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AN APPROXIMATION APPROACH TO EIGENVALUE INTERVALS FOR SINGULAR BOUNDARY VALUE PROBLEMS WITH SIGN CHANGING NONLINEARITIES

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Abstract. This paper presents new existence results for the singular boundary value problem

$$\begin{cases} -u'' = g(t,u) + \lambda h(t,u), \ t \in (0,1) \\ u(0) = 0 = u(1). \end{cases}$$

In particular our nonlinearity may be singular at $t = 0, 1$ and $u = 0$ and is allowed to change sign. Existence in this paper will be established by obtaining a sequence of upper and lower solutions which in turn will generate a sequence of approximate solutions.

1. Introduction

In this paper we study positive solutions of the second order boundary value problem

$$\begin{cases} -u'' = g(t,u) + \lambda h(t,u), \ t \in (0,1) \\ u(0) = 0 = u(1) \end{cases} \tag{1.1}$$

where $\lambda \geqslant \lambda_0 > 0$ is a constant; an estimate on λ_0 will also be provided. Here $g : (0,1) \times (0,\infty) \to R$ and $h : (0,1) \times [0,\infty) \to [0,\infty)$ are continuous so as a result our nonlinearity may be singular at $t = 0, 1$ and $u = 0$. Also our nonlinearity may change sign. A function u is a solution of the boundary value problem (1.1) if $u : [0,1] \to R$, u satisfies the differential equation (1.1) on $(0,1)$ and the stated boundary data.

We let $C[0,1]$ denote the class of maps u continuous on $[0,1]$, with norm $|u|_\infty = \max_{t\in[0,1]} |u(t)|$. Let

$$M = \left\{ h \in C(0,1) : \begin{array}{l} \int_0^1 |h(s)|\, ds < \infty \ \text{ with} \\ \lim_{t\to 0^+} t\,|h(t)| < \infty \text{ and } \lim_{t\to 1^-} (1-t)\,|h(t)| < \infty \end{array} \right\}.$$

The main result of the paper is the following.

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THEOREM 1.1. *Suppose the following conditions hold:*

$(G1)$ *there exist* $g_i : (0,1)\times(0,\infty)\to(0,\infty)$ $(i=1,2)$ *continuous functions such that*

$$\begin{cases} g_i(t,\cdot) \text{ is strictly decreasing for } t\in(0,1),\\ -g_1(t,r)\leqslant g(t,r)\leqslant g_2(t,r) \text{ for } (t,r)\in(0,1)\times(0,\infty),\\ \quad \text{for all } r_1>r_2>0 \text{ there exists } \gamma(\cdot)\in M \text{ such that}\\ g_2(t,r)+\gamma(t)\,r \text{ is increasing in } (r_1,r_2); \end{cases}$$

$(G2)$ $g_1(\cdot, r\phi_1(\cdot)), g_2(\cdot,r)\in M$ *for all* $r>0$;

$(H1)$ *there exist* $h_i : (0,1)\times[0,\infty)\to(0,\infty)$ $(i=1,2)$ *continuous functions such that*

$$\begin{cases} h_i(t,\cdot) \text{ is increasing for } t\in(0,1),\\ h_1(t,r)\leqslant h(t,r)\leqslant h_2(t,r) \text{ for } (t,r)\in(0,1)\times(0,\infty),\\ \lim\limits_{r\to\infty}\dfrac{h_2(t,r)}{r}=0 \text{ for } t\in(0,1) \text{ and}\\ \text{there exists } \overline{r}>0 \text{ such that } h_1(t,\overline{r})>0 \text{ for all } t\in(0,1); \end{cases}$$

$(H2)$ $h_1(\cdot,r), h_2(\cdot,r)\in M$ *for all* $r>0$.

Then there exists $\lambda_0>0$ *such that for every* $\lambda\geqslant\lambda_0$ *problem* (1.1) *has at least one solution* $u\in C[0,1]\cap C^1(0,1)$ *and* $u>0$ *for* $t\in(0,1)$. *Moreover, there exists* $c_i=c_i(\lambda,g,h,\phi_1)>0$ $(i=1,2)$ *such that*

$$c_1\phi_1(t)\leqslant u(t)\leqslant c_2(\phi_1(t)+1) \text{ for } t\in[0,1],$$

where ϕ_1 *is defined in Lemma* 2.1.

REMARK 1.1. In [3], the authors consider the boundary value problem

$$\begin{cases} -u''=f(t,u),\ t\in(0,1)\\ u(0)=0=u(1) \end{cases} \tag{1.2}$$

under the conditions

(i_1) there exists a constant $L>0$ such that for any compact set $K\subset(0,1)$, there is $\varepsilon=\varepsilon_K>0$ such that

$$f(t,r)\geqslant L \text{ for all } t\in K,\ r\in(0,\varepsilon];$$

(i_2) for any $\delta>0$ there is $h_\delta\in C((0,1),R^+)$ with

$$|f(t,r)|\leqslant h_\delta(t) \text{ for all } t\in(0,1),\ r\geqslant\delta,$$

and

$$\int_0^1 t(1-t)\,h_\delta(t)\,dt<\infty.$$

Then problem (1.2) has at least one solution $u\in C([0,1],R^+)\cap C^2((0,1),R^+)$.

In Section 3 we give an example (see Example 3.1) which satisfies the conditions in Theorem 1.1 but it does not satisfy the conditions in Remark 1.1.

REMARK 1.2. An estimate for λ_0 will be given in the proof of Lemma 2.7.

2. Proof of Theorem 1.1

We first give some lemmas which will help us to prove Theorem 1.1.

LEMMA 2.1. *Consider the following eigenvalue problem*

$$\begin{cases} -u'' = \lambda u(t), \ t \in (0,1) \\ u(0) = u(1) = 0. \end{cases} \tag{2.1}$$

Then the eigenvalues are

$$\lambda_m = (m\pi)^2 \ \textit{for}\ m = 1,2,\cdots$$

and the corresponding eigenfunctions are

$$\phi_m(t) = \sin m\pi t \ \textit{for}\ m = 1,2,\cdots.$$

Let $G(t,s)$ be the Green's function for the BVP

$$\begin{cases} -u'' = 0 \text{ for } t \in (0,1) \\ u(0) = u(1) = 0. \end{cases}$$

Then

$$G(t,s) = \begin{cases} s(1-t), & 0 \leqslant s < t \leqslant 1 \\ t(1-s), & 0 \leqslant t < s \leqslant 1. \end{cases} \tag{2.2}$$

Also for all $(t,s) \in [0,1] \times [0,1]$ define

$$N(t,s) = \begin{cases} \dfrac{G(t,s)}{\phi_1(t)} & \text{if } t \neq 0,1 \\ \dfrac{1-s}{\pi} & \text{if } t = 0 \\ \dfrac{s}{\pi} & \text{if } t = 1. \end{cases} \tag{2.3}$$

It follows easily that

$$\begin{aligned} &0 < G(t,s) \leqslant t(1-t) && \text{for } t \in (0,1) \text{ and } s \in (0,1); \\ &\frac{s(1-s)}{2\pi} \leqslant N(t,s) \leqslant \frac{1}{2} && \text{for all } t \in (0,1) \text{ and } s \in (0,1). \end{aligned} \tag{2.4}$$

Define the operator $A, B : M \to C[0,1]$ by

$$Ax(t) = \int_0^1 G(t,s)\,x(s)\,ds. \tag{2.5}$$

and

$$Bx(t) = \int_0^1 N(t,s)\,x(s)\,ds. \tag{2.6}$$

LEMMA 2.2. *Let* $E \subset M$ *and* $\beta \in M$. *If* $|x(t)| \leqslant \beta(t),\ t \in (0,1)$ *for all* $x \in E$, *then* $A(E)$ *and* $B(E)$ *are relatively compact with respect to the topology of* $C[0,1]$.

