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SINGULAR INTEGRALS AND FRACTIONAL INTEGRALS IN TRIEBEL–LIZORKIN SPACES AND IN WEIGHTED L^p SPACES

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Abstract. We study the hypersingular integral

$$T_{h,\alpha}f(x) = \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{b(|y|)e^{ih(|y|)}\Omega(y')}{|y|^{n+\alpha}} f(x-y)\,dy$$

and the fractional integral

$$I_{h,\alpha}f(x) = \int_{R^n} \frac{b(|y|)e^{ih(|y|)}\Omega(y')}{|y|^{n-\alpha}} f(x-y)\,dy$$

in Triebel-Lizorkin spaces and weighted L^p spaces. Here $\Omega \in H^r(S^{n-1})$, and $b(|y|)$ and $h(|y|)$ are measurable radial functions which satisfy some suitable conditions. We also consider the above integrals along some surfaces of revolution. The results in this paper extend some known results about hypersingular integrals and fractional integrals.

Introduction

The subject of singular integral operators is well known for many years. It is initially pioneered by Calderón and Zygmund (see [2, 3]), and is subsequently studied by many other authors. For instance, the reader may view [4-6, 10-11, 16-18] among many other references for a good survey. In this note, we are particularly interested in some variations of singular integrals, i.e., fractional integrals and singular integrals that are strongly singular at infinity and at the origin respectively. Recently the authors in [4] proved that the singular integral operator

$$T_\alpha f(x) = \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{b(|y|)\Omega(y')}{|y|^{n+\alpha}} f(x-y)\,dy$$

$$\left(\Omega \in H^r(S^{n-1}),\ r = (n-1)/(n-1+\alpha),\ \alpha > 0\right)$$

is a bounded map from $\dot{F}_p^{s+\alpha,q}(\mathbb{R}^n)$ to $\dot{F}_p^{s,q}(\mathbb{R}^n)$ for $s \in \mathbb{R}$, $1 < p, q < \infty$, where $\dot{F}_p^{s,q}(\mathbb{R}^n)$ is the Triebel-Lizorkin space. By introducing an oscillating factor $e^{i|y|^{-\beta}}$ in

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the kernel of the singular integral (that is motivated by the Bochner-Riesz means), the author in [12-13] showed that the singular integral operator along surface

$$T_{\Gamma,\alpha}f(x,x_{n+1}) = \lim_{\epsilon\to 0}\int_{|y|>\epsilon} \frac{b(|y|)e^{i|y|^{-\beta}}\Omega(y')}{|y|^{n+\alpha}} f(x-y, x_{n+1}-\Gamma(|y|))\,dy \qquad (\Omega\in H^1(S^{n-1}))$$

is bounded in $L^p(\mathbb{R}^{n+1})$ for $\beta/(\beta-\alpha) < p < \beta/\alpha,\ \beta > 2\alpha > 0$.

These two results motivated us to investigate this subject in further detail. As a consequence, we have obtained new results (with $\Omega\in H^r(S^{n-1})$) which extend both of the results above. It should be noted that by introducing the oscillating factor $e^{ih(|y|)}$ in the kernel, we can obtain parallel results for the fractional integral $I_{h,\alpha}f(x)$ defined in the abstract. Moreover, we also consider these integrals both in Triebel-Lizorkin spaces and in weighted L^p spaces, with $\Omega\in L^r(S^{n-1})$ $(r\geqslant 1)$ which does not satisfy the mean value zero property. We divide this paper in three sections. The first section deals with singular integrals in Triebel-Lizorkin spaces. Fractional integrals in Triebel-Lizorkin spaces are discussed in the second section. Finally, the third section involves fractional integrals in weighted L^p spaces.

1. Singular integrals in Triebel-Lizorkin spaces

We briefly review some function spaces.

The Hardy Space $H^r(S^{n-1})$. The Poisson kernel on S^{n-1} is defined by $P_{ty'}(x') = \dfrac{(1-t^2)}{|ty'-x'|^n}$, where $0\leqslant t<1$ and $x',y'\in S^{n-1}$. Let $\mathscr{S}'(S^{n-1})$ stand for the space of Schwartz distributions on S^{n-1}. For any $\Omega\in\mathscr{S}'(S^{n-1})$, we define the radial maximal function $P^+\Omega(x')$ by $P^+\Omega(x') = \sup_{0\leqslant t<1} |<P_{ty'},\Omega>|$, where $<P_{ty'},\Omega>$ denotes the pairing between $P_{ty'}$ and Ω. The Hardy space $H^r(S^{n-1})$, $0<r<\infty$, is the linear space of distributions $\Omega\in\mathscr{S}'(S^{n-1})$ with the finite norm $||\Omega||_{H^r(S^{n-1})} = ||P^+\Omega||_{L^r(S^{n-1})}<\infty$. See [7-8] for more details.

The Triebel-Lizorkin space $\dot F_p^{s,q}(\mathbb{R}^n)$. Fix a radial function $\phi\in\mathscr{S}(\mathbb{R}^n)$ such that $\mathrm{supp}(\hat\phi)\subset\{\xi\in\mathbb{R}^n : 1/2\leqslant|\xi|\leqslant 2\}$, $0\leqslant\hat\phi(\xi)\leqslant 1$, $\hat\phi(\xi)\geqslant c>0$ if $3/5\leqslant|\xi|\leqslant 5/3$, and $\sum_{j=-\infty}^{\infty}\hat\phi_{2^j}^2(|\xi|)=1$ for all $\xi\neq 0$, where $\hat\phi_{2^j}(\xi)=\hat\phi(2^j\xi)$. Note that $\phi_{2^j}(x)=2^{-jn}\phi(2^{-j}x)$, $x\in\mathbb{R}^n$. For $1<p,q<\infty$, $s\in\mathbb{R}$, the Triebel-Lizorkin space $\dot F_p^{s,q}(\mathbb{R}^n)$ is the space of all distributions f with the norm defined by

$$||f||_{\dot F_p^{s,q}(\mathbb{R}^n)} = \left\|\left(\sum_j |2^{-js}\phi_{2^j}*f|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)} < \infty.$$

It is well known that $\mathscr{S}(\mathbb{R}^n)$ is dense in $\dot F_p^{s,q}(\mathbb{R}^n)$ for $s\in\mathbb{R}$, $1<p,q<\infty$. See [20-21] for more information on this subject.

For $\alpha > 0$, we denote the following singular integrals by

$$Tf(x) = p.v. \int_{\mathbb{R}^n} \frac{b(|y|)\Omega(y')}{|y|^n} f(x-y)\,dy,$$

$$T_\alpha f(x) = \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{b(|y|)\Omega(y')}{|y|^{n+\alpha}} f(x-y)\,dy,$$

$$T_{h,\alpha} f(x) = \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{b(|y|)e^{ih(|y|)}\Omega(y')}{|y|^{n+\alpha}} f(x-y)\,dy,$$

and

$$T_{h,\alpha,\Gamma} f(x, x_{n+1}) = \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{b(|y|)e^{ih(|y|)}\Omega(y')}{|y|^{n+\alpha}} f(x-y, x_{n+1} - \Gamma(|y|))\,dy,$$

where $x, y \in \mathbb{R}^n$, $x_{n+1} \in \mathbb{R}$, and $\Gamma(|y|)$ is a measurable radial function defined on $\mathbb{R}^n$. For the rest of this paper, the letter C will denote a positive constant which may vary at each occurence, but it is independent of the essential variables.

