{p',(a,t+],\mu}\right) \|u\|^{-r}_{q,(a,x_M+],\nu},$$

and, in view of (3.10) and (4.6), we arrive at

$$A_2^r \lesssim \sum_{k=N}^{M} \left(\sum_{i=N}^{k} \int_{J_i} d\left(\|w\|^r_{p',(a,t+],\mu}\right)\right) \|u\|^{-r}_{q,(a,x_k+],\nu}. \tag{4.48}$$

The fact that $\{\|u\|^{-r}_{q,(a,x_k+],\nu}\}_{k=N}^{M}$ is almost geometrically decreasing, Lemma 2.4, (4.4) or (4.32) and (3.10) imply that

$$\begin{aligned} A_2^r &\approx \sum_{k=N}^{M} \int_{J_k} d\left(\|w\|^r_{p',(a,t+],\mu}\right) \|u\|^{-r}_{q,(a,x_k+],\nu} \\ &\lesssim \sum_{k=N}^{M} \int_{J_k} d\left(\|w\|^r_{p',(a,t+],\mu}\right) \|u\|^{-r}_{q,(a,x_{k+1}),\nu} \\ &\lesssim \sum_{k=N}^{M} \int_{J_k} \|u\|^{-r}_{q,(a,t),\nu}\, d\left(\|w\|^r_{p',(a,t+],\mu}\right) \\ &\lesssim A_3^r. \end{aligned} \tag{4.49}$$

Suppose now that $\lim_{t\to a+} \|w\|_{p',(a,t),\mu} \neq 0$. The condition $A_3 < +\infty$ implies that $\lim_{t\to a+} \|u\|_{q,(a,t),\nu} \neq 0$ (therefore $x_N = a$ and $N > -\infty$). By (4.47), (4.24) and (4.22),

$$\begin{aligned} A_2^r \lesssim \sum_{k=N}^{M-1} &\left[\int_{(a,x_{k+1}]} d\left(\|w\|^r_{p',(a,t+],\mu}\right) + \lim_{t\to a+} \|w\|^r_{p',(a,t),\mu} \right] \|u\|^{-r}_{q,(a,x_k+],\nu} \\ &+ \left[\int_{(a,b)} d\left(\|w\|^r_{p',(a,t+],\mu}\right) + \lim_{t\to a+} \|w\|^r_{p',(a,t),\mu} \right] \|u\|^{-r}_{q,(a,x_M+],\nu}, \end{aligned}$$

and, in view of (3.10) and (4.6), we arrive at

$$
\begin{aligned}
A_2^r &\lesssim \sum_{k=N}^{M} \left(\sum_{i=N}^{k} \int_{J_i} d\left(\|w\|^r_{p',(a,t+],\mu} \right) \right) \|u\|^{-r}_{q,(a,x_k+],\nu} \\
&\qquad + \left(\lim_{t\to a+} \|w\|^r_{p',(a,t),\mu} \right) \sum_{k=N}^{M} \|u\|^{-r}_{q,(a,x_k+],\nu} \\
&=: I_1 + I_2. \qquad (4.50)
\end{aligned}
$$

Analogously as in (4.49), we obtain

$$
I_1 \lesssim A_3^r. \qquad (4.51)
$$

Furthermore, as a consequence of Lemma 2.3, we get

$$
I_2 \lesssim \lim_{t\to a+} \|w\|^r_{p',(a,t),\mu} \lim_{t\to a+} \|u\|^{-r}_{q,(a,t),\nu} \lesssim A_3^r. \qquad (4.52)
$$

Thus, the estimate $A_2 \lesssim A_3$ follows from (4.50)–(4.52).

(i-2) Assume now that $A_2 < +\infty$ and the weight w satisfies (4.40). Then (cf. the proof of Theorem 4.4) (4.6) holds. Together with (3.10), this shows that

$$
\begin{aligned}
A_3^r &\approx \sum_{k=N}^{M} \int_{J_k} \|u\|^{-r}_{q,(a,t),\nu} \, d\left(\|w\|^r_{p',(a,t+],\mu} \right) + \frac{\lim_{t\to a+} \|w\|^r_{p',(a,t),\mu}}{\lim_{t\to a+} \|u\|^r_{q,(a,t),\nu}} \\
&=: V_1 + V_2. \qquad (4.53)
\end{aligned}
$$

In view of (3.10), (4.24), (4.22) and (4.40),

$$
\begin{aligned}
V_1 &\lesssim \sum_{k=N}^{M} \|u\|^{-r}_{q,(a,x_k+],\nu} \int_{J_k} d\left(\|w\|^r_{p',(a,t+],\mu} \right) \\
&\leqslant \sum_{k=N}^{M-1} \|u\|^{-r}_{q,(a,x_k+],\nu} \|w\|^r_{p',(a,x_{k+1}+],\mu} + \|u\|^{-r}_{q,(a,x_M+],\nu} \|w\|^r_{p',(a,b),\mu} \\
&\lesssim \sum_{k=N}^{M-1} \|u\|^{-r}_{q,(a,x_k+],\nu} \|w\|^r_{p',(a,x_{k+1}],\mu} + \|u\|^{-r}_{q,(a,x_M+],\nu} \|w\|^r_{p',(a,b),\mu}. \qquad (4.54)
\end{aligned}
$$

Using (4.6) and (3.10), we arrive at

$$
V_1 \lesssim \sum_{k=N}^{M} \|u\|^{-r}_{q,(a,x_k+],\nu} \left(\sum_{i=N}^{k} \|w\|^{p'}_{p',J_i,\mu} \right)^{\frac{r}{p'}} \quad \text{if} \quad 0 < p < 1,
$$

and

$$
V_1 \lesssim \sum_{k=N}^{M} \|u\|^{-r}_{q,(a,x_k+],\nu} \left(\sup_{N\leqslant i\leqslant k} \|w\|_{p',J_i,\mu} \right)^{r} \quad \text{if} \quad p = 1.
$$

Now, we deduce from the fact that $\{\|u\|^{-1}_{q,(a,x_k+],\nu}\}_{k=N}^{M}$ is an almost geometrically decreasing sequence and from Lemma 2.4 that

$$V_1 \lesssim \sum_{k=N}^{M} \|u\|^{-r}_{q,(a,x_k+],\nu} \|w\|^{r}_{p',J_k,\mu}.$$

This estimate and (4.27) if ($q < +\infty$) or (4.36) if ($q = +\infty$) imply that

$$V_1 \lesssim A_2^r. \tag{4.55}$$

If $x_N > a$, then $\lim_{t\to a+} \|w\|_{p',(a,t),\mu} = 0$ by (4.6). This means that $V_2 = 0$ and the estimate $A_3 \lesssim A_2$ follows from (4.53) and (4.55).