Proof. Now since

$$Ax(t) \leqslant \left| \int_0^1 G(t,s)\, x(s)\, ds \right| \leqslant \int_0^1 s(1-s)\, \beta(s)\, ds,$$

we know that $A(E)$ is uniformly bounded. On the other hand, $\forall x \in E,$ we find

$$\begin{aligned}
\left|(Ax)'(t)\right| &= \left| \frac{d}{dt} \int_0^t s(1-t)\, x(s)\, ds + \frac{d}{dt} \int_t^1 t(1-s)\, x(s)\, ds \right| \\
&\leqslant \int_0^t s\,|x(s)|\, ds + \int_t^1 (1-s)\,|x(s)|\, ds \\
&\leqslant \int_0^t s\beta(s)\, ds + \int_t^1 (1-s)\, \beta(s)\, ds \\
&\equiv \gamma(t) \ \text{ for } t \in (0,1).
\end{aligned}$$

We now prove that $\gamma \in L^1((0,1), R)$. This is sufficient to ensure the compactness of the image $A(E)$ via the Arzela-Ascoli theorem.

A simple computation yields

$$\begin{aligned}
\int_0^1 |\gamma(s)|\, ds &= \int_0^1 \gamma(s)\, ds \\
&\leqslant \lim_{t\to 1^-} (1-t) \int_0^t s\beta(s)ds + \lim_{t\to 0^+} t \int_t^1 (1-s)\beta(s)ds + 2\int_0^1 s(1-s)\beta(s)ds \\
&\leqslant 4 \int_0^1 s(1-s)\, \beta(s)\, ds < \infty.
\end{aligned}$$

Thus $A(E)$ is relatively compact with respect to the topology of $C[0,1]$. We next prove $B(E)$ is relatively compact with respect to the topology of $C[0,1]$. Now since

$$Bx(t) \leqslant \left| \int_0^1 N(t,s)\, x(s)\, ds \right| \leqslant \frac{1}{2} \int_0^1 \beta(s)\, ds,$$

we know that $B(E)$ is uniformly bounded. On the other hand, $\forall x \in E,$ we let

$$Bx(t) = h(t)\, Vx(t)$$

where

$$h(t) = \begin{cases} \dfrac{t(1-t)}{\phi_1(t)} & \text{for } t \in (0,1) \\[2ex] \dfrac{1}{\pi} & \text{for } t = 0,1 \end{cases}$$

and

$$Vx(t) = \frac{1}{h(t)} Bx(t).$$

Now, $\forall x \in E,$ we find

$$\begin{aligned} \left|(Vx)'(t)\right| &= \left|\frac{d}{dt}\int_0^t \frac{s(1-t)}{t(1-t)}x(s)\,ds + \frac{d}{dt}\int_t^1 \frac{t(1-s)}{t(1-t)}x(s)\,ds\right| \\ &= \left|-\frac{1}{t^2}\int_0^t sx(s)\,ds + \frac{1}{(1-t)^2}\int_t^1 (1-s)\,x(s)\,ds\right| \\ &\leqslant \frac{1}{t^2}\int_0^t s\beta(s)\,ds + \frac{1}{(1-t)^2}\int_t^1 (1-s)\,\beta(s)\,ds \\ &\equiv \tau(t) \ \text{ for } t \in (0,1). \end{aligned}$$

We now prove that $\tau \in L^1((0,1), R)$. This is sufficient to ensure the compactness of the image $V(E)$ via the Arzela-Ascoli theorem.

A simple computation yields

$$\begin{aligned} \int_0^1 |\tau(s)|\,ds &= \int_0^1 \tau(s)\,ds \\ &\leqslant \int_0^1 \beta(s) + \lim_{s\to 0^+} s\beta(s) + \lim_{s\to 1^-}(1-s)\,\beta(s) < \infty. \end{aligned}$$

Thus $V(E)$ is relatively compact with respect to the topology of $C[0,1]$. Finally, for any $0 \leqslant t_1, t_2 \leqslant 1,$ note

$$\begin{aligned} |Bx(t_1) - Bx(t_2)| &= |h(t_1)\,Vx(t_1) - h(t_2)\,Vx(t_2)| \\ &\leqslant h(t_1)\,|Vx(t_1) - Vx(t_2)| + |Vx(t_2)|\,|h(t_1) - h(t_2)| \\ &\leqslant c_1\,|Vx(t_1) - Vx(t_2)| + c_2\,|h(t_1) - h(t_2)| \end{aligned}$$

where $c_i\,(i = 1,2) > 0$ are constants. Thus $B(E)$ is relatively compact with respect to the topology of $C[0,1]$.

Next we consider the boundary value problem

$$\begin{cases} -u'' + a(t)\,u(t) = f(t), & t \in (0,1) \\ u(0) = 0 = u(1) \end{cases} \tag{2.7}$$

where $a, f \in M,$ $a(t) \geqslant 0$ for $t \in (0,1)$.

LEMMA 2.3. (1 , pp 69) *The following statements hold:*

(i) *for any* $f \in M,$ (2.7) *is uniquely solvable and*

$$u + A(au) = A(f);$$

(ii) *if* $f(t) \geqslant 0$ *for* $t \in (0,1)$, *then the solution of* (2.7) *is nonnegative.*

COROLLARY 2.1. *Let* $\Phi : M \to C[0,1] \cap C^1(0,1)$ *be the operator such that* $\Phi(f)$ *is the solution of* (2.7). *Then we have*

(i) *if* $f_1(t) \leqslant f_2(t)$ *for* $t \in (0,1)$, *then* $\Phi(f_1)(t) \leqslant \Phi(f_2)(t)$ *for* $t \in [0,1]$;

(ii) *let* $E \subset M$ *and* $\beta \in M$. *If* $|f(t)| \leqslant \beta(t)$, $t \in (0,1)$ *for all* $f \in E,$ *then* $\Phi(E)$ *is relatively compact with respect to the topology of* $C[0,1]$.