THEOREM 1. *Let $\Omega \in H^r(S^{n-1})$, $0 < r = (n-1)/(n-1+\gamma) \leqslant 1$, $\gamma \geqslant 0$. Let N denote the smallest non-negative integer such that $4(N+1) > \tilde{p}\gamma\tilde{q}$, where $\tilde{p} = max\{p, p/(p-1)\}$ and $\tilde{q} = max\{q, q/(q-1)\}$. Suppose that $<\Omega, Y_m>= 0$ for all spherical polynomials Y_m defined on S^{n-1} with degree $m \leqslant N$. Assume that $b(|y|)$ is a bounded measurable function on $\mathbb{R}^+$ ($= [0,\infty)$) such that either $b(t)$ is monotone on $\mathbb{R}^+$ or $b'(t) \in L^1(\mathbb{R}^+)$. Suppose that $h''(t) \geqslant Ct^{-\beta-2}$ for all $t \in (0,\infty)$ and for some fixed $\beta > 0$. Then we have*

$$||T_{h,\alpha} f||_{\dot{F}_p^{s,q}(\mathbb{R}^n)} \leqslant C||f||_{\dot{F}_p^{s+\gamma,q}(\mathbb{R}^n)}$$

for $\beta/(\beta+\gamma-\alpha) < p,\ q < \beta/(\alpha-\gamma)$, $s \in \mathbb{R}$, provided that $\beta > 2(\alpha-\gamma) \geqslant 0$ and $0 < \gamma \leqslant \alpha$.
The above result also holds if $\gamma = 0$, $s = 0$ and $q = 2$. That is,

$$||T_{h,\alpha} f||_{\dot{F}_p^{0,2}(\mathbb{R}^n)} \leqslant C||f||_{\dot{F}_p^{0,2}(\mathbb{R}^n)}$$

for $\beta/(\beta-\alpha) < p < \beta/\alpha$, and $\beta > 2\alpha > 0$.
Moreover, if $b(|y|)$ is merely a bounded function, then

$$||T_\alpha f||_{\dot{F}_p^{s,q}(\mathbb{R}^n)} \leqslant C||f||_{\dot{F}_p^{s+\alpha,q}(\mathbb{R}^n)}$$

for $1 < p,\ q < \infty$, $s \in \mathbb{R}$ and $\alpha > 0$.

Proof of Theorem 1. It suffices to prove the theorem by considering $\Omega(y')$ as an (r,∞) atom $a(y')$ on S^{n-1} (see [5, 7-8]). We may assume without of loss of generality that $\operatorname{supp} a(y') \subset B(\mathbf{1},\rho) \bigcap S^{n-1}$, where $\mathbf{1} = (1,0,...,0)$. Consider a family of analytic operators T_z defined on $\mathscr{S}(\mathbb{R}^n)$ by

$$T_z f(x) = p.v. \int_{\mathbb{R}^n} \frac{a(y')e^{ih(|y|)}b(|y|)f(x-y)}{|y|^{n+\alpha+z}}\,dy.$$

We decompose the operator T_z as $T_zf(x) = \sum_k T_kf(x) \equiv \sum_k \sigma_k * f(x)$, where

$$\hat{\sigma}_k(\zeta) = \int_{|y|\cong 2^k} \frac{a(y')b(|y|)e^{-i\zeta\cdot y}e^{ih(|y|)}}{|y|^{n+\alpha+z}}\,dy.$$

We have the following estimates for $\hat{\sigma}_k(\zeta)$. $\square$

LEMMA 1. *If* $\Re z = \gamma - \alpha \leqslant 0,\ 0 \leqslant \gamma \leqslant \alpha,$ *then*

$$|\hat{\sigma}_k(\zeta)| \leqslant C2^{-k\gamma}(2^k|A_\rho\zeta|)^{N+1}\rho^{-\gamma}, \tag{1}$$

$$|\hat{\sigma}_k(\zeta)| \leqslant C2^{-k\gamma}\rho^{-\gamma}, \tag{2}$$

$$|\hat{\sigma}_k(\zeta)| \leqslant C2^{-k\gamma}(2^k|A_\rho\zeta|)^{-1/2}\rho^{-\gamma}. \tag{3}$$

If $\Re z = \beta/2 - \alpha + \gamma > 0$ *and* $0 < \gamma \leqslant \alpha,$ *then*

$$|\hat{\sigma}_k(\zeta)| \leqslant C_z2^{-k\gamma}(2^k|A_\rho\zeta|)^{N+1}\rho^{-\gamma}, \tag{4}$$

$$|\hat{\sigma}_k(\zeta)| \leqslant C_z2^{-k\gamma}\rho^{-\gamma}. \tag{5}$$

If $\gamma = 0$ *and* $0 < \Re z < \beta/2 - \alpha$*, then*

$$|\hat{\sigma}_k(\zeta)| \leqslant C_z min\left\{2^{-k(\alpha+\Re z)},\ 2^{k(\beta/2-\alpha-\Re z)}\right\}. \tag{6}$$

Here $A_\rho\zeta = (\rho^2\zeta_1, \rho\zeta_2, ..., \rho\zeta_n),\ C_z = C_\gamma(1+|z|)$ *and* $C_\gamma = C/\gamma.$

Proof of Lemma 1. We will prove Lemma 1 for the case $n \geqslant 3,$ since the proof of the case $n = 2$ is essentially similar. For any fixed $\zeta \in \mathbb{R}^n,\ \zeta \neq 0,$ choose a rotation θ such that $\theta(\zeta) = |\zeta|\mathbf{1} = |\zeta|(1,0,...,0).$ For $x' \in S^{n-1},$ denote $x' = (s, x_2', ..., x_n').$ Then we have

$$\hat{\sigma}_k(\zeta) = \int_{2^k}^{2^{k+1}} b(t)e^{ih(t)}t^{-1-\alpha-z}\int_{S^{n-1}} a(\theta^{-1}(y'))e^{-i|\zeta|ts}\,d\sigma(y')dt,$$

where θ^{-1} is the inverse of $\theta.$ Observe that $a(\theta^{-1}(y'))$ is again an (r,∞) atom with support in $B(\zeta',\rho)\bigcap S^{n-1},\ (\zeta' = \zeta/|\zeta|)$ since $\operatorname{supp} a(y') \subset B(\mathbf{1},\rho)\bigcap S^{n-1}.$ Thus

$$\hat{\sigma}_k(\zeta) = \int_{2^k}^{2^{k+1}} b(t)e^{ih(t)}t^{-1-\alpha-z}\int_{\mathbb{R}} F_a(s)e^{-i|\zeta|ts}\,dsdt,$$

where $F_a(s) = (1-s^2)^{(n-3)/2}\chi_{(-1,1)}(s)\int_{S^{n-2}} a(s,(1-s^2)^{1/2}\tilde{y})\,d\sigma(\tilde{y})$. Note that $F_a(s)$ has support in $(-2r(\zeta'), 2r(\zeta'))$, and $r(\zeta') = A_\rho\zeta/|\zeta|$ (see [5, Lemma 2.1] for properties of $F_a(s)$). We now consider the estimates of $\hat{\sigma}_k(\zeta)$ in several cases.