If $x_N = a$, we use the definition of V_2 and (4.22) to get

$$\begin{aligned} V_2 &= \left(\lim_{t\to a+} \|w\|^r_{p',(a,t),\mu}\right)\left(\int_{(a,b)} d\left(-\|u\|^{-r}_{q,(a,t+],\nu}\right) + \|u\|^{-r}_{q,(a,b),\nu}\right) \\ &\leqslant \left(\int_{(a,b)} \|w\|^r_{p',(a,t],\mu}\, d\left(-\|u\|^{-r}_{q,(a,t+],\nu}\right) + \|w\|^r_{p',(a,b),\mu}\|u\|^{-r}_{q,(a,b),\nu}\right) \\ &\lesssim A_2^r, \end{aligned} \tag{4.56}$$

and the estimate $A_3 \lesssim A_2$ follows from (4.53), (4.55) and (4.56).

(*ii*-1) Assume now that $A_4 < +\infty$ and the weight w satisfies (4.41). A similar idea to that used in part (*i*-1) shows that $\|w\|_{p',(a,t+],\mu} = 0$ for all $t \in (a, x_N)$. This implies that $\|w\|_{p',(a,x_N),\mu} = 0$, and, on using (4.41) (with $t = x_N$), we arrive at (4.6). Therefore, as in part (i-1), we see that (4.47) is satisfied. The condition $A_4 < +\infty$ also implies that $\|w\|^r_{p',(a,t+],\mu} < +\infty$ for all $t \in (a,b)$.

First assume that $\lim_{t\to a+} \|w\|_{p',(a,t),\mu} = 0$. Using (4.47), (4.41) and (4.22), we obtain

$$\begin{aligned} A_2^r &\lesssim \sum_{k=N}^{M-1} \|w\|^r_{p',(a,x_{k+1}),\mu}\|u\|^{-r}_{q,(a,x_k+],\nu} + \|w\|^r_{p',(a,b),\mu}\|u\|^{-r}_{q,(a,x_M+],\nu} \\ &\lesssim \sum_{k=N}^{M-1} \int_{(a,x_{k+1})} d\left(\|w\|^r_{p',(a,t+],\mu}\right)\|u\|^{-r}_{q,(a,x_k+],\nu} + \int_{(a,b)} d\left(\|w\|^r_{p',(a,t+],\mu}\right)\|u\|^{-r}_{q,(a,x_M+],\nu}, \end{aligned}$$

and, in view of (4.6), we arrive at

$$A_2^r \lesssim \sum_{k=N}^{M}\left(\sum_{i=N}^{k}\int_{I\cap[x_i,x_{i+1})} d\left(\|w\|^r_{p',(a,t+],\mu}\right)\right)\|u\|^{-r}_{q,(a,x_k+],\nu}. \tag{4.57}$$

The fact that $\{\|u\|^{-r}_{q,(a,x_k+],\nu}\}_{k=N}^{M}$ is almost geometrically decreasing, Lemma 2.4, (4.4) or (4.32) imply that

$$
\begin{aligned}
A_2^r &\lesssim \sum_{k=N}^{M} \int_{I\cap[x_k,x_{k+1})} d\left(\|w\|^r_{p',(a,t+],\mu}\right) \|u\|^{-r}_{q,(a,x_k+],\nu} \\
&\lesssim \sum_{k=N}^{M} \int_{I\cap[x_k,x_{k+1})} d\left(\|w\|^r_{p',(a,t+],\mu}\right) \|u\|^{-r}_{q,(a,x_{k+1}),\nu} \\
&\lesssim \sum_{k=N}^{M} \int_{I\cap[x_k,x_{k+1})} \|u\|^{-r}_{q,(a,t+],\nu}\, d\left(\|w\|^r_{p',(a,t+],\mu}\right) \\
&\lesssim A_4^r. \qquad (4.58)
\end{aligned}
$$

Suppose now that $\lim_{t\to a+}\|w\|_{p',(a,t),\mu} \neq 0$. The condition $A_4 < +\infty$ implies that $\lim_{t\to a+}\|u\|_{q,(a,t),\nu} \neq 0$ (therefore $x_N = a$ and $N > -\infty$). By (4.47), (4.41) and (4.22),

$$
\begin{aligned}
A_2^r \lesssim \sum_{k=N}^{M-1} &\left[\int_{(a,x_{k+1})} d\left(\|w\|^r_{p',(a,t+],\mu}\right) + \lim_{t\to a+}\|w\|^r_{p',(a,t),\mu}\right] \|u\|^{-r}_{q,(a,x_k+],\nu} \\
&+ \left[\int_{(a,b)} d\left(\|w\|^r_{p',(a,t+],\mu}\right) + \lim_{t\to a+}\|w\|^r_{p',(a,t),\mu}\right] \|u\|^{-r}_{q,(a,x_M+],\nu},
\end{aligned}
$$

and, in view of (4.6), we arrive at

$$
\begin{aligned}
A_2^r &\lesssim \sum_{k=N}^{M} \left(\sum_{i=N}^{k} \int_{I\cap[x_i,x_{i+1})} d\left(\|w\|^r_{p',(a,t+],\mu}\right)\right) \|u\|^{-r}_{q,(a,x_k+],\nu} \\
&+ \left(\lim_{t\to a+}\|w\|^r_{p',(a,t),\mu}\right) \sum_{k=N}^{M} \|u\|^{-r}_{q,(a,x_k+],\nu} \\
&=: I_1 + I_2. \qquad (4.59)
\end{aligned}
$$

Analogously as in (4.58), we obtain

$$
I_1 \lesssim A_4^r. \qquad (4.60)
$$

Furthermore, as a consequence of Lemma 2.3, we get

$$
I_2 \lesssim \lim_{t\to a+}\|w\|^r_{p',(a,t),\mu} \lim_{t\to a+}\|u\|^{-r}_{q,(a,t),\nu} \lesssim A_4^r. \qquad (4.61)
$$

Thus, the estimate $A_2 \lesssim A_4$ follows from (4.59)–(4.61).