LEMMA 2.4. *Suppose* $(G1)$, $(G2)$, $(H1)$ *and* $(H2)$ *hold. Let* $n_0 \in N$. *Assume for every* $n > n_0$, *there exist* $a_n, \delta_n, \delta \in M$ *such that*

$$0 \leqslant a_n(t), \ |\delta_n(t)| \leqslant \delta(t) \ \text{and} \ \lim_{n\to\infty} \delta_n(t) = 0 \ \text{for} \ t \in (0,1)$$

and there exist $\overline{u}, \overline{u}_n, \widehat{u}_n, \widehat{u} \in C[0,1]$ *such that*

$$0 < \overline{u}(t) \leqslant \overline{u}_n(t) \leqslant \widehat{u}_n(t) \leqslant \widehat{u}(t) \ \text{for} \ t \in (0,1)$$

and $\widehat{u}(0) = \widehat{u}(1) = 0$. *If*

$$-\overline{u}_n''(t) + a_n(t)\overline{u}_n(t) \leqslant g\left(t, \frac{1}{n} + v\right) + \lambda h(t,v) + \delta_n(t) + a_n(t)v(t) \quad \text{for} \ t \in (0,1) \tag{2.8}$$

and

$$-\widehat{u}_n''(t) + a_n(t)\widehat{u}_n(t) \geqslant g\left(t, \frac{1}{n} + v\right) + \lambda h(t,v) + \delta_n(t) + a_n(t)v(t) \quad \text{for} \ t \in (0,1) \tag{2.9}$$

where $\lambda \geqslant 0$ *and* $v \in [\overline{u}_n, \widehat{u}_n] = \{u \in C[0,1],\ \overline{u}_n(t) \leqslant u(t) \leqslant \widehat{u}_n(t) \ \text{for} \ t \in [0,1]\}$, *then problem* (1.1) *has a solution* $u \in C[0,1] \cap C^1(0,1)$ *such that* $\overline{u}(t) \leqslant u(t) \leqslant \widehat{u}(t)$ *for* $t \in [0,1]$.

Proof. Fix $v \in [\overline{u}, \widehat{u}]$. From Lemma 2.3, there exists $\Psi(v) \in C[0,1]$ such that

$$\begin{cases} -\Psi''(v)(t) + a_n(t)\Psi(v)(t) = g\left(t, \frac{1}{n} + v\right) + \lambda h(t,v) + a_n(t)v(t) + \delta_n(t) \\ \qquad \text{for } t \in (0,1) \\ \Psi(v)(0) = \Psi(v)(1) = 0. \end{cases}$$

Then

$$\Psi(v)(t) + A(a_n\Psi(v))(t) = A\left(g\left(\cdot, \frac{1}{n} + v\right) + \lambda h(\cdot, v) + a_n v + \delta_n\right)(t) \ \text{for } t \in (0,1).$$

By Corollary 2.1, we have

$$\overline{u}_n(t) \leqslant \Psi(v)(t) \leqslant \widehat{u}_n(t) \ \text{for } t \in [0,1].$$

Also

$$\begin{aligned} \left| g\left(t, \frac{1}{n} + v\right) + \lambda h(t,v) + a_n v + \delta_n \right| &\leqslant g_1\left(t, \frac{\phi_1(t)}{n}\right) + g_2\left(t, \frac{1}{n}\right) \\ &\quad + \lambda h_2\left(t, |\widehat{u}_n|_\infty\right) + a_n |\widehat{u}_n|_\infty + |\delta_n(t)| \\ &\equiv \beta(t) \in M \ \text{for } t \in (0,1). \end{aligned}$$

By Corollary 2.1 we have that $\Psi : [\overline{u}_n, \widehat{u}_n] \to [\overline{u}_n, \widehat{u}_n]$ is relatively compact. From Schauder's fixed point theorem, there exists $u_n \in C[0,1]$ such that $\overline{u}_n(t) \leqslant u_n(t) \leqslant \widehat{u}_n(t)$ and $\Psi(u_n)(t) = u_n(t)$ for $t \in (0,1)$. Note

$$\begin{cases} -u_n''(t) = g\left(t, \frac{1}{n} + u_n\right) + \lambda h(t, u_n) + \delta_n(t) \ \text{for } t \in (0,1) \\ u_n(0) = u_n(1) = 0. \end{cases}$$

Fix $m_0 \in \{2,3,\cdots\}$. Let us look at the interval $\left[1/2^{m_0+1}, 1-1/2^{m_0+1}\right]$. Let

$$R_{m_0} = \sup\{|g(t,u)| + \lambda\,|h(t,u)| + |\delta(t)| \\ : t \in \left[1/2^{m_0+1}, 1-1/2^{m_0+1}\right] \text{ and } \overline{u}(t) \leqslant u(t) \leqslant \widehat{u}(t)+1\}.$$

The mean value theorem implies that there exists $\tau \in \left(1/2^{m_0+1}, 1-1/2^{m_0+1}\right)$ with $|u_n'(\tau)| \leqslant 2\sup_{t\in[0,1]} \widehat{u}(t)$. As a result

$$\{u_n\}_{n=m_0+1}^{\infty} \text{ is bounded, equicontinuous family on } \left[1/2^{m_0+1}, 1-1/2^{m_0+1}\right].$$

The Arzela-Ascoli theorem guarantees the existence of subsequence N_{m_0} of integers and a function $z_{m_0} \in \left[1/2^{m_0+1}, 1-1/2^{m_0+1}\right]$ with u_n converging uniformly to z_{m_0} on $\left[1/2^{m_0+1}, 1-1/2^{m_0+1}\right]$ as $n \to \infty$ through N_{m_0}. Similarly

$$\{u_n\}_{n=m_0+1}^{\infty} \text{ is bounded, equicontinuous family on } \left[1/2^{m_0+2}, 1-1/2^{m_0+2}\right]$$

so there is a subsequence N_{m_0+1} of N_{m_0} and a function $z_{m_0+1} \in C\left[1/2^{m_0+2}, 1-1/2^{m_0+2}\right]$ with u_n converging uniformly to z_{m_0+1} on $\left[1/2^{m_0+2}, 1-1/2^{m_0+2}\right]$ as $n \to \infty$ through N_{m_0+1}. Note $z_{n_0+1} = z_{n_0}$ on $\left[1/2^{m_0+1}, 1-1/2^{m_0+1}\right]$ since $N_{m_0+1} \subseteq N_{m_0}$. Proceed inductively to obtain subsequences of integers $N_{m_0} \supseteq N_{m_0+1} \supseteq \cdots \supseteq N_k \supseteq \cdots$ and functions $z_k \in C\left[1/2^{k+1}, 1-1/2^{k+1}\right]$ with u_n converging uniformly to z_k on $\left[1/2^{k+1}, 1-1/2^{k+1}\right]$ as $n \to \infty$ through N_k, and $z_k = z_{k-1}$ on $\left[1/2^{k}, 1-1/2^{k}\right]$.