Case 1. $\Re z = \gamma - \alpha \leqslant 0,\ 0 \leqslant \gamma \leqslant \alpha.$

By the cancellation and support conditions of $F_a(s)$ we obtain inequality (1). By a direct integration, we get inequality (2). On the other hand, we write

$$\begin{aligned}|\hat{\sigma}_k(\zeta)| &\leqslant C||b||_\infty 2^{-k\gamma}\int_{2^k|\zeta|}^{2^{k+1}|\zeta|} t^{-1}\left|\int_{\mathbb{R}} F_a(s)e^{-its}ds\right| dt\\ &\leqslant C2^{-k\gamma}(2^k|\zeta|)^{-1/2}||\hat{F}_a(t)||_{L^2(\mathbb{R})} \leqslant C2^{-k\gamma}(2^k|\zeta|)^{-1/2}|A_\rho\zeta'|^{-1/2}\rho^{-\gamma}\\ &= C2^{-k\gamma}|2^kA_\rho\zeta|^{-1/2}\rho^{-\gamma},\end{aligned}$$

which is the desired inequality (3).

Case 2. $\Re z = \beta/2 - \alpha + \gamma > 0$ and $0 < \gamma \leqslant \alpha$.

Using the cancellation condition of $F_a(s)$ inherited from that of Ω, we can write

$$\hat{\sigma}_k(\zeta) = \int_{\mathbb{R}} F_a(s)\int_{2^k}^{2^{k+1}} \frac{b(t)e^{ih(t)}}{t^{1+\alpha+z}}\left\{e^{-it|\zeta|s} - \sum_{k=0}^{N}\frac{(-i|\zeta|st)^k}{k!}\right\} dt\, ds \equiv \int_{\mathbb{R}} F_a(s)I_1(s)ds,$$

where $I_1(s)$ denotes the inner integral in the double integral above. We write

$$I_1(s) = \int_{2^k}^{2^{k+1}} G'(t)U(t)dt,$$

where $G(t) = \int_{2^k}^{t} b(\tau)e^{ih(\tau)}\tau^{-1-\alpha-z}d\tau$ and $U(t) = e^{-i|\zeta|st} - \sum_{k=0}^{N}\frac{(-i|\zeta|st)^k}{k!}$. Note that $|U(t)| \leqslant (|\zeta|st)^{N+1}$ and $|U'(t)| \leqslant |s\zeta|^{N+1}t^N$.

We claim that $|G(t)| \leqslant C_\gamma(1+|z|)2^{-k\gamma}$ ($C_\gamma = C/\gamma$) for $2^k \leqslant t \leqslant 2^{k+1}$. To see this, write

$$G(t) = \int_{2^k}^{t}\Psi'(\tau)b(\tau)d\tau,\ \text{where } \Psi(\tau) = \int_{2^k}^{\tau} e^{ih(v)}v^{-1-\alpha-z}dv \equiv \int_{2^k}^{\tau} g'(v)v^{-1-\alpha-z}dv,$$

with $g(v) = \int_{2^k}^{v} e^{ih(r)}dr,\ 2^k \leqslant r \leqslant v \leqslant \tau \leqslant t \leqslant 2^{k+1}$. By van der Corput's lemma, $|g(v)| \leqslant C\,v^{(\beta+2)/2}$ for $2^k \leqslant v \leqslant \tau \leqslant t \leqslant 2^{k+1}$. Integrating $\Psi(\tau)$ by parts yields for $2^k \leqslant \tau \leqslant t \leqslant 2^{k+1}$, $|\Psi(\tau)| \leqslant C_\gamma(1+|z|)2^{k(\beta/2-\alpha-\Re z)} \equiv C_z 2^{-k\gamma}$, where $C_z = C_\gamma(1+|z|)$ and $C_\gamma = C/\gamma$. By applying the hypothesis of $b(t)$ and by integrating $G(t)$ by parts, we obtain $|G(t)| \leqslant C_z 2^{-k\gamma}$ for $2^k \leqslant t \leqslant 2^{k+1}$. Recall that $F_a(s)$ has support in $(-2r(\zeta'), 2r(\zeta'))$, and $r(\zeta') = A_\rho\zeta/|\zeta|$ (see [5, Lemma 2.1]). Thus integrating $I_1(s)$ by parts yields

$$|I_1(s)| \leqslant C_z 2^{-k\gamma}(2^k|\zeta\, s|)^{N+1} \leqslant C_z 2^{-k\gamma}(2^k|\zeta\, r(\xi')|)^{N+1} = C_z 2^{-k\gamma}(2^k|A_\rho\zeta|)^{N+1},$$

which leads to inequality (4).

Observe that we can also write $\hat{\sigma}_k(\zeta)$ as

$$\hat{\sigma}_k(\zeta) = \int_{\mathbb{R}} F_a(s)\int_{2^k}^{2^{k+1}} b(t)t^{-1-\alpha-z}e^{i(h(t)-t|\zeta|s)}dt\, ds \equiv \int_{\mathbb{R}} F_a(s)I_2(s)ds.$$

Now write $I_2(s) = \int_{2^k}^{2^{k+1}} b(t)\psi'(t)dt$ where $\psi(t) = \int_{2^k}^{t} \tau^{-1-\alpha-z}e^{i\phi(\tau)}d\tau$, with $\phi(\tau) = h(\tau) - \tau|\zeta|s$. Let $g(\tau) = \int_{2^k}^{\tau} e^{i\phi(r)}dr$. Then $\psi(t) = \int_{2^k}^{t} g'(\tau)\tau^{-1-\alpha-z}d\tau$. Again, by van der Corput's lemma, $|g(\tau)| \leqslant C\tau^{(\beta+2)/2}$ for $2^k \leqslant \tau \leqslant t \leqslant 2^{k+1}$. Using integration by parts technique for the integrals $\psi(t)$, $I_2(s)$, we obtain $|\psi(t)| \leqslant C_z 2^{-k\gamma}$ for $2^k \leqslant t \leqslant 2^{k+1}$ and $|I_2(s)| \leqslant C_z 2^{-k\gamma}$, whence inequality (5) is obtained.

Case 3. $\gamma = 0$ and $0 < \Re z < \beta/2 - \alpha$.

It's clear that $|\hat{\sigma}_k(\zeta)| \leqslant C2^{-k(\alpha+\Re z)}$. By employing similar techniques as in the previous case, we get $|\hat{\sigma}_k(\zeta)| \leqslant C_z 2^{k(\beta/2-\alpha-\Re z)}$. These two inequalities yield the desired inequality (6). Lemma 1 is proved. □

Let us choose a radial function $\phi \in S(\mathbb{R}^n)$ so that $\operatorname{supp}\hat{\phi} \subset \{\zeta : 1/2 \leqslant |\zeta| \leqslant 2\}$, $0 \leqslant \hat{\phi}(|\zeta|) \leqslant 1$, $\hat{\phi}(|\zeta|) \geqslant c > 0$ if $3/5 \leqslant |\zeta| \leqslant 5/3$, and $\sum_k \hat{\phi}^2(2^k|\zeta|) = 1$ for all $\zeta \neq 0$.