(*ii*-2) Assume now that $A_2 < +\infty$. Then (cf. the proof of Theorem 4.4) (4.6) holds. This shows that

$$
\begin{aligned}
A_4^r &\approx \sum_{k=N}^{M} \int_{I\cap[x_k,x_{k+1})} \|u\|^{-r}_{q,(a,t+],\nu}\, d\left(\|w\|^r_{p',(a,t+],\mu}\right) + \frac{\lim_{t\to a+}\|w\|^r_{p',(a,t),\mu}}{\lim_{t\to a+}\|u\|^r_{q,(a,t),\nu}} \\
&=: V_1 + V_2. \qquad (4.62)
\end{aligned}
$$

In view of (4.22),

$$V_1 \lesssim \sum_{k=N}^{M} \|u\|_{q,(a,x_k+],\nu}^{-r} \int_{(a,x_{k+1})} d\left(\|w\|_{p',(a,t+],\mu}^{r}\right)$$
$$\leqslant \sum_{k=N}^{M} \|u\|_{q,(a,x_k+],\nu}^{-r} \|w\|_{p',(a,x_{k+1}),\mu}^{r}. \tag{4.63}$$

Using (4.6) and (3.10), we arrive at

$$V_1 \lesssim \sum_{k=N}^{M} \|u\|_{q,(a,x_k+],\nu}^{-r} \left(\sum_{i=N}^{k} \|w\|_{p',J_i,\mu}^{p'}\right)^{\frac{r}{p'}} \quad \text{if} \quad 0<p<1,$$

and

$$V_1 \lesssim \sum_{k=N}^{M} \|u\|_{q,(a,x_k+],\nu}^{-r} \left(\sup_{N\leqslant i\leqslant k} \|w\|_{p',J_i,\mu}\right)^{r} \quad \text{if} \quad p=1.$$

Now, we deduce from the fact that $\{\|u\|_{q,(a,x_k+],\nu}^{-1}\}_{k=N}^{M}$ is an almost geometrically decreasing sequence and from Lemma 2.4 that

$$V_1 \lesssim \sum_{k=N}^{M} \|u\|_{q,(a,x_k+],\nu}^{-r} \|w\|_{p',J_k,\mu}^{r}.$$

This estimate and (4.27) if ($q<+\infty$) or (4.36) if ($q=+\infty$) imply that

$$V_1 \lesssim A_2^r. \tag{4.64}$$

If $x_N > a$, then $\lim_{t\to a+} \|w\|_{p',(a,t),\mu} = 0$ by (4.6). This means that $V_2 = 0$ and the estimate $A_4 \lesssim A_2$ follows from (4.62) and (4.64).

If $x_N = a$, we use the definition of V_2 and (4.22) to get

$$V_2 = \lim_{t\to a+} \|w\|_{p',(a,t),\mu}^{r} \left(\int_{(a,b)} d\left(-\|u\|_{q,(a,t+],\nu}^{-r}\right) + \|u\|_{q,(a,b),\nu}^{-r}\right)$$
$$\leqslant \left(\int_{(a,b)} \|w\|_{p',(a,t],\mu}^{r}\, d\left(-\|u\|_{q,(a,t+],\nu}^{-r}\right) + \|w\|_{p',(a,b),\mu}^{r} \|u\|_{q,(a,b),\nu}^{-r}\right)$$
$$\lesssim A_2^r. \tag{4.65}$$

The estimate $A_4 \lesssim A_2$ follows from (4.62), (4.64) and (4.65). □

REMARK 4.8. One can see from the proof of Theorem 4.6 that the implication $A_3 < +\infty \Rightarrow A_2 \lesssim A_3$ holds without the additional assumption (4.40). Similarly, the implication $A_2 < +\infty \Rightarrow A_4 \lesssim A_2$ holds without the additional assumption (4.41). Consequently, under the assumptions of Theorem 4.4,

$$A_3 < +\infty \Rightarrow A_4 \lesssim A_2 \lesssim A_3.$$

Moreover, if

$$\|u\|_{q,(a,t+],\nu} \lesssim \|u\|_{q,(a,t),\nu}, \tag{4.66}$$

then $A_3 \lesssim A_4$. Consequently, if (4.66) holds, then under the assumptions of Theorem 4.4, $A_2 \approx A_3 \approx A_4$.

To characterize the validity of inequality (4.1) for all $g \in B^+(I)$, in Theorems 4.1 and 4.4 we have supposed that the weight function u satisfies

$$\|u\|_{q,(a,t],\nu} < +\infty \quad \text{for all} \quad t \in I. \tag{4.67}$$

The next theorem concerns the case when (4.67) is violated.

THEOREM 4.9. *Assume that* $0 < p \leqslant 1$ *and* $0 < q \leqslant +\infty$. *Let* μ *and* ν *be non-negative Borel measures on* $I = (a,b) \subseteq \mathbb{R}$. *Let* $w \in B^+(I)$ *and let* $u \in B^+(I)$ *be such that* (4.67) *does not hold. Put*

$$T := \inf\{t \in I : \|u\|_{q,(a,t],\nu} = +\infty\}. \tag{4.68}$$

(i) *If* $T = a$, *then the inequality* (4.1) *holds for all* $g \in B^+(I)$.