Define a function $u : [0,1] \to [0,\infty)$ by $u(t) = z_k(t)$ on $\left[1/2^{k+1}, 1-1/2^{k+1}\right]$ and $u(0) = u(1) = 0$. Notice u is well defined and $\overline{u}(t) \leqslant u(t) \leqslant \widehat{u}(t)$ for $t \in (0,1)$. Next fix $t \in (0,1)$ (without loss of generality assume $t \neq \frac{1}{2}$) and let $m \in \{m_0, m_0+1,\cdots\}$ be such that $1/2^{m+1} < t < 1-1/2^{m+1}$. Let $N_m^* = \{n \in N_m : n \geqslant m\}$. Now $u_n,\ n \in N_m^*$ satisfies the integral equation

$$u_n(t) = u_n\left(\frac{1}{2}\right) + u_n'\left(\frac{1}{2}\right)\left(t-\frac{1}{2}\right) + \int_{1/2}^{t}(s-t)\left(g\left(s,\frac{1}{n}+u_n\right)+\lambda h(s,u_n)+\delta_n(s)\right)ds$$

for $t \in \left[1/2^{m+1}, 1-1/2^{m+1}\right]$. Notice (take $t = 2/3$ say) that $\{u_n'(1/2)\},\ n \in N_m^*$ is a bounded sequence since $\overline{u}(t) \leqslant u_n(t) \leqslant \widehat{u}(t)$ for $t \in [0,1]$. Thus $\{u_n'(1/2)\}_{n\in N_m^*}$ has a convergent subsequence; for convenience we will let $\{u_n'(1/2)\}_{n\in N_m^*}$ denote this subsequence also, and let $\tau \in R$ be its limit. Now for the above fixed t, and let $n \to \infty$ through N_m^* to obtain

$$z_m(t) = z_m\left(\frac{1}{2}\right) + \tau\left(t-\frac{1}{2}\right) + \int_{1/2}^{t}(s-t)\left(g(s,z_m)+\lambda h(s,z_m)+\delta_m\right)ds,$$

i.e,

$$u(t) = u\left(\frac{1}{2}\right) + \tau\left(t-\frac{1}{2}\right) + \int_{1/2}^{t}(s-t)\left(g(s,u)+\lambda h(s,u)\right)ds.$$

We can do this argument for each $t \in (0,1)$ and so

$$-u''(t) = g(t,u) + \lambda h(t,u) \text{ for } t \in (0,1).$$

It remains to show u is continuous at 0 and 1.

Let $\varepsilon > 0$ be given. Since $\widehat{u} \in C[0,1]$ there exists $\delta > 0$ with $\widehat{u}(t) < \varepsilon/2$ for $t \in [0,\delta]$. As a result $\overline{u}(t) \leqslant u_n(t) \leqslant \widehat{u}(t) < \varepsilon/2$ for $t \in [0,\delta]$. Consequently $\overline{u}(t) \leqslant u(t) \leqslant \varepsilon/2 < \varepsilon$ for $t \in [0,\delta]$ and so u is continuous at 0. Similarly u is continuous at 1. As a result $u \in C[0,1]$ and

$$\begin{cases} -u'' = g(t,u) + \lambda h(t,u) \text{ for } t \in (0,1) \\ u(0) = u(1) = 0. \end{cases}$$

LEMMA 2.5. *Let $\psi : (0,1) \times (0,\infty) \to (0,\infty)$ be a continuous function with*

$$\begin{cases} \psi(t,\cdot) \text{ is strictly decreasing} \\ \psi(\cdot,r) \in M \text{ for all } r > 0. \end{cases}$$

Then the problem

$$\begin{cases} -\omega''(t) = \psi\left(t, \omega(t) + \dfrac{1}{n}\right) \text{ for } t \in (0,1) \\ \omega(0) = \omega(1) = 0 \end{cases} \tag{2.10}$$

has a solution $\omega_n \in C[0,1]$ such that

$$\omega_n(t) \leqslant \omega_{n+1}(t) \leqslant 1 + \omega_1(t) \text{ for } t \in [0,1] \text{ and } n \in N.$$

If we let $\omega(t) = \lim_{n\to\infty} \omega_n(t)$ for $t \in [0,1]$, then

$$\omega \in C[0,1],\ \omega(t) > 0 \text{ for } t \in (0,1)$$

and

$$\begin{cases} -\omega''(t) = \psi(t,\omega(t)) \text{ for } t \in (0,1) \\ \omega(0) = \omega(1) = 0. \end{cases}$$

Proof. There exists $\chi_1 \in C[0,1]$ such that

$$\begin{cases} -\chi_1''(t) = \psi(t,1) \\ \chi_1(0) = \chi_1(1) = 0 \\ \chi_1(t) > 0 \text{ for } t \in (0,1). \end{cases}$$

Notice

$$-\chi_1''(t) = \psi(t,1) \geqslant \psi(t, 1 + \chi_1(t))$$

and

$$-0''(t) = 0 \leqslant \psi(t, 1+0).$$

By a standard upper and lower solution method [1, 3], there exists $\omega_1 \in C[0,1]$ such that

$$\begin{cases} -\omega_1''(t) = \psi(t, 1 + \omega_1(t)) \text{ for } t \in (0,1) \\ \omega_1(0) = \omega_1(1) = 0. \end{cases}$$

Suppose there exists $\omega_n \in C[0,1]$ such that

$$\begin{cases} -\omega_n''(t) = \psi\left(t, \dfrac{1}{n} + \omega_n(t)\right) \\ \omega_n(0) = \omega_n(1) = 0 \\ \omega_n(t) > 0 \text{ for } t \in (0,1). \end{cases}$$

We know there exist $\chi_{n+1} \in C[0,1]$ such that

$$\begin{cases} -\chi_{n+1}''(t) = \psi\left(t, \dfrac{1}{n+1}\right) \\ \chi_{n+1}(0) = \chi_{n+1}(1) = 0 \\ \chi_{n+1}(t) > 0 \text{ for } t \in (0,1). \end{cases}$$

Then

$$\begin{aligned} -\chi_{n+1}''(t) &= \psi\left(t, \frac{1}{n+1}\right) \\ &\geqslant \psi\left(t, \frac{1}{n+1} + \chi_{n+1}(t)\right), \end{aligned}$$

$$\begin{cases} -\omega_n''(t) = \psi\left(t, \dfrac{1}{n} + \omega_n(t)\right) \leqslant \psi\left(t, \dfrac{1}{n+1} + \omega_n(t)\right) \text{ for } t \in (0,1) \\ \omega_n(0) = \omega_n(1) = 0 \end{cases}$$

and

$$\begin{aligned} \omega_n(t) &= \int_0^1 G(t,s)\, \psi\left(s, \frac{1}{n} + \omega_n(s)\right) ds \\ &\leqslant \int_0^1 G(t,s)\, \psi\left(s, \frac{1}{n+1} + \omega_n(s)\right) ds \\ &\leqslant \int_0^1 G(t,s)\, \psi\left(s, \frac{1}{n+1}\right) ds \\ &= \chi_{n+1}(t) \text{ for } t \in (0,1). \end{aligned}$$