Define ϕ_{2^k} by $\phi_{2^k}(x) = 2^{-kn}\phi(2^{-k}x)$. Then $\hat{\phi}_{2^k}(\zeta) = \hat{\phi}(2^k\zeta) = \hat{\phi}(|2^k\zeta|)$. Define Ψ by $\hat{\Psi}(\zeta) = \hat{\phi}(\rho\zeta)$. Then Ψ is also a Schwartz radial function with $\operatorname{supp}\hat{\Psi} \subset \{\zeta : 1/2 \leqslant \rho|\zeta| \leqslant 2\}$, $0 \leqslant \hat{\Psi}(|\zeta|) \leqslant 1$, $\hat{\Psi}(\zeta) \geqslant c > 0$ if $3/5 \leqslant \rho|\zeta| \leqslant 5/3$, and $\sum_k \hat{\Psi}^2(2^k|\zeta|) = 1$ for all $\zeta \neq 0$. Define Ψ_{2^k} by $\Psi_{2^k}(x) = 2^{-kn}\Psi(2^{-k}x)$. Then $\Psi_{2^k}(x) = \phi_{2^k\rho}(x)$ and $\hat{\Psi}_{2^k}(\zeta) = \hat{\phi}_{2^k\rho}(\zeta)$. For $f \in \mathscr{S}(\mathbb{R}^n)$, define the operator S_k ($k \in \mathbb{Z}$) by $S_k f(x) = \Psi_{2^k} * f(x)$. We now decompose the operator T_z as follows.

$$T_z f = \sum_k \sigma_k * \left(\sum_j S_{j+k}S_{j+k}f\right) = \sum_j \sum_k S_{k+j}\left(\sigma_k * S_{k+j}f\right) \equiv \sum_j \tilde{T}_j f,$$

where $\tilde{T}_j f = \sum_k S_{k+j}\left(\sigma_k * S_{k+j}f\right)$ and recall that $S_k f = \Psi_{2^k} * f$.
Let S_k^* be the dual operator of S_k. There exists an $m \in \mathbb{Z}$ such that $2^m \leqslant \rho \leqslant 2^{m+1}$. Let c_1 be a fixed constant such that $\rho = c_1 2^m$, $1 \leqslant c_1 \leqslant 2$. Now observe that

$$\begin{aligned}
&\left\| \left(\sum_j |(2^j\rho)^{-s} S_j^* f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
&\cong \left\| \left(\sum_j |(2^j\rho)^{-s} S_j f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} = \left\| \left(\sum_j |(c_1 2^{j+m})^{-s} \phi_{c_1 2^{j+m}} * f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
&= \left\| \left(\sum_j |(c_1 2^j)^{-s} \phi_{c_1 2^j} * f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \cong \|f\|_{\dot{F}_p^{s,q}(\mathbb{R}^n)}. \qquad (7)
\end{aligned}$$

Thus for any $g \in \dot{F}_{p'}^{-s,q'}(\mathbb{R}^n)$, we have

$$\begin{aligned}
&|< \tilde{T}_j f, g >| \\
&= \left| \int_{\mathbb{R}^n} \sum_k S_{k+j}\left(\sigma_k * S_{k+j}f\right)(x)g(x)dx \right| = \left| \int_{\mathbb{R}^n} \sum_k \left(\sigma_k * S_{k+j}f\right)(x) S_{k+j}^* g(x)dx \right|
\end{aligned}$$

$$\leqslant \int_{\mathbb{R}^n} \left(\sum_k |(2^{k+j}\rho)^{-s}\sigma_k * S_{k+j}f(x)|^q \right)^{1/q} \left(\sum_k |(2^{k+j}\rho)^s S^*_{k+j}g(x)|^{q'} \right)^{1/q'} dx$$

$$\leqslant \left\| \left(\sum_k |(2^{k+j}\rho)^{-s}\sigma_k * S_{k+j}f(x)|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \left\| \left(\sum_k |(2^{k+j}\rho)^s S^*_{k+j}g(x)|^{q'} \right)^{1/q'} \right\|_{L^{p'}(\mathbb{R}^n)}.$$

Taking the supremum over all g with $||g||_{\dot{F}^{-s,q'}_{p'}(\mathbb{R}^n)} \leqslant 1$, we obtain

$$||\tilde{T}_j f||_{\dot{F}^{s,q}_p(\mathbb{R}^n)} \leqslant C \left\| \left(\sum_k |(2^{k+j}\rho)^{-s}\sigma_k * S_{k+j}f(x)|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}. \tag{8}$$

In particular, when $p = q = 2$, inequality (8) implies

$$\begin{aligned} ||\tilde{T}_j f||^2_{\dot{F}^{s,2}_2} &\leqslant C \sum_k \int_{\mathbb{R}^n} (2^{k+j}\rho)^{-2s} |\hat{\sigma}_k(\zeta)\hat{\phi}(2^{k+j}\rho\zeta)\hat{f}(\zeta)|^2 d\zeta \\ &\leqslant C \sum_k \int_{D_{k+j}} (2^{k+j}\rho)^{-2s} |\hat{\sigma}_k(\zeta)\hat{f}(\zeta)|^2 d\zeta, \end{aligned} \tag{9}$$

where $D_{k+j} = \left\{ \zeta \in \mathbb{R}^n : 1/2 \leqslant |2^{k+j}\rho\zeta| \leqslant 2 \right\}$. If $\Re z = \beta/2 - \alpha + \gamma > 0$ and $0 < \gamma \leqslant \alpha$, then inequalities (4), (5) and (9) imply

$$\begin{aligned} ||\tilde{T}_j f||_{\dot{F}^{s,2}_2(\mathbb{R}^n)} &\leqslant C_z ||f||_{L^2_{s+\gamma}(\mathbb{R}^n)} \min\left\{ 2^{-j(N+1-\gamma)}, 2^{j\gamma} \right\} \\ &\equiv C_z ||f||_{\dot{F}^{s+\gamma,2}_2(\mathbb{R}^n)} \min\left\{ 2^{-j(N+1-\gamma)}, 2^{j\gamma} \right\} \end{aligned} \tag{10}$$

whenever $s \geqslant 0$. By duality, we also obtain inequality (10) for all $s \in \mathbb{R}$. Since $T_z f = \sum_j \tilde{T}_j f$, inequality (10) implies that

$$||T_z f||_{\dot{F}^{s,2}_2(\mathbb{R}^n)} \leqslant C_z ||f||_{\dot{F}^{s+\gamma,2}_2(\mathbb{R}^n)} \text{ for } \Re z = \beta/2 - \alpha + \gamma > 0 \quad \text{and} \quad 0 < \gamma \leqslant \alpha. \tag{11}$$

For the case $\Re z = \gamma - \alpha \leqslant 0$, we use inequalities (1), (2) and (9) to obtain the following inequalities

$$||\tilde{T}_j f||_{\dot{F}^{s,2}_2(\mathbb{R}^n)} \leqslant C 2^{-j(N+1-\gamma)} ||f||_{\dot{F}^{s+\gamma,2}_2(\mathbb{R}^n)} \tag{12}$$

$$\text{and} \quad ||\tilde{T}_j f||_{\dot{F}^{s,2}_2(\mathbb{R}^n)} \leqslant C 2^{j\gamma} ||f||_{\dot{F}^{s+\gamma,2}_2(\mathbb{R}^n)}. \tag{13}$$

Note that $|\sigma_k * S_{k+j}f(x)| \leqslant ||b||_\infty 2^{-k\gamma}\rho^{-\gamma} \int_{|y|\cong 2^k} |A(y')||y|^{-n}|S_{k+j}f(x-y)|dy$, where $A(y') = \rho^\gamma a(y')$ is a $(1,\infty)$ atom. By Hölder's inequality, we have

$$|\sigma_k * S_{k+j}f(x)|^q \leqslant C 2^{-k\gamma q}\rho^{-\gamma q} ||A||^{q/q'}_{L^1(S^{n-1})} \int_{|y|\cong 2^k} |A(y')||y|^{-n}|S_{k+j}f(x-y)|^q dy$$

and therefore

$$||\sigma_k * S_{k+j}f||_q \leqslant C 2^{-k\gamma}\rho^{-\gamma} ||S_{k+j}f||_q. \tag{14}$$