(ii) *If* $T \in (a,b)$ *and* $\mu(\{T\}) = 0$, *then the inequality* (4.1) *holds for all* $g \in B^+(I)$ *if and only if the inequality*

$$\|gw\|_{p,(a,T),\mu} \lesssim \left\| u(x) \int_{(x,T)} g(y)\, d\mu \right\|_{q,(a,T),\nu} \tag{4.69}$$

holds for all $g \in B^+((a,T))$.

(iii) *If* $T \in (a,b)$ *and* $\mu(\{T\}) > 0$, *then the inequality* (4.1) *holds for all* $g \in B^+(I)$ *if and only if*

$$w(T)(\mu(\{T\}))^{1/p} \lesssim \mu(\{T\}) \|u\|_{q,(a,T),\nu} \tag{4.70}$$

and the inequality (4.69) *is satisfied for all* $g \in B^+((a,T))$.

REMARK 4.10. To characterize the validity of inequality (4.69) on (a,T) if $T \in (a,b)$, one can use Theorems 4.1 and 4.4 since $\|u\|_{q,(a,t],\nu} < +\infty$ for all $t \in (a,T)$.

Note also that condition (4.70) holds if and only if

$$\begin{aligned}
&\text{either} && 0 < \|u\|_{q,(a,T),\nu} < +\infty && \text{and} && w(T) < +\infty, \\
&\text{or} && 0 < \|u\|_{q,(a,T),\nu} < +\infty && \text{and} && w(T) = +\infty \quad \text{and} \quad \mu(\{T\}) = +\infty, \\
&\text{or} && \|u\|_{q,(a,T),\nu} = 0 && \text{and} && w(T) = 0, \\
&\text{or} && \|u\|_{q,(a,T),\nu} = +\infty. && &&
\end{aligned}$$

Proof of Theorem 4.9. We prove only part (iii) since proofs of parts (i) and (ii) are similar.

Let $T \in I$ and $\mu(\{T\}) > 0$. Take $g \in S^+(I)$, where

$$S^+(I) = \{\, g \in B^+(I) : \exists E = E_g \subseteq (T,b),\ \mu(E) > 0, \text{ and } g > 0 \text{ on } E \,\}, \tag{4.71}$$

and choose $n \in \mathbb{N}$ such that $\mu((T + 1/n, b) \cap E) > 0$. Together with (4.68), this implies that

$$\left\| u(x) \int_{(x,b)} g(y)\, d\mu \right\|_{q,(a,b),\nu} \geqslant \left(\int_{(T+1/n,b)} g(y)\, d\mu \right) \|u\|_{q,(a,T+1/n],\nu} = +\infty.$$

Thus, (4.1) holds trivially for $g \in S^+(I)$. Consequently, (4.1) is satisfied for all $g \in B^+(I)$ if and only if (4.1) holds for all $g \in B^+(I) \setminus S^+(I)$.

If $g \in B^+(I) \setminus S^+(I)$, then $\int_{(x,b)} g(y)\,d\mu = 0$ for all $x \in [T,b)$. Hence,

$$\begin{aligned}\left\| u(x) \int_{(x,b)} g(y)\,d\mu \right\|_{q,(a,b),\nu} &= \left\| u(x) \int_{(x,b)} g(y)\,d\mu \right\|_{q,(a,T),\nu} \\ &\approx \left\| u(x) \int_{(x,T)} g(y)\,d\mu \right\|_{q,(a,T),\nu} + g(T)\mu(\{T\})\|u\|_{q,(a,T),\nu}.\end{aligned}$$

Since also

$$\|gw\|_{p,(a,b),\mu} \approx \|gw\|_{p,(a,T),\mu} + g(T)w(T)(\mu(\{T\}))^{1/p},$$

we see that (4.1) holds for $g \in B^+(I) \setminus S^+(I)$ if and only if the inequality

$$\begin{aligned}&\|gw\|_{p,(a,T),\mu} + g(T)w(T)(\mu(\{T\}))^{1/p} \\ &\qquad \lesssim \left\| u(x) \int_{(x,T)} g(y)\,d\mu \right\|_{q,(a,T),\nu} + g(T)\mu(\{T\})\|u\|_{q,(a,T),\nu}\end{aligned} \tag{4.72}$$

is satisfied for such g. However, (4.72) holds on $B^+(I) \setminus S^+(I)$ if and only if both (4.69) and

$$g(T)w(T)(\mu(\{T\}))^{1/p} \lesssim g(T)\mu(\{T\})\|u\|_{q,(a,T),\nu}$$

are satisfied on $B^+(I) \setminus S^+(I)$. Consequently, (4.1) holds for all $g \in B^+(I) \setminus S^+(I)$ if and only if both (4.69) holds on $B^+((a,T))$ and (4.70) is satisfied. $\square$

5. The reverse Hardy inequality for the dual operator

Let $I = (a,b) \subseteq \mathbb{R}$ and let μ be a non-negative Borel measure on I. The aim of this section is to characterize the validity of the reverse Hardy inequality involving the operator H^* given by

$$(H^*g)(x) := \int_{(a,x)} g(y)\,d\mu, \quad g \in B^+(I), \quad x \in I,$$

which is the dual operator to that one given by

$$(Hg)(x) := \int_{(x,b)} g(y)\,d\mu, \quad g \in B^+(I), \quad x \in I.$$

To this end, we are going to make use of the results for the Hardy operator H proved in Section 4. Our next assertion is a counterpart of Theorem 4.1.

THEOREM 5.1. *Assume that $0 < q \leqslant p \leqslant 1$. Let μ and ν be non-negative Borel measures on $I = (a,b) \subseteq \mathbb{R}$. Let $w \in B^+(I)$ and let $u \in B^+(I)$ satisfy $\|u\|_{q,[t,b),\nu} < +\infty$ for all $t \in I$. Then the inequality*

$$\|gw\|_{p,(a,b),\mu} \leqslant c \left\| u(x) \int_{(a,x)} g(y)\,d\mu \right\|_{q,(a,b),\nu} \tag{5.1}$$

holds for all $g \in B^+(I)$ *if and only if*

$$B_1 := \sup_{x\in(a,b)} \|w\|_{p',[x,b),\mu} \|u\|^{-1}_{q,(x,b),\nu} < +\infty. \tag{5.2}$$

The best possible constant c *in* (5.1) *satisfies* $c \approx B_1$.