By a standard upper and lower solution method, there exist $\omega_{n+1} \in C[0,1]$ such that

$$\begin{cases} -\omega_{n+1}''(t) = \psi\left(t, \dfrac{1}{n+1} + \omega_{n+1}(t)\right) \text{ for } t \in (0,1) \\ \omega_{n+1}(0) = \omega_{n+1}(1) = 0 \end{cases}$$

and

$$\omega_n(t) \leqslant \omega_{n+1}(t) \text{ for } t \in [0,1].$$

Next we prove

$$\omega_{n+1}(t) + \frac{1}{n+1} \leqslant w_n(t) + \frac{1}{n} \text{ for } t \in [0,1]. \tag{2.11}$$

To see this we consider the problem

$$\begin{cases} -v''(t) = \psi(t, v(t)) \text{ for } t \in (0,1) \\ v(0) = v(1) = \dfrac{1}{n}. \end{cases} \tag{2.12_n}$$

Then $v_n(t) = \frac{1}{n} + w_n(t)$, $t \in [0,1]$ is a solution of (2.12_n). We now prove

$$v_{n+1}(t) \leqslant v_n(t) \text{ for } t \in [0,1].$$

Since $v_{n+1}(0) = \frac{1}{n+1} < \frac{1}{n} = v_n(0)$, $v_{n+1}(1) = \frac{1}{n+1} < \frac{1}{n} = v_n(1)$, we need only prove

$$v_{n+1}(t) \leqslant v_n(t) \text{ for } t \in (0,1). \tag{2.13}$$

Suppose (2.13) is not true. Let $y(t) = v_{n+1}(t) - v_n(t)$ and $\sigma \in (0,1)$ be the point where $y(t)$ attains its maximum over $(0,1)$. We have

$$y(\sigma) > 0 \text{ and } y''(\sigma) \leqslant 0.$$

On the other hand, since $v_{n+1}(\sigma) > v_n(\sigma)$ we have

$$\begin{aligned} y''(\sigma) &= v''_{n+1}(\sigma) - v''_n(\sigma) \\ &= -\psi(\sigma, v_{n+1}(\sigma)) + \psi(\sigma, v_n(\sigma)) \\ &= \psi(\sigma, v_n(\sigma)) - \psi(\sigma, v_{n+1}(\sigma)) \\ &> 0. \end{aligned}$$

This is a contradiction. Thus $v_{n+1}(t) \leqslant v_n(t)$ for $t \in (0,1)$ and so

$$0 < \frac{1}{n+1} + \omega_{n+1} \leqslant w_n + \frac{1}{n}.$$

Thus

$$\begin{aligned} \omega_1(t) &\leqslant \omega_n(t) \\ &\leqslant \omega_{n+1}(t) \\ &\leqslant 1 + \omega_1(t) \text{ for } t \in [0,1] \text{ and } n \in N. \end{aligned} \tag{2.14}$$

Put

$$\begin{aligned} \omega(t) &= \lim_{n\to\infty} \omega_n(t) \\ &= \sup_{n\in N} \omega_n(t) \text{ for } t \in [0,1]. \end{aligned}$$

By (2.14), we have

$$0 < \omega_1(t) \leqslant \omega(t) \leqslant 1 + \omega_1(t) \text{ for } t \in (0,1)$$

and

$$\omega(0) = \omega(1) = 0.$$

Now it is easy to prove (see the ideas in the proof of Lemma 2.4) that

$$\begin{cases} -\omega''(t) = \psi(t, \omega(t)) \text{ for } t \in (0,1) \\ \omega(0) = \omega(1) = 0. \end{cases}$$

LEMMA 2.6. *Suppose* $m : (0,1) \times [0,\infty) \to [0,\infty)$ *is a continuous function with*

$$\begin{cases} m(\cdot, r) \in M \text{ for all } r \geqslant 0 \\ m(t, \cdot) \text{ is increasing} \end{cases}$$

and there exist $b \in M$, $b(t) > 0$ *for* $t \in (0,1)$ *with*

$$\lim_{r \to +\infty} \frac{m(t,r)}{b(t)\, r} = 0 \text{ uniformly with respect to } t \in (0,1). \tag{2.15}$$

Then there exist $R_0 > 0$ *and* $\widetilde{v} \in C[0,1]$, $0 \leqslant \widetilde{v} \leqslant R_0\phi_1$ *such that*

$$\begin{cases} -\widetilde{v}''(t) = m(t, \widetilde{v}) \text{ for } t \in (0,1) \\ \widetilde{v}(0) = \widetilde{v}(1) = 0. \end{cases}$$

Proof. We first prove that

$$\lim_{R \to \infty} \frac{\int_0^1 N(t,s)\, m(s,v)\, ds}{R} = 0 \text{ uniformly with respect to } t \in (0,1), \tag{2.16}$$

$\forall v \in C[0,1]$ with $0 \leqslant v(t) \leqslant R\phi_1(t)$ for $t \in [0,1]$; here $N(t,s)$ is defined in (2.3).

From (2.15), for all $\sigma > 0$, there exist $s_\sigma > 0$ such that

$$m(t,s) \leqslant \sigma b(t)\, s \text{ for } t \in (0,1) \text{ and } s_\sigma \leqslant s.$$

As a result

$$\begin{aligned} m(t, v(t))\,|_{0 \leqslant v(t) \leqslant R\phi_1(t)} &\leqslant m(t, s_\sigma) + \sigma b(t)\, v(t) \\ &\leqslant m(t, s_\sigma) + \sigma R b(t)\, \phi_1(t) \text{ for } t \in (0,1), \end{aligned}$$

so

$$\int_0^1 N(t,s)\, m(s,v)\, ds \leqslant \int_0^1 N(t,s)\, m(s,s_\sigma)\, ds + R\sigma \int_0^1 N(t,s)\, b(s)\, \phi_1(s)\, ds.$$

Consequently,

$$\begin{aligned} \frac{1}{R}\int_0^1 N(t,s)\, m(s,v)\, ds &\leqslant \frac{1}{R}\int_0^1 N(t,s)\, m(s,s_\sigma)\, ds + \sigma \int_0^1 N(t,s)\, b(s)\, \phi_1(s)\, ds \\ &\leqslant \frac{1}{2R}\int_0^1 m(s,s_\sigma)\, ds + \frac{\sigma}{2}\int_0^1 b(s)\, \phi_1(s)\, ds, \end{aligned}$$

and (2.16) is proved. There exist $R_0 > 0$ (independent of $t \in (0,1)$) such that if $v \in C[0,1]$ and $0 \leqslant v(t) \leqslant R_0\phi_1(t)$ for $t \in [0,1]$ then

$$\frac{1}{R_0}\int_0^1 N(t,s)\, m(s,v)\, ds \leqslant 1 \text{ for } t \in (0,1),$$

and so

$$0 \leqslant \int_0^1 G(t,s)\, m(s,v)\, ds \leqslant R_0\phi_1(t) \text{ for } t \in [0,1].$$