When $p = q$, inequalities (8) and (14) imply

$$\begin{aligned} \|\tilde{T}_j f\|_{\dot{F}_q^{s,q}(\mathbb{R}^n)} &\leqslant C\left\{\sum_k (2^{k+j}\rho)^{-sq}\int_{\mathbb{R}^n}|\sigma_k * S_{k+j}f(x)|^q dx\right\}^{1/q} \\ &\leqslant C2^{j\gamma}\left\{\int_{\mathbb{R}^n}\sum_k (2^{k+j}\rho)^{-(s+\gamma)q}|S_{k+j}f(x)|^q dx\right\}^{1/q} \\ &\leqslant C2^{j\gamma}\|f\|_{\dot{F}_q^{s+\gamma,q}(\mathbb{R}^n)}. \end{aligned} \tag{15}$$

If $p > q$, we infer from inequality (8) that there exists a function $h \in L^{(p/q)'}(\mathbb{R}^n)$ with $\|h\|_{(p/q)'} = 1$ such that

$$\|\tilde{T}_j f\|^q_{\dot{F}_p^{s,q}(\mathbb{R}^n)} \leqslant C\sum_k \int_{\mathbb{R}^n}|(2^{k+j}\rho)^{-s}\sigma_k * S_{k+j}f(x)|^q h(x)dx. \tag{16}$$

Recall that $|\sigma_k * S_{k+j}f(x)|^q \leqslant C2^{-k\gamma q}\rho^{-\gamma q}L_k(|S_{k+j}f|^q)(x)$, where

$$L_k(f)(x) = \int_{|y|\cong 2^k}|A(y')||y|^{-n}f(x-y)dy.$$

Denote L_k^* to be the dual operator of L_k and let $N_A f(x) = \sup_{k\in\mathbb{Z}} L_k^*(|f|)(x)$. By the method of rotation and by the L^p boundedness of the Hardy-Littlewood maximal function, we have $\|N_A f\|_{L^p} \leqslant C\|f\|_p$ for $1 < p < \infty$. Thus inequality (16) becomes

$$\begin{aligned} \|\tilde{T}_j f\|^q_{\dot{F}_p^{s,q}(\mathbb{R}^n)} &\leqslant C2^{j\gamma q}\sum_k \int_{\mathbb{R}^n}(2^{k+j}\rho)^{-(s+\gamma)q}L_k(|S_{k+j}f|^q)(x)h(x)dx \\ &\leqslant C2^{j\gamma q}\int_{\mathbb{R}^n}\left\{\sum_k |(2^{k+j}\rho)^{-(s+\gamma)}S_{k+j}f(x)|^q\right\}N_A h(x)dx \\ &\leqslant C2^{j\gamma q}\left\|\sum_k (2^{k+j}\rho)^{-(s+\gamma)}S_{k+j}f\right\|_p^q \|h\|_{(p/q)'}. \end{aligned}$$

Therefore $\|\tilde{T}_j f\|_{\dot{F}_p^{s,q}(\mathbb{R}^n)} \leqslant C2^{j\gamma}\|f\|_{\dot{F}_p^{s+\gamma,q}(\mathbb{R}^n)}$, which together with inequality (15) yields

$$\|\tilde{T}_j f\|_{\dot{F}_p^{s,q}(\mathbb{R}^n)} \leqslant C2^{j\gamma}\|f\|_{\dot{F}_p^{s+\gamma,q}(\mathbb{R}^n)} \quad \text{for} \quad p \geqslant q. \tag{17}$$

Now set $q = 2$ in (17) and by duality, we obtain

$$\|\tilde{T}_j f\|_{\dot{F}_p^{s,2}(\mathbb{R}^n)} \leqslant C2^{j\gamma}\|f\|_{\dot{F}_p^{s+\gamma,2}(\mathbb{R}^n)} \quad \text{for} \quad 1 < p < \infty,\ j \in \mathbb{Z}. \tag{18}$$

Interpolating between (12)-(13) and (18) for a fixed $q = 2$ (see [21, p. 185]) yields

$$\|\tilde{T}_j f\|_{\dot{F}_p^{s,2}(\mathbb{R}^n)} \leqslant C2^{-\theta j}\|f\|_{\dot{F}_p^{s+\gamma,2}(\mathbb{R}^n)} \tag{19}$$

$$\text{and} \quad \|\tilde{T}_j f\|_{\dot{F}_p^{s,2}(\mathbb{R}^n)} \leqslant C2^{j\gamma}\|f\|_{\dot{F}_p^{s+\gamma,2}(\mathbb{R}^n)} \tag{20}$$

where $\theta \cong (2(N+1) - \gamma\tilde{p})/\tilde{p} > 0$.

Recall that $\mathscr{S}(\mathbb{R}^n)$ is dense in $\dot{F}_p^{s,q}(\mathbb{R}^n)$ for $s \in \mathbb{R}$, $1 < p, q < \infty$. Thus we may view the tempered distribution f as a Schwartz function $f \in \mathscr{S}(\mathbb{R}^n)$ and apply Riesz-Thorin interpolation theory (for a fixed q) to obtain the same results above.

Interpolating (19)-(20) and (17) yields

$$||\tilde{T}_j f||_{\dot{F}_p^{s,q}(\mathbb{R}^n)} \leqslant C2^{-\delta|j|}||f||_{\dot{F}_p^{s+\gamma,q}(\mathbb{R}^n)} \tag{21}$$

for $1 < q \leqslant p < \infty$, and $\delta = \min\{\gamma, (4(N+1) - \gamma\tilde{p}\tilde{q})/\tilde{p}\tilde{q})\} > 0$. Since $T_z f = \sum_j \tilde{T}_j f$, we infer from inequality (21) that

$$||T_z f||_{\dot{F}_p^{s,q}(\mathbb{R}^n)} \leqslant C||f||_{\dot{F}_p^{s+\gamma,q}(\mathbb{R}^n)} \tag{22}$$

for $1 < q \leqslant p < \infty$, and by duality we obtain (22) for $1 < p, q < \infty$, $s \in \mathbb{R}$. Finally, an application of complex interpolation between (11) and (22) yields

$$||T_{h,\alpha} f||_{\dot{F}_p^{s,q}(\mathbb{R}^n)} \leqslant C||f||_{\dot{F}_p^{s+\gamma,q}(\mathbb{R}^n)} \tag{23}$$

for $\beta/(\beta+\gamma-\alpha) < p, q < \beta/(\alpha-\gamma)$, $s \in \mathbb{R}$, provided that $\beta/2 > \alpha - \gamma \geqslant 0$ and $0 < \gamma \leqslant \alpha$.