Proof. If λ is a non-negative Borel measure on I, we denote by $\tilde{\lambda}$ a non-negative Borel measure on $\tilde{I} := (-b,-a)$ defined by

$$\tilde{\lambda}(E) := \lambda(-E), \quad \text{where} \quad -E := \{-x : \ x \in E\}.$$

Similarly, if $h \in B^+(I)$, then the function $\tilde{h} \in B^+(\tilde{I})$ is given by

$$\tilde{h}(x) := h(-x), \quad x \in \tilde{I}.$$

Now it is clear that

$$\int_E h\,d\lambda = \int_{-E} \tilde{h}\,d\tilde{\lambda} \tag{5.3}$$

for any Borel subset E of I. (Indeed, this is a consequence of the fact that (5.3) holds for any step function $h \in B^+(I)$.) In particular,

$$\|gw\|_{p,(a,b),\mu} = \|\tilde{g}\tilde{w}\|_{p,(-b,-a),\tilde{\mu}}$$

and

$$\left\| u(x) \int_{(a,x)} g(y)\,d\mu \right\|_{q,(a,b),\nu} = \left\| \tilde{u}(x) \int_{(a,-x)} g\,d\mu \right\|_{q,(-b,-a),\tilde{\nu}}.$$

Moreover, since (by (5.3))

$$\int_{(a,-x)} g\,d\mu = \int_{(x,-a)} \tilde{g}\,d\tilde{\mu} \quad \text{if} \quad x \in (-b,-a),$$

we arrive at

$$\left\| u(x) \int_{(a,x)} g(y)\,d\mu \right\|_{q,(a,b),\nu} = \left\| \tilde{u}(x) \int_{(x,-a)} \tilde{g}\,d\tilde{\mu} \right\|_{q,(-b,-a),\tilde{\nu}}.$$

Consequently, the inequality (5.1) holds for all $g \in B^+(I)$ if and only if the inequality

$$\|\tilde{g}\tilde{w}\|_{p,(-b,-a),\tilde{\mu}} \leqslant c \left\| \tilde{u}(x) \int_{(x,-a)} \tilde{g}\,d\tilde{\mu} \right\|_{q,(-b,-a),\tilde{\nu}} \tag{5.4}$$

holds for all $\tilde{g} \in B^+(\tilde{I})$.

As $\|\tilde{u}\|_{q,(-b,x],\tilde{\nu}} = \|u\|_{q,[-x,b),\nu} < +\infty$ if $x \in (-b,-a)$, we deduce from Theorem 4.1 that the inequality (5.1) holds on $B^+(I)$ if and only if

$$\sup_{x\in(-b,-a)} \|\tilde{w}\|_{p',(-b,x],\tilde{\mu}} \|\tilde{u}\|^{-1}_{q,(-b,x),\tilde{\nu}} < +\infty. \tag{5.5}$$

However, using (5.3) and its analogue

$$\|h\|_{\infty,E,\lambda} = \|\tilde{h}\|_{\infty,-E,\tilde{\lambda}}, \tag{5.6}$$

we see that the condition (5.5) coincides with (5.2). □

REMARK 5.2. Let B_1 be the number defined in Theorem 5.1. If $p = 1$, then

$$B_1 = \left\| w(x)\|u\|_{q,(x,b),\nu}^{-1} \right\|_{\infty,(a,b),\mu}.$$

Indeed, using the idea of the proof of Theorem 5.1, we obtain the result from Remark 4.2.

Consider now the inequality (5.1) on $B^+(I)$ in the case when $0 < p \leqslant 1$, $p < q \leqslant +\infty$ and define r by

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{q}. \tag{5.7}$$

As in Section 4, in such a case we shall write a condition characterizing the validity of the inequality (5.1) on $B^+(I)$ in a compact form involving the Lebesgue-Stieltjes integral $\int_{(a,b)} f\, dh$, say. In contrast to Section 4, now the Lebesgue-Stieltjes integral $\int_{(a,b)} f\, dh$ will be defined by a *non-decreasing* and *left-continuous function* h on I. We shall see in our next theorem that $f(t) = \|w\|_{p',[t,b),\mu}^r$ and $h(t) = \|u\|_{q,[t-,b),\nu}^{-r} := \lim_{s\to t-} \|u\|_{q,[s,b),\nu}^{-r}$, $t \in (a,b)$. However, it can happen that $\|u\|_{q,[t-,b),\nu} = 0$ for all $t \in (c,b)$ with some $c \in (a,b)$ (provided that we omit the trivial case when $u = 0$ ν-a.e. on (a,b)). Then we have to explain what is the meaning of the Lebesgue-Stieltjes integral since in such a case the function $h = +\infty$ on (c,b). To this end, we adopt the following convention.

CONVENTION 5.3. Let $I = (a,b) \subseteq \mathbb{R}$, $f : I \to [0,+\infty]$ and $h : I \to [0,+\infty]$. Assume that h is non-decreasing and left-continuous on I. If $h : I \to [0,+\infty)$, then the symbol $\int_I f\, dh$ means the usual Lebesgue-Stieltjes integral (the measure λ associated to h is given by $\lambda([\alpha,\beta)) = h(\beta) - h(\alpha)$ if $[\alpha,\beta) \subset (a,b)$ – cf. (4.17)). However, if $h = +\infty$ on some subinterval (c,b) with $c \in I$, then we define $\int_I f\, dh$ only if $f = 0$ on $[c,b)$ and we put

$$\int_I f\, dh = \int_{(a,c)} f\, dh.$$

THEOREM 5.4. *Assume that $0 < p \leqslant 1$, $p < q \leqslant +\infty$ and r is given by (5.7). Let μ and ν be non-negative Borel measures on $I = (a,b) \subseteq \mathbb{R}$. Let $w \in B^+(I)$ and let $u \in B^+(I)$ satisfy $\|u\|_{q,[t,b),\nu} < +\infty$ for all $t \in I$ and $u \neq 0$ ν-a.e. on I. Then the inequality (5.1) holds for all $g \in B^+(I)$ if and only if*

$$B_2 := \left(\int_{(a,b)} \|w\|_{p',[t,b),\mu}^r \, d\left(\|u\|_{q,[t-,b),\nu}^{-r} \right) \right)^{\frac{1}{r}} + \frac{\|w\|_{p',(a,b),\mu}}{\|u\|_{q,(a,b),\nu}} < +\infty.$$

The best possible constant c in (5.1) satisfies $c \approx B_2$.