Let $\Phi : C[0,1] \to C[0,1]$ be the operator defined by

$$(\Phi v)(t) := \int_0^1 G(t,s)\, m(s,v)\, ds \text{ for } v \in C[0,1] \text{ and } t \in [0,1].$$

It is easy to see that Φ is a completely continuous operator. Also if $0 \leqslant v(t) \leqslant R_0\phi_1(t)$ for $t \in [0,1]$ then $0 \leqslant \Phi(v)(t) \leqslant R_0\phi_1(t)$ for $t \in [0,1]$, so Schauder's fixed point theorem guarantees that there exists $\widetilde{v} \in [0, R_0\phi_1]$ such that $\Phi(\widetilde{v}) = \widetilde{v}$, i.e.

$$\begin{cases} -\widetilde{v}''(t) = m(t, \widetilde{v}(t)) \\ \widetilde{v}(0) = \widetilde{v}(1) = 0. \end{cases}$$

COROLLARY 2.2. *Let* $\psi(t,s)$, $m(t,s)$, $(\omega_n)_{n\in N}$ *and* $R_0 > 0$ *be as in Lemma* 2.5 *and Lemma* 2.6. *There exists* $\{\widetilde{v}_n\}_{n\in N} \subset C[0,1]$ *and* $0 \leqslant \widetilde{v}_n \leqslant R_0\phi_1$ *such that*

$$\begin{cases} -\widetilde{v}_n''(t) = m(t, \omega_n + \widetilde{v}_n) \text{ for } t \in (0,1) \\ \widetilde{v}_n(0) = \widetilde{v}_n(1) = 0 \end{cases} \tag{2.17}$$

and

$$-(w_n + \widetilde{v}_n)''(t) \geqslant \psi\left(t, \frac{1}{n} + \omega_n + \widetilde{v}_n\right) + m(t, \omega_n + \widetilde{v}_n) \text{ for } t \in (0,1).$$

Proof. Let $n \in N$ be fixed. Then $m(t, \omega_n + s)$ satisfies the conditions of Lemma 2.6, so there exists $\widetilde{v}_n \in C[0,1]$ with $0 \leqslant \widetilde{v}_n \leqslant R_0\,\phi_1$ such that (2.17) holds and

$$\begin{aligned} -(w_n + \widetilde{v}_n)''(t) &= -w_n''(t) - \widetilde{v}_n''(t) \\ &= \psi\left(t, \frac{1}{n} + \omega_n\right) + m(t, \omega_n + \widetilde{v}_n) \\ &\geqslant \psi\left(t, \frac{1}{n} + \omega_n + \widetilde{v}_n\right) + m(t, \omega_n + \widetilde{v}_n) \text{ for } t \in (0,1). \end{aligned}$$

LEMMA 2.7. *Suppose* $(G1)$, $(G2)$, $(H1)$ *and* $(H2)$ *hold. Then there exists* $\lambda_0 > 0$, $c > 0$ *such that for all* $\lambda \geqslant \lambda_0$ *there exist* $R_c > c$, $\overline{u} \in C([0,1])$ *with* $c\phi_1(t) \leqslant \overline{u}(t) \leqslant R_c\phi_1(t)$ *and*

$$\begin{cases} -\overline{u}''(t) = -g_1(t, \overline{u}(t)) + \lambda h_1(t, \overline{u}(t)) \text{ for } t \in (0,1) \\ \overline{u}(0) = \overline{u}(1) = 0 \end{cases} \tag{2.18}$$

with

$$g_1(\cdot, \overline{u}(\cdot)), h_1(\cdot, \overline{u}(\cdot)) \in M.$$

Proof. Let us consider the operator $T_\lambda : C[0,1] \to C[0,1]$ given by

$$T_\lambda(v)(t) := \int_0^1 N(t,s)\left[-g_1(s, v\phi_1) + \lambda h_1(s, v\phi_1)\right] ds \text{ for } t \in (0,1).$$

By $(H1)$, there exists $\overline{r} \geqslant 0$ such that $0 < h_1(t, \overline{r})$ for $t \in (0,1)$. We let

$$c = 2(\overline{r} + 1),\ \Theta = \left\{ t \in (0,1) : \frac{1}{2} < \phi_1(t) \right\}.$$

Note that Θ is nonempty. If $t \in \Theta$, $v \in C[0,1]$ and $c \leqslant v$, we have

$$\overline{r} = \frac{c}{2} - 1 \leqslant \frac{c}{2} \leqslant c\phi_1(t) \leqslant v(t)\,\phi_1(t),$$

so

$$h_1(t, \overline{r}) \leqslant h_1(t, v\phi_1), \tag{2.19}$$

for all $v \in C[0,1]$ with $c \leqslant v$. Let

$$\rho = \frac{1}{2\pi} \int_{s \in \Theta} s(1-s)\,h_1(s, \overline{r})\,ds > 0,$$

and note for $v \in C[0,1]$ with $c \leqslant v$ that

$$\begin{aligned}
\int_0^1 N(t,s)\,h_1(s, v\phi_1)\,ds &\geqslant \int_{s \in \Theta} N(t,s)\,h_1(s, v\phi_1)\,ds \\
&\geqslant \int_{s \in \Theta} N(t,s)\,h_1(s, \overline{r})\,ds \text{ (see (2.19))} \\
&\geqslant \frac{1}{2\pi} \int_{s \in \Theta} s(1-s)\,h_1(s, \overline{r})\,ds \text{ (see (2.4))} \\
&= \rho > 0 \text{ for all } t \in (0,1),
\end{aligned}$$

i.e.

$$\frac{1}{\int_0^1 N(t,s)\,h_1(s, v\phi_1)\,ds} \leqslant \frac{1}{\rho} \text{ for } t \in (0,1). \tag{2.20}$$

On the other hand, for $\forall v \in C[0,1]$ with $v \geqslant c$, we have

$$c + \int_0^1 N(t,s)\,g_1(s, v\phi_1)\,ds \leqslant c + \int_0^1 N(t,s)\,g_1(s, c\phi_1)\,ds \text{ for } t \in (0,1),$$

so for $t \in (0,1)$ we have

$$\begin{aligned}
\frac{c + \int_0^1 N(t,s)\,g_1(s, v\phi_1)\,ds}{\int_0^1 N(t,s)\,h_1(s, v\phi_1)\,ds} &\leqslant \frac{1}{\rho}\left(c + \frac{1}{2}\int_0^1 N(t,s)\,g_1(s, c\phi_1)\,ds \right) \\
&\leqslant \frac{1}{\rho}\left(c + \frac{1}{2}\int_0^1 g_1(s, c\phi_1)\,ds \right) \quad \text{(independent of } t\text{)}.
\end{aligned}$$