We now consider the case $\gamma = 0$. Recall the Schwartz functions ϕ and ϕ_{2^k} defined earlier. We change the definition of Ψ (which was defined earlier) by redefining Ψ as $\hat{\Psi}(\zeta) = \phi(A_\rho \zeta)$, where again $A_\rho \zeta = (\rho^2\zeta_1, \rho\zeta_2, ..., \rho\zeta_n)$. We then define Ψ_{2^k} by $\Psi_{2^k}(x) = 2^{-kn}\Psi(2^{-k}x)$. Let $S_k f = \Psi_{2^k} * f$, and let S_k^* denote the dual operator of S_k. Now observe that for $q = 2$ and $s = 0$, we have

$$\begin{aligned}
\left\| (\sum_k |S_k^* f|^2)^{1/2} \right\|_{L^p(\mathbb{R}^n)} &\cong \left\| (\sum_k |S_k f|^2)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \\
&\cong \left\| (\sum_k |\Psi_k * f|^2)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \\
&\cong ||f||_{L^p(\mathbb{R}^n)} \text{ (by Littlewood-Paley theory)} \\
&\cong ||f||_{\dot{F}_p^{0,2}(\mathbb{R}^n)} \\
&\cong \left\| (\sum_k |\phi_k * f|^2)^{1/2} \right\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

That is, the identity (7) remains valid for our new operator S_k if $q = 2$ and $s = 0$. Now for $0 < \Re z < \beta/2 - \alpha$, we use the fact that $T_z f = \sum_k \sigma_k * f$ and inequality (6) to obtain $||T_z f||_{\dot{F}_2^{0,2}(\mathbb{R}^n)} \leqslant C_z ||f||_{\dot{F}_2^{0,2}(\mathbb{R}^n)}$. Note that this inequality is the same as inequality (11) with $s = 0$ and $\gamma = 0$. By applying inequalities (1) and (3), we obtain

$$||\tilde{T}_j f||_{\dot{F}_2^{0,2}(\mathbb{R}^n)} \leqslant C2^{-j}||f||_{\dot{F}_2^{0,2}(\mathbb{R}^n)} \quad \text{and}$$

$$||\tilde{T}_j f||_{\dot{F}_2^{0,2}(\mathbb{R}^n)} \leqslant C2^{j/2}||f||_{\dot{F}_2^{0,2}(\mathbb{R}^n)}.$$

Note again that the two inequalities above are similar to the inequalities (12) and (13) with $s = 0$ and $\gamma = 0$. Thus if we run parallel arguments as in the previous case in which $\gamma > 0$, we obtain inequality (23) for $s = 0$, $\gamma = 0$, and $q = 2$. That is, $||T_{h,\alpha}f||_{\dot{F}_p^{0,2}(\mathbb{R}^n)} \leqslant C||f||_{\dot{F}_p^{0,2}(\mathbb{R}^n)}$ for $\beta/(\beta-\alpha) < p < \beta/\alpha$, provided that $\beta > 2\alpha$. It remains to show the boundedness of $T_\alpha f$ in Triebel-Lizorkin spaces. Note that the oscillating factor $e^{ih(|y|)}$ in $T_z f$ is not essentially involved in the proof of the case $\Re z = \gamma - \alpha \leqslant 0$. Also, the proof of this case only requires the function $b(|y|)$ to be bounded. Thus, the result follows from inequality (22) with $\gamma = \alpha$ and $T_\alpha f = T_z f$ with $z = 0$. That is, $||T_\alpha f||_{\dot{F}_p^{s,q}(\mathbb{R}^n)} \leqslant C||f||_{\dot{F}_p^{s+\alpha,q}(\mathbb{R}^n)}$ for $1 < p, q < \infty$, $s \in \mathbb{R}$ and $\alpha > 0$. The proof of Theorem 1 is complete. □

REMARK 1. If $\Gamma(|y|)$ is a C^1 increasing function on its compact support and if $\Gamma'(t)$ is increasing on its support, then

$$||T_{h,\alpha,\Gamma}f||_{L^p(\dot{F}_p^{s,q}(\mathbb{R}^n),\mathbb{R})} \leqslant C||f||_{L^p(\dot{F}_p^{s+\gamma,q}(\mathbb{R}^n),\mathbb{R})}$$

for $\beta/(\beta+\gamma-\alpha) < p,q < \beta/(\alpha-\gamma)$, $s \in \mathbb{R}$, provided that $\beta > 2(\alpha-\gamma) \geqslant 0$ and $0 < \gamma \leqslant \alpha$. Also,

$$||T_{h,\alpha,\Gamma}f||_{L^p(\mathbb{R}^{n+1})} \leqslant C||f||_{L^p(\mathbb{R}^{n+1})}$$

for $\beta/(\beta-\alpha) < p < \beta/\alpha$, with $\beta > 2\alpha > 0$.

The proof of the above results is similar to the proof of Theorem 1, with some slight modifications. For instance, $S_k f(x)$ in the proof of Theorem 1 should be replaced by $S_k f(x, x_{n+1}) = (\Psi_{2^k} \otimes \delta) * f(x, x_{n+1})$, where δ is the Dirac distribution acting on the variable $x_{n+1} \in \mathbb{R}$.

Note also that $\hat{\sigma}_k(\zeta, \zeta_{n+1}) = \displaystyle\int_{|y|\cong 2^k} \frac{a(y')b(|y|)e^{-i\zeta\cdot y}e^{-i\zeta_{n+1}\Gamma(|y|)}e^{ih(|y|)}}{|y|^{n+\alpha+z}}\,dy.$

By the same integration techniques as in the proof of Theorem 1 (with $b(|y|)$ being replaced by $b(|y|)e^{-i\zeta_{n+1}\Gamma(|y|)}$) and by the hypothesis of $\Gamma(|y|)$, we see that $\hat{\sigma}_k(\zeta, \zeta_{n+1})$ also satisfies inequalities (1)-(6) in Lemma 1, with the constants that are independent of ζ_{n+1} (see [12, Lemma 1]). Also, the two dimensional maximal function

$$M_\Gamma f(x_1, x_2) = \sup_{k\in\mathbb{Z}} \left\{ \frac{1}{2^k} \int_{2^k}^{2^{k+1}} |f(x_1 - t, x_2 - \Gamma(t))|\,dt \right\}$$

is bounded in $L^p(\mathbb{R}^2)$ for all $p > 1$ (see [14, Corollary 1]). This result together with the method of rotation imply that $N_A f(x, x_{n+1})$ is bounded in $L^p(\mathbb{R}^{n+1})$ for all $p > 1$, where $N_A f(x, x_{n+1})$ is obviously defined in a similar manner as $N_A f(x)$ in the proof of Theorem 1.

If $\Omega(y') \in H^1(S^{n-1})$, then the condition that $\Gamma(|y|)$ has compact support can be relaxed. We state the following theorem.

THEOREM 2. *Let $\Omega(y') \in H^1(S^{n-1})$ with the mean value zero property and let $b(|y|)$ be given as in Theorem 1. Suppose that $h'(t)$ is increasing on $(0,\infty)$, $h''(t)$ is decreasing on $(0,\infty)$ with $|h''(t)| \geqslant Ct^{-\beta-2}$ and $|h^{(3)}(t)| \geqslant Ct^{-\beta-3}$ for all*

$t \in (0,\infty)$ and some $\beta > 0$, $\beta > 3\alpha > 0$. If Γ, Γ', and Γ'' are increasing on $(0,\infty)$ with $\Gamma(0) = 0 = \Gamma'(0)$, then

$$||T_{h,\alpha,\Gamma} f||_{L^p(\mathbb{R}^{n+1})} \leqslant C||f||_{L^p(\mathbb{R}^{n+1})}$$

for $2\beta/(2\beta - 3\alpha) < p < 2\beta/3\alpha$, with $\beta > 3\alpha > 0$.

Proof of Theorem 2. From remark 1, it suffices to show that the two dimensional maximal function $M_\Gamma f(x_1, x_2)$ is bounded in $L^p(\mathbb{R}^2)$ for all $p > 1$, and

$$|\hat{\sigma}_k(\zeta, \zeta_{n+1})| \leqslant C \min(|2^k A_\rho \zeta|, |2^k A_\rho \zeta|^{-1/2}) \text{ when } \Re z = -\alpha \leqslant 0,$$

and $|\hat{\sigma}(\zeta, \zeta_{n+1})| \leqslant C$ uniformly for all $(\zeta, \zeta_{n+1}) \in \mathbb{R}^{n+1}$ when $0 < \Re z < \beta/3 - \alpha$.