REMARK 5.5. Let $q < +\infty$ in Theorem 5.4. Then

$$\|u\|_{q,[t-,b),v} = \|u\|_{q,[t,b),v} \quad \text{for all} \quad t \in I,$$

which implies that

$$B_2 = \left(\int_{(a,b)} \|w\|^r_{p',[t,b),\mu} \, d\left(\|u\|^{-r}_{q,[t,b),v} \right) \right)^{\frac{1}{r}} + \frac{\|w\|_{p',(a,b),\mu}}{\|u\|_{q,(a,b),v}}.$$

Proof of Theorem 5.4. As in the proof of Theorem 5.1, one can show that the inequality (5.1) holds on $B^+(I)$ if and only if the inequality (5.4) is satisfied for all $\tilde{g} \in B^+(\tilde{I})$. Thus, by Theorem 4.4, the inequality (5.1) holds on $B^+(I)$ if and only if

$$\left(\int_{(-b,-a)} \|\tilde{w}\|^r_{p',(-b,t],\tilde{\mu}} \, d\left(-\|\tilde{u}\|^{-r}_{q,(-b,t+],\tilde{v}} \right) \right)^{\frac{1}{r}} + \frac{\|\tilde{w}\|_{p',(-b,-a),\tilde{\mu}}}{\|\tilde{u}\|_{q,(-b,-a),\tilde{v}}} < +\infty. \tag{5.8}$$

It is clear that

$$\|\tilde{w}\|_{p',(-b,-a),\tilde{\mu}} = \|w\|_{p',(a,b),\mu} \quad \text{and} \quad \|\tilde{u}\|_{q,(-b,-a),\tilde{v}} = \|u\|_{q,(a,b),v}. \tag{5.9}$$

Moreover, by the definition of the Lebesgue-Stieltjes integral,

$$\int_{(-b,-a)} \|\tilde{w}\|^r_{p',(-b,t],\tilde{\mu}} \, d\left(-\|\tilde{u}\|^{-r}_{q,(-b,t+],\tilde{v}} \right) = \int_{(-b,-a)} \|\tilde{w}\|^r_{p',(-b,t],\tilde{\mu}} \, d\tilde{\lambda} =: I, \tag{5.10}$$

where $\tilde{\lambda}$ is the non-negative Borel measure associated to the non-decreasing and right-continuous function $\tilde{\varphi} := -\|\tilde{u}\|^{-r}_{q,(-b,t+],\tilde{v}}$, $t \in (-b,-a)$, that is,

$$\tilde{\lambda}((\tilde{\alpha},\tilde{\beta}]) = \tilde{\varphi}(\tilde{\beta}) - \tilde{\varphi}(\tilde{\alpha}) \quad \text{for any} \quad (\tilde{\alpha},\tilde{\beta}] \subset (-b,-a).$$

Since, by (5.3) or (5.6),

$$\|\tilde{w}\|^r_{p',(-b,t],\tilde{\mu}} = \|w\|^r_{p',[-t,b),\mu} \quad \text{for all} \quad t \in (-b,-a),$$

we obtain from (5.3) that

$$I = \int_{(-b,-a)} \|\tilde{w}\|^r_{p',(-b,t],\tilde{\mu}} \, d\tilde{\lambda} = \int_{(a,b)} \|w\|^r_{p',[t,b),\mu} \, d\lambda, \tag{5.11}$$

where $\lambda(E) = \tilde{\lambda}(-E)$ if E is a Borel subset of I. In particular, if $[\alpha,\beta) \subset (a,b)$, then

$$\begin{aligned} \lambda([\alpha,\beta)) &= \tilde{\lambda}((-\beta,-\alpha]) = \tilde{\varphi}(-\alpha) - \tilde{\varphi}(-\beta) \\ &= -\|\tilde{u}\|^{-r}_{q,(-b,-\alpha+],\tilde{v}} + \|\tilde{u}\|^{-r}_{q,(-b,-\beta+],\tilde{v}} \\ &= -\|u\|^{-r}_{q,[\alpha-,b),v} + \|u\|^{-r}_{q,[\beta-,b),v} \end{aligned}$$

(the last equality follows from (5.3) and (5.6)). That means (cf. (4.17)) that the non-negative Borel measure λ is associated to the non-decreasing and left-continuous function φ given on (a,b) by

$$\varphi(t) := \|u\|^{-r}_{q,[t-,b),v}, \quad t \in (a,b).$$

Consequently,

$$\int_{(a,b)} \|w\|^r_{p',[t,b),\mu}\, d\lambda = \int_{(a,b)} \|w\|^r_{p',[t,b),\mu}\, d\left(\|u\|^{-r}_{q,[t-,b),\nu}\right). \tag{5.12}$$

The result now follows from (5.8)–(5.12). □

The following assertion is a counterpart of Theorem 4.6.