Let

$$\lambda_0 := \sup\left\{ \left| \frac{c + \int_0^1 N(t,s)\,g_1(s, v\phi_1)\,ds}{\int_0^1 N(t,s)\,h_1(s, v\phi_1)\,ds} \right|_\infty : v \in C[0,1],\ c \leqslant v \right\} < \infty.$$

Then, for all $\lambda \geqslant \lambda_0$ with $v \in C[0,1]$ with $c \leqslant v$ we have

$$\frac{c + \int_0^1 N(t,s)\, g_1(s, v\phi_1)\, ds}{\int_0^1 N(t,s)\, h_1(s, v\phi_1)\, ds} \leqslant \lambda \text{ for } t \in (0,1)$$

i.e.

$$c + \int_0^1 N(t,s)\, g_1(s, v\phi_1)\, ds \leqslant \lambda \int_0^1 N(t,s)\, h_1(s, v\phi_1)\, ds \text{ for } t \in (0,1),$$

so

$$\begin{aligned} c &\leqslant \int_0^1 N(t,s)\left(-g_1(s, v\phi_1) + \lambda h_1(s, v\phi_1)\right) ds \\ &= T_\lambda(v)(t) \text{ for } t \in (0,1). \end{aligned}$$

On the other hand, for $\forall v \in C[0,1]$ with $v \geqslant c,$ we have

$$\begin{aligned} 0 &\leqslant \int_0^1 N(t,s)\, g_1(s, v\phi_1)\, ds \leqslant \int_0^1 N(t,s)\, g_1(s, c\phi_1)\, ds \\ &\leqslant \frac{1}{2}\int_0^1 g_1(s, c\phi_1)\, ds \text{ for } t \in [0,1]. \end{aligned}$$

Thus

$$\begin{aligned} 0 &\leqslant \lim_{R\to\infty} \frac{1}{R}\left[\int_0^1 N(t,s)\, g_1(s, v\phi_1)\, ds\right] \\ &\leqslant \lim_{R\to\infty} \frac{1}{2R}\int_0^1 g_1(s, c\phi_1)\, ds = 0, \end{aligned}$$

uniformly with respect to $t \in [0,1]$.

Essentially the same reasoning as in the proof of (2.16) yields for $v \in C[0,1]$ with $0 \leqslant v(t) \leqslant R$ for $t \in [0,1]$,

$$\lim_{R\to\infty} \frac{1}{R}\left[\int_0^1 N(t,s)\, h_1(s, v\phi_1)\, ds\right] = 0$$

uniformly with respect to $t \in [0,1]$. Then there exists $R_c > c$ so that $T_\lambda([c, R_c]) \subset [c, R_c]$.

It is easy to see that $T_\lambda : [c, R_c] \to [c, R_c]$ is a completely continuous operator, so Schauder's fixed point theorem guarantees that there exists $\overline{v} \in [c_0, R_c]$ such that $T_\lambda(\overline{v}) = \overline{v}$ i.e.

$$\overline{v}(t)\, \phi_1(t) = \int_0^1 G(t,s)\left(-g_1(s, \overline{v}\phi_1) + \lambda h_1(s, \overline{v}\phi_1)\right) ds.$$

The function $\overline{u}(t) = \phi_1(t)\, \overline{v}(t)$ for $t \in [0,1]$ satisfies (2.18). Moreover, we have $c_0\phi_1(t) \leqslant \overline{u}(t) \leqslant R_c\phi_1(t)$ for $t \in [0,1]$ and

$$g_1(\cdot, \overline{u}(\cdot)),\, h_1(\cdot, \overline{u}(\cdot)) \in M.$$

Proof of theorem 1.1. Let $\lambda_0 > 0$, $c > 0$ and $\overline{u} \in (C[0,1])$ be as defined in Lemma 2.7. Define

$$\psi(t,r) = g_2(t,r) + \lambda h_1(t, \overline{u}(t)) \text{ for } t \in (0,1),$$

and

$$m(t,r) = \lambda h_2(t,r)$$

where $\lambda \geqslant \lambda_0$.

From $(G1)$ notice ψ satisfies the assumptions of Lemma 2.5, so there exists ω, $\omega_n \in C[0,1]$ such that

$$\begin{cases} -\omega_n''(t) = g_2\left(t, \dfrac{1}{n} + \omega_n\right) + \lambda h_1(t, \overline{u}) \text{ for } t \in (0,1) \\ \omega_n(0) = \omega_n(1) = 0 \end{cases}$$

and

$$\omega(t) = \lim_{n\to\infty} \omega_n(t) \text{ for } t \in [0,1].$$

From $(H1)$ notice m satisfies the assumption of Lemma 2.6, so by Corollary 2.2, there exists $R_0 > 0$ and $\widetilde{v}_n \in C([0,1])$, $0 \leqslant \widetilde{v}_n(t) \leqslant R_0\phi_1(t)$ for $t \in [0,1]$ such that

$$\begin{cases} -\widetilde{v}_n''(t) = \lambda h_2(t, \omega_n + \widetilde{v}_n) \text{ for } t \in (0,1) \\ \widetilde{v}_n(0) = \widetilde{v}_n(1) = 0 \end{cases}$$

and

$$-(\omega_n + \widetilde{v}_n)''(t) \geqslant g_2\left(t, \frac{1}{n} + \omega_n + \widetilde{v}_n\right) + \lambda h_1(t, \overline{u}) + \lambda h_2(t, \omega_n + \widetilde{v}_n) \text{ for } t \in (0,1).$$

Let

$$\widehat{u}_n(t) = \omega_n(t) + \widetilde{v}_n(t) \text{ for } t \in [0,1].$$

Then $\widehat{u}_n \in C[0,1]$ and $\widehat{u}_n(1) = \widehat{u}_n(1) = 0$.