Here $\hat{\sigma}(\zeta, \zeta_{n+1}) = \int_{\mathbb{R}^n} \frac{a(y')b(|y|)e^{-i\zeta\cdot y}e^{-i\zeta_{n+1}\Gamma(|y|)}e^{ih(|y|)}}{|y|^{n+\alpha+z}} dy$. For the L^p bounds of $M_\Gamma f(x_1, x_2)$, see [10, Corollary 5.3]. The first estimate of $|\hat{\sigma}_k(\zeta, \zeta_{n+1})|$ is easily obtained by the same techniques as in the proof of inequalities (1) and (3) in Lemma 1.

It remains to prove the second estimate of $|\hat{\sigma}_k(\zeta, \zeta_{n+1})|$. We write

$$\hat{\sigma}_k(\zeta, \zeta_{n+1}) = \int_{\mathbb{R}} F_a(s) \int_{\mathbb{R}} b(t) e^{i\phi(t)} t^{-1-\alpha-z} dt ds \equiv \int_{\mathbb{R}} F_a(s) I_3(s) ds,$$

where $\phi(t) = -|\zeta|ts - \zeta_{n+1}\Gamma(t) + h(t)$.

Denote $I_3(s) = \int_0^1 ...dt + \int_1^\infty ...dt \equiv I_4(s) + I_5(s)$. It is clear that $I_5(s) \leqslant C$ if $0 < \Re z < \beta/3 - \alpha$. Now write $I_4(s) = \int_0^1 b(t) G'(t) dt$, where $G(t) = \int_0^t e^{i\phi(\tau)} \tau^{-1-\alpha-z} d\tau$. If $\zeta_{n+1} < 0$, then $\phi''(r) = h''(r) - \zeta_{n+1}\Gamma''(r) \geqslant h''(r) \geqslant Cr^{-\beta-2} \geqslant C\tau^{-\beta-2}$ for $0 < r \leqslant \tau \leqslant t \leqslant 1$. By van der Corput's lemma, $|G(\tau)| \leqslant C\tau^{(\beta+2)/2}$ for $0 \leqslant \tau \leqslant t \leqslant 1$. If $\zeta_{n+1} \geqslant 0$, then $|\phi^{(3)}(r)| \geqslant |h^{(3)}(r)| \geqslant Cr^{-\beta-3} \geqslant C\tau^{-\beta-3}$ for $0 < r \leqslant \tau \leqslant t \leqslant 1$. Hence, $|G(\tau)| \leqslant C\tau^{(\beta+3)/3}$ for $0 \leqslant \tau \leqslant t \leqslant 1$. In either case, $|G(\tau)| \leqslant C\tau^{(\beta+3)/3}$ for $0 \leqslant \tau \leqslant t \leqslant 1$. Finally, by integrating by parts, we obtain $|I_4(s)| \leqslant C$, and consequently $|\hat{\sigma}(\zeta, \zeta_{n+1})| \leqslant C$ uniformly for all $(\zeta, \zeta_{n+1}) \in \mathbb{R}^{n+1}$, with $0 < \Re z < \beta/3 - \alpha$. Theorem 2 is proved. □

THEOREM 3. *Consider the singular integral $T_{h,\alpha}$ defined in Theorem 1, but with $\Omega \in L^r(S^{n-1})$ $(r > 1)$, and Ω needs not satisfy the cancellation condition. Then*

$$||T_{h,\alpha} f||_{\dot{F}_p^{s_1,q}(\mathbb{R}^n)} \leqslant C||f||_{\dot{F}_p^{s_2,q}(\mathbb{R}^n)}$$

for $\beta/(\beta - \alpha - \gamma) < p, q < \beta/(\alpha + \gamma)$, $s_2 = s_1 - \gamma(\beta - 2\alpha - 2\gamma)/\beta$, $s_1 \in \mathbb{R}$, provided that $\beta > 2(\alpha + \gamma)$, $\alpha \geqslant 0$, $0 < \gamma < 2/(r'\tilde{p}\tilde{q})$, and r' is the conjugate of r.

Proof of Theorem 3. The proof of this theorem is essentially similar to the proof of Theorem 1. Recall the analytic operator $T_z f$ defined in Theorem 1. We need to prove the following inequalities.

$$||T_z f||_{L^2(\mathbb{R}^n)} \leqslant C_z||f||_{L^2(\mathbb{R}^n)} \quad \text{when} \quad \Re z = \beta/2 - \alpha - \gamma > 0, \quad \text{and} \qquad (24)$$

$$||T_z f||_{\dot{F}_p^{s,q}(\mathbb{R}^n)} \leqslant C||f||_{\dot{F}_p^{s-\gamma,q}(\mathbb{R}^n)} \quad \text{for} \quad 1 < p,\, q < \infty, s \in \mathbb{R}, \tag{25}$$

when $\Re z = -\gamma - \alpha < 0$.
Observe that if $\Re z = \beta/2 - \alpha - \gamma > 0$, then $|\hat{\sigma}_k(\zeta)| \leqslant C_z \min\left\{2^{k\gamma},\, 2^{-k(\beta/2-\gamma)}\right\}$. On the other hand, if $\Re z = -\gamma - \alpha < 0$, then $|\hat{\sigma}_k(\zeta)| \leqslant C\min\left\{2^{k\gamma},\, 2^{k\gamma}(2^k|\zeta|)^{-\delta}\right\}$ where $0 < \delta < 1/2r'$ and $\frac{1}{r} + \frac{1}{r'} = 1$. Therefore, it is straightforward to see that inequalities (24)-(25) hold. Finally, an application of interpolation yields the desired result. Theorem 3 is proved. □

2. Fractional integrals in Triebel-Lizorkin spaces

We denote the following fractional integrals by

$$I_\alpha f(x) = \int_{R^n} \frac{b(|y|)\Omega(y')}{|y|^{n-\alpha}} f(x-y)\,dy,$$

$$I_{h,\alpha} f(x) = \int_{R^n} \frac{b(|y|)e^{ih(|y|)}\Omega(y')}{|y|^{n-\alpha}} f(x-y)\,dy,$$

and

$$I_{h,\alpha,\Gamma} f(x, x_{n+1}) = \int_{R^n} \frac{b(|y|)e^{ih(|y|)}\Omega(y')}{|y|^{n-\alpha}} f(x-y, x_{n+1} - \Gamma(|y|))\,dy,$$

where $x, y \in \mathbb{R}^n$, $x \in \mathbb{R}$, and $\Gamma(|y|)$ is a measurable radial function defined on $\mathbb{R}^n$.

THEOREM 4. *Let $b(|y|)$ be given as in Theorem 1. Assume that $|h''(t)| \geqslant Ct^{\beta-2}$ for all $t > 0$ and some $\beta > 0$.*

If Ω is given as in Theorem 1, then

$$||I_{h,\alpha} f||_{\dot{F}_p^{s,q}(\mathbb{R}^n)} \leqslant C||f||_{\dot{F}_p^{s+\gamma,q}(\mathbb{R}^n)}$$

for $\beta/(\beta-\gamma-\alpha) < p,\, q < \beta/(\alpha+\gamma)$, $s \in \mathbb{R}$, provided that $\beta > 2(\alpha+\gamma)$ and $\gamma > 0$. Moreover, if $\gamma = 0$ and $\beta > 2\alpha > 0$, then

$$||I_{h,\alpha} f||_{\dot{F}_p^{0,2}(\mathbb{R}^n)} \leqslant C||f||_{\dot{F}_p^{0,2}(\mathbb{R}^n)}$$

for $\beta/(\beta-\alpha) < p < \beta/\alpha$.