THEOREM 5.6. *Suppose that all the assumptions of Theorem* 5.4 *are satisfied.*
(i) *Let*

$$\|w\|_{p',[t-,b),\mu} \lesssim \|w\|_{p',[t,b],\mu} \quad \textit{for all} \quad t \in (a,b). \tag{5.13}$$

Then $B_2 \approx B_3$, *where*

$$B_3 := \left(\int_{(a,b)} \|u\|^{-r}_{q,(t,b),\nu}\, d\left(-\|w\|^r_{p',[t-,b),\mu}\right)\right)^{\frac{1}{r}} + \frac{\lim_{t\to b-} \|w\|_{p',(t,b),\mu}}{\lim_{t\to b-} \|u\|_{q,(t,b),\nu}}.$$

(ii) *Let*

$$\|w\|_{p',[t,b),\mu} \lesssim \|w\|_{p',(t,b),\mu} \quad \textit{for all} \quad t \in (a,b). \tag{5.14}$$

Then $B_2 \approx B_4$, *where*

$$B_4 := \left(\int_{(a,b)} \|u\|^{-r}_{q,[t-,b),\nu}\, d\left(-\|w\|^r_{p',[t-,b),\mu}\right)\right)^{\frac{1}{r}} + \frac{\lim_{t\to b-} \|w\|_{p',(t,b),\mu}}{\lim_{t\to b-} \|u\|_{q,(t,b),\nu}}.$$

Proof. It is left to the reader. □

We conclude this section with counterparts of Remarks 4.7, 4.8, 4.10 and Theorem 4.9 (proofs are left to the reader).

REMARK 5.7. Theorems 5.4 and 5.6 imply that if the weight w satisfies (5.13), then the conditions $B_2 < +\infty$ and $B_3 < +\infty$ are equivalent. Similarly, if w satisfies (5.14), then the conditions $B_2 < +\infty$ and $B_4 < +\infty$ are equivalent.

Note also that if $p' < +\infty$ and $\|w\|_{p',[t,b),\mu} < +\infty$ for all $t \in (a,b)$, then $\|w\|_{p',[t-,b),\mu} = \|w\|_{p',[t,b),\mu}$ for all $t \in (a,b)$ and therefore w satisfies (5.13). In this case the condition $B_3 < +\infty$ reduces to

$$\left(\int_{(a,b)} \|u\|^{-r}_{q,(t,b),\nu}\, d\left(-\|w\|^r_{p',[t-,b),\mu}\right)\right)^{\frac{1}{r}} < +\infty \tag{5.15}$$

since $\lim_{t\to b-} \|w\|_{p',(t,b),\mu} = 0$.

Suppose now that $p' = +\infty$, the weight w satisfies (5.13), and $B_3 < +\infty$. Moreover, let $\lim_{t\to b-} \|u\|_{q,(t,b),\nu} = 0$. Then the second term in B_3 has to be finite which can happen only if $\lim_{t\to b-} \|w\|_{\infty,(t,b),\mu} = 0$ (cf. our convention that $0/0 = 0$).

Let μ be a non-negative Borel measure on (a,b) which has no atoms. Then it is clear that the condition (5.14) holds.

Suppose now that $p' = +\infty$, the weight w satisfies (5.14), and $B_4 < +\infty$. Moreover, let $\lim_{t\to b-} \|u\|_{q,(t,b),\nu} = 0$. Then the second term in B_4 has to be finite which can happen only if $\lim_{t\to b-} \|w\|_{\infty,(t,b),\mu} = 0$ (cf. our convention that $0/0 = 0$).

REMARK 5.8. The implication $B_3 < +\infty \Rightarrow B_2 \lesssim B_3$ holds without the additional assumption (5.13). (This follows from Remark 4.8.) Similarly, the implication $B_2 < +\infty \Rightarrow B_4 \lesssim B_2$ holds without the additional assumption (5.14). Consequently, under the assumptions of Theorem 5.4,

$$B_3 < +\infty \Rightarrow B_4 \lesssim B_2 \lesssim B_3.$$

Moreover, if

$$\|u\|_{q,[t-,b),\nu} \lesssim \|u\|_{q,(t,b),\nu}, \tag{5.16}$$

then $B_3 \lesssim B_4$. Consequently, if (5.16) holds, then under the assumptions of Theorem 5.4, $B_2 \approx B_3 \approx B_4$.

To characterize the validity of inequality (5.1) on $B^+(I)$, in Theorems 5.1 and 5.4 we have supposed that the weight function u satisfies

$$\|u\|_{q,[t,b),\nu} < +\infty \quad \text{for all} \quad t \in I. \tag{5.17}$$

The next theorem concerns the case when (5.17) is violated.

THEOREM 5.9. *Assume that* $0 < p \leqslant 1$ *and* $0 < q \leqslant +\infty$. *Let* μ *and* ν *be non-negative Borel measures on* $I = (a,b) \subseteq \mathbb{R}$. *Let* $w \in B^+(I)$ *and let* $u \in B^+(I)$ *be such that* (5.17) *does not hold. Put*

$$T := \sup\{t \in I : \|u\|_{q,[t,b),\nu} = +\infty\}.$$

(*i*) *If* $T = b$, *then the inequality* (5.1) *holds for all* $g \in B^+(I)$.

(*ii*) *If* $T \in (a,b)$ *and* $\mu(\{T\}) = 0$, *then the inequality* (5.1) *holds for all* $g \in B^+(I)$ *if and only if the inequality*

$$\|gw\|_{p,(T,b),\mu} \lesssim \left\| u(x) \int_{(T,x)} g(y)\, d\mu \right\|_{q,(T,b),\nu} \tag{5.18}$$

holds for all $g \in B^+((T,b))$.

(*iii*) *If* $T \in (a,b)$ *and* $\mu(\{T\}) > 0$, *then the inequality* (5.1) *holds for all* $g \in B^+(I)$ *if and only if*

$$w(T)(\mu(\{T\}))^{1/p} \lesssim \mu(\{T\}) \|u\|_{q,(T,b),\nu} \tag{5.19}$$

and the inequality (5.18) *is satisfied for all* $g \in B^+((a,T))$.

REMARK 5.10. To characterize the validity of inequality (5.18) on (T,b) if $T \in (a,b)$, one can use Theorems 5.1 and 5.4 since $\|u\|_{q,[t,b),\nu} < +\infty$ for all $t \in (T,b)$.