Let

$$\widehat{u}(t) = \omega(t) + R_0\phi_1(t) \text{ for } t \in [0,1],$$

so

$$0 \leqslant \widehat{u}_n(t) \leqslant \widehat{u}(t) \text{ for } t \in [0,1]. \tag{2.21}$$

From Lemma 2.7 we obtain

$$\begin{aligned} -\overline{u}''(t) &= -g_1(t, \overline{u}) + \lambda h_1(t, \overline{u}) \\ &\leqslant \lambda h_1(t, \overline{u}) \\ &\leqslant \lambda h_1(t, \overline{u}) + g_2\left(t, \frac{1}{n} + \widehat{u}_n\right) + \lambda h_2(t, \widehat{u}_n) \\ &\leqslant -\widehat{u}_n''(t) \text{ for } t \in (0,1) \end{aligned}$$

i.e

$$-(\overline{u} - \widehat{u}_n)''(t) \leqslant 0 \text{ for } t \in (0,1).$$

A standard argument yields

$$\overline{u}(t) \leqslant \widehat{u}_n(t) \text{ for } t \in [0,1]. \tag{2.22}$$

From $(G1)$, there exists $\gamma \in M$ such that $r \to g_2\left(t, \frac{1}{n}+r\right)+\gamma(t)\,r$ is increasing. Let $u_n = \overline{u}$. From (2.21) and (2.22), we have

$$0 < \overline{u}(t) \leqslant \overline{u}_n(t) \leqslant \widehat{u}_n(t) \leqslant \widehat{u}(t) \text{ for } t \in (0,1).$$

Also for $v \in C[0,1]$ with $\overline{u}_n(t) \leqslant v(t) \leqslant \widehat{u}_n(t)$, $t \in [0,1]$, we have

$$\begin{aligned}
-\overline{u}_n''(t) + \gamma(t)\,\overline{u}_n(t) &= -g_1(t,\overline{u}_n) + \lambda h_1(t,\overline{u}_n) + \gamma(t)\,\overline{u}_n(t) \\
&\leqslant -g_1(t,v) + \lambda h_1(t,v) + \gamma(t)\,v(t) \\
&\leqslant -g_1\left(t,\frac{1}{n}+v\right) + \lambda h_1(t,v) + \gamma(t)\,v(t) \\
&\leqslant g\left(t,\frac{1}{n}+v\right) + \lambda h(t,v) + \gamma(t)\,v(t) \text{ for } t \in (0,1),
\end{aligned}$$

so (2.8) holds with $\delta_n(t) \equiv 0$ for $t \in [0,1]$, $n \in N$.

Also for $v \in C[0,1]$ with $\overline{u}_n(t) \leqslant v(t) \leqslant \widehat{u}_n(t)$, $t \in [0,1]$ we have

$$\begin{aligned}
-\widehat{u}_n''(t) + a_n(t)\,\widehat{u}_n(t) &\geqslant g_2\left(t,\frac{1}{n}+\widehat{u}_n\right) + \lambda h_1(t,\overline{u}) + \lambda h_2(t,\widehat{u}_n) + \gamma(t)\,\widehat{u}_n(t) \\
&\geqslant g_2\left(t,\frac{1}{n}+\widehat{u}_n\right) + \gamma(t)\,\widehat{u}_n(t) + \lambda h_2(t,\widehat{u}_n) \\
&\geqslant g_2\left(t,\frac{1}{n}+v\right) + \gamma(t)\,v(t) + \lambda h_2(t,v(t)) \\
&\geqslant g\left(t,\frac{1}{n}+v\right) + \lambda h(t,v) + \gamma(t)\,v(t) \text{ for } t \in (0,1),
\end{aligned}$$

so (2.9) holds. Lemma 2.4 guarantees that there exists a solution $u \in C[0,1]$ to (1.1) with

$$\overline{u}(t) \leqslant u(t) \leqslant \widehat{u}(t) \text{ for } t \in [0,1].$$

Moreover, because $\widehat{u}(t) \leqslant |\omega|_\infty + R_0\phi_1(t) \leqslant (|\omega|_\infty + R_0)(1+\phi_1(t))$ and $c\phi_1(t) < \overline{u}(t)$ (see Lemma 2.7), the estimates asserted in the theorem follow.

3. Example

EXAMPLE 3.1. Consider the boundary value problem

$$\begin{cases} -u''(t) = g(t,u) + \lambda h(t,u) \text{ for } t \in (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{3.1}$$

where

$$g(t,r) = -\frac{1}{\sqrt{r}} \text{ for } t \in (0,1) \text{ and } r \in (0,\infty)$$

$$h(t,r) = \sqrt{r} \text{ for } t \in (0,1) \text{ and } r \in [0,\infty)$$

and $\lambda > 199.05$. We will show that Theorem 1.1 guarantees that (3.1) has at least a solution $u \in C[0,1] \cap C^1(0,1)$ with $u(t) > 0$ for $t \in [0,1]$.

To see this we first prove that g satisfies $(G1)$, $(G2)$. Let

$$g_1(t,r) = g_2(t,r) = \frac{1}{\sqrt{r}}.$$

For every $r_1, r_2\ (0 < r_1 < r_2)$, let

$$\gamma(t) = 1 + \frac{1}{2r_1^{\frac{3}{2}}} \text{ for } t \in (0,1).$$

Now

$$\left|\frac{\partial g_2}{\partial r}\right| \leqslant \left|-\frac{1}{2r^{\frac{3}{2}}}\right| < 1 + \frac{1}{2r_1^{\frac{3}{2}}} \text{ for } t \in (0,1),\ r \in (r_1, r_2),$$

so

$$\frac{\partial}{\partial r}\left(g_2(t,r) + \gamma(t)\, r\right) > 0 \text{ for } t \in (0,1),\ r \in (r_1, r_2).$$

Consequently the function

$$g_2(t,r) + \gamma(t)\, r$$

is increasing in $[r_1, r_2]$, so condition $(G1)$ holds. Also

$$g_1(\cdot, r\phi_1(\cdot)), g_2(\cdot, r) \in M \text{ for all } r > 0,$$

so $(G2)$ is satisfied.

Also with

$$h_1(t,r) = h_2(t,r) = \sqrt{r},$$

$(H1)$ and $(H2)$ are satisfied. We next obtain an upper bound for λ_0 in Theorem 1.1.

In $(H2)$, let $\overline{r} = 1$, so in Lemma 2.7, we have $c = 4$, $\Theta = \left(\frac{1}{6}, \frac{5}{6}\right)$ and

$$\begin{aligned} \rho &= \frac{1}{2\pi}\int_{s\in\Theta} s(1-s)\, h_1(s,\overline{r})\, ds \\ &= \frac{1}{2\pi}\int_{\frac{1}{6}}^{\frac{5}{6}} s(1-s)\, ds = \frac{23}{324\pi}. \end{aligned}$$

Then

$$\frac{324\pi}{23}\left|4 + \int_0^1 N(t,s)\frac{1}{\sqrt{4\sin \pi s}} ds\right|_\infty \leqslant \frac{324\pi}{23} \times 4.5 \approx 199.05,$$

so

$$0 < \lambda_0 \leqslant 199.05.$$

Then Theorem 1.1 guarantees that (3.1) has at least a solution $u \in C[0,1] \cap C^1(0,1)$ with $u(t) > 0$ for $t \in [0,1]$.

On the other hand, for $\lambda > 199.05$,

$$\lim_{r\to\infty} g(t,r) + \lambda h(t,r) = -\infty \text{ for } t \in (0,1),$$

so the conditions in Remark 1.1 are not satisfied.

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