Note also that condition (5.19) holds if and only if

$$\begin{aligned}
&\text{either} && 0 < \|u\|_{q,(T,b),\nu} < +\infty && \text{and} && w(T) < +\infty, \\
&\text{or} && 0 < \|u\|_{q,(T,b),\nu} < +\infty && \text{and} && w(T) = +\infty \quad \text{and} \quad \mu(\{T\}) = +\infty, \\
&\text{or} && \|u\|_{q,(T,b),\nu} = 0 && \text{and} && w(T) = 0, \\
&\text{or} && \|u\|_{q,(T,b),\nu} = +\infty. &&
\end{aligned}$$

6. Reverse Hardy inequalities involving three measures

So far we have studied the reverse Hardy inequalities of the form

$$\|gw\|_{p,(a,b),\mu} \leqslant c \left\| u(x) \int_{S_x} g(y)\, d\mu \right\|_{q,(a,b),\nu}, \quad g \in B^+(I), \tag{6.1}$$

where $0 < p \leqslant 1$, $0 < q \leqslant +\infty$, μ and ν are non-negative Borel measures on $I := (a,b) \subseteq \mathbb{R}$, w and u are weight functions on I and either $S_x = (a,x)$ or $S_x = (x,b)$ for all $x \in I$.

Now, we replace the left-hand side of inequality (6.1) by $\|g\|_{p,(a,b),\lambda}$, where λ is a non-negative Borel measure on I, that is, we consider the inequality

$$\|g\|_{p,(a,b),\lambda} \leqslant c \left\| u(x) \int_{S_x} g(y)\, d\mu \right\|_{q,(a,b),\nu}, \quad g \in B^+(I). \tag{6.2}$$

We claim that to characterize the validity of (6.2) on $B^+(I)$ it is enough to characterize the validity of (6.1) on $B^+(I)$. To see it, assume that (6.2) holds for all $g \in B^+(I)$. Let $E \subseteq I$ be such that $\mu(E) = 0$ and put $g = \chi_E$. Then

$$0 \leqslant \int_{S_x} g(y)\, d\mu = \mu(S_x \cap E) \leqslant \mu(E) = 0 \quad \text{for all} \quad x \in I.$$

Therefore, the right-hand side of (6.2) is zero, which implies that

$$0 = \|g\|_{p,(a,b),\lambda} = \left(\int_E d\lambda \right)^{1/p} = (\lambda(E))^{1/p},$$

that is, $\lambda(E) = 0$. Hence, the measure λ is absolutely continuos with respect to μ, and, by the Radon-Nikodym theorem, there is $v \in B^+(I)$ such that $d\lambda = v\, d\mu$. Putting $w = v^{1/p}$, we have $d\lambda = w^p\, d\mu$. Consequently, for any $g \in B^+(I)$, we can rewrite the left-hand side of (6.2) as

$$\|g\|_{p,(a,b),\lambda} = \left(\int_{(a,b)} g^p\, d\lambda \right)^{1/p} = \left(\int_{(a,b)} (gw)^p\, d\mu \right)^{1/p} = \|gw\|_{p,(a,b),\mu},$$

and our claim follows.

COROLLARY 6.1. *Assume that* $0 < p \leqslant 1$, $0 < q \leqslant +\infty$. *Let* λ, μ *and* ν *be non-negative Borel measures on* $I := (a,b) \subseteq \mathbb{R}$ *and let* $u \in B^+(I)$. *Then the inequality* (6.2) *holds for all* $g \in B^+(I)$ *if and only if the measure* λ *is absolutely continuous with respect to* μ *and the inequality* (6.1) *with* $w := \left(\frac{d\lambda}{d\mu}\right)^{1/p}$ *holds for all* $g \in B^+(I)$.

REFERENCES

[1] P. R. BEESACK, H. P. HEINIG, *Hardy's inequalities with indices less than 1*, Proc. Amer. Math. Soc., **83** (1981), 532–536.

[2] G. BENNETT, *Factorizing the classical inequalities*, Mem. Amer. Math. Soc., **120** (1996), no. 576.

[3] G. A. BLISS, *An integral inequality*, J. London Math. Soc., **5** (1930), 40–46.

[4] E. T. COPSON, *Note on a series of positive terms*, J. London Math. Soc., **2** (1927), 9–12.

[5] E. T. COPSON, *Note on a series of positive terms*, J. London Math. Soc., **3** (1928), 49–51.

[6] A. GOGATISHVILI, L. PICK, *Discretization and anti-discretization of rearrangement-invariant norms*, Publ. Mat., **47**, (2003) 311–358.

[7] K.-G. GROSSE-ERDMANN, *The Blocking Technique, Weighted Mean Operators and Hardy's Inequality*, Lect. Notes Math. 1679, Springer, Berlin, 1998.

[8] G.H. HARDY, *Note on a theorem of Hilbert*, Math. Zeitschr., **6** (1920), 314–317.

[9] A. KUFNER, L.-E. PERSSON, *Weighted inequalities of Hardy type*, World Scientific Publishing Co, Singapore, 2003.

[10] E. LANDAU, *A note on a theorem concerning series of positive terms*, J. London Math. Soc., **1** (1926), 38–39.

[11] L. LEINDLER, *Inequalities of Hardy and Littlewood type*, Acta Sci. Math. (Szeged), **2** (1976), 117–123.

[12] L. LEINDLER, *On the converses of inequalities of Hardy and Littlewood*, Acta Sci. Math. (Szeged), **58** (1993), 191–196.

[13] B. OPIC, A. KUFNER, *Hardy–type inequalities*, Pitman Reaserch Notes in Mathematics Series, 219, Longman Scientific & Technical, Harlow, 1990.

[14] D. V. PROKHOROV, *Weighted Hardy's inequalities for negative indices*, Publ. Mat., **48** (2004), 423–443.

[15] W. RUDIN, *Principles of Mathematical Analysis*, McGraw-Hill Book Company, New York, 2nd ed., 1964.

[16] E. SAWYER, *Weighted Lebesgue and Lorentz norm inequalities for the Hardy operator*, Trans. Amer. Math. Soc., **281** (1984), 329–337.

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