On the other hand, if $\Omega \in L^r(S^{n-1})$ $(r > 1)$ without satisfying the cancellation condition, then

$$||I_{h,\alpha} f||_{\dot{F}_p^{s_1,q}(\mathbb{R}^n)} \leqslant C||f||_{\dot{F}_p^{s_2,q}(\mathbb{R}^n)}$$

for $(\beta-4\gamma)/(\beta-\alpha-3\gamma) < p,\, q < (\beta-4\gamma)/(\alpha-\gamma)$, $s_2 = s_1 - \gamma(\beta-2\alpha-2\gamma)/(\beta-4\gamma)$, $s_1 \in \mathbb{R}$, provided that $\beta > 2(\alpha+\gamma)$, $\alpha \geqslant \gamma > 0$, $0 < \gamma < 2/(r'\tilde{p}\tilde{q})$, and r' is the conjugate of r. Also, if $b(|y|)$ is merely a bounded function, then

$$||I_\alpha f||_{\dot{F}_p^{s,q}(\mathbb{R}^n)} \leqslant C||f||_{\dot{F}_p^{s-\alpha,q}(\mathbb{R}^n)}$$

for $1 < p,\, q < \infty$ and $s \in \mathbb{R}$.

Proof of Theorem 4. We consider a family of analytic operators I_z defined by

$$I_z f(x) = p.v. \int \frac{\Omega(y') e^{ih(|y|)} b(|y|) f(x-y)}{|y|^{n-\alpha+z}}\, dy.$$

Since the proof of this theorem is analogous to the proof of Theorem 1, we will outline some necessary steps in this proof, and omit the details. If $\Omega \in H^r(S^{n-1})$, then we must show that

$$\begin{aligned} ||I_z f||_{\dot{F}_p^{s,q}(\mathbb{R}^n)} &\leqslant C||f||_{\dot{F}_p^{s+\gamma,q}(\mathbb{R}^n)} \quad \text{for } 1 < p,q < \infty, s \in \mathbb{R}, \gamma > 0 && (26)\\ &\quad\quad \text{and } \Re z = \gamma+\alpha. \\ ||I_z f||_{\dot{F}_p^{0,2}(\mathbb{R}^n)} &\leqslant C||f||_{\dot{F}_p^{0,2}(\mathbb{R}^n)} \quad \text{for } 1 < p < \infty, \Re z = \alpha, \text{ and } \gamma = 0. && (27)\\ ||I_z f||_{\dot{F}_2^{s,2}(\mathbb{R}^n)} &\leqslant C_z||f||_{\dot{F}_2^{s+\gamma,2}(\mathbb{R}^n)} \quad \text{for } \Re z = \gamma + \alpha - \beta/2 < 0 \text{ and } \gamma > 0. && (28)\\ ||I_z f||_{\dot{F}_2^{0,2}(\mathbb{R}^n)} &\leqslant C_z||f||_{\dot{F}_2^{0,2}(\mathbb{R}^n)} \quad \text{for } \Re z = \alpha - \beta/2 < 0 \text{ and } \gamma = 0. && (29)\end{aligned}$$

If $\Omega \in L^r(S^{n-1})$ $(r > 1)$ instead of $\Omega \in H^r(S^{n-1})$, then we need to prove the following inequalities

$$\begin{aligned} ||I_z f||_{\dot{F}_p^{s,q}(\mathbb{R}^n)} &\leqslant C||f||_{\dot{F}_p^{s-\gamma,q}(\mathbb{R}^n)} \text{ for } 1 < p,q < \infty, s \in \mathbb{R}, 0 < \gamma \leqslant \alpha && (30)\\ &\quad\quad \text{and } \Re z = \alpha - \gamma \geqslant 0, \\ \text{and} \quad ||I_z f||_{\dot{F}_2^{0,2}(\mathbb{R}^n)} &\leqslant C_z||f||_{\dot{F}_2^{0,2}(\mathbb{R}^n)} \quad \text{for } \Re z = \alpha + \gamma - \beta/2 < 0. && (31)\end{aligned}$$

Note also that by setting $\gamma = \alpha$ (i.e., $\Re z = 0$) in inequality (30), we will obtain the results for the fractional integral $I_\alpha f$. Theorem 4 is proved. □

REMARK 2. If $\Omega \in H^r(S^{n-1})$ and Γ satisfies the same conditions mentioned in Remark 1, then

$$||I_{h,\alpha,\Gamma} f||_{L^p(\dot{F}_p^{s,q}(\mathbb{R}^n),\mathbb{R})} \leqslant C||f||_{L^p(\dot{F}_p^{s+\gamma,q}(\mathbb{R}^n),\mathbb{R})}$$

for $\beta/(\beta-\gamma-\alpha) < p,\, q < \beta/(\alpha+\gamma)$, $s \in \mathbb{R}$, provided that $\beta > 2(\alpha+\gamma)$ and $\gamma > 0$. Also,

$$||I_{h,\alpha,\Gamma} f||_{L^p(\mathbb{R}^{n+1})} \leqslant C||f||_{L^p(\mathbb{R}^{n+1})}$$

for $\beta/(\beta-\alpha) < p < \beta/\alpha$, with $\beta > 2\alpha > 0$.

3. Fractional integrals in weighted L^p spaces

We now consider fractional integrals in weighted L^p spaces with the special case $h(|y|) = e^{i|y|^\beta}$. That is,

$$I_{\alpha,\beta} f(x) = \int_{R^n} \Omega(y')|y|^{\alpha-n} e^{i|y|^\beta} f(x-y)\, dy,$$

where $\alpha, \beta > 0$. A quick review of the elementary properties of A_p weights can be found in [18]. For the rest of this paper, $\Omega \in L^r(S^{n-1})$, $1 \leqslant r < \infty$, and Ω needs not satisfy the mean value zero property. We denote r' to be the conjugate of r, i.e., $\frac{1}{r}+\frac{1}{r'}=1$. We consider two cases: $\beta=1$, and $\beta>0$, $\beta\neq 1$. We have the following results.

THEOREM 5. *Let* $\Omega\in L^r(S^{n-1})$, $r>1$, $0<\alpha<1/r'$, $\beta=1$. *Then*

$$||I_{\alpha,\beta}f||_p \leqslant C||f||_p \text{ for } 2/(2-\alpha r')<p<2/(\alpha r').$$

Moreover, if $1<r\leqslant 2$, $0<\alpha<2/(r'^2)$, *and* $\omega(x)$ *is a positive function in* $A_{(p/r')}$, $p>r'$, *then*

$$||I_{\alpha,\beta}f||_{L^p(\omega^t dx)} \leqslant C||f||_{L^p(\omega^t dx)} \text{ for } r'<p<2/(\alpha r') \text{ and } 0<t<1-\alpha r'p/2.$$

Proof of Theorem 5. We write $I_{\alpha,\beta}f(x)=\int_{|y|\leqslant 1}...dy+\int_{|y|>1}...dy\equiv J_1f(x)+J_2f(x)$. Observe that

$$||J_1f||_p\leqslant C||\Omega||_{L^1(S^{n-1})}||f||_p \text{ for } 1<p<\infty. \tag{32}$$

Now write $J_2f(x)=\sum_{j=0}^{\infty}I_jf(x)$, where $I_jf(x)=\int_{|y|\cong 2^j}\Omega(y')|y|^{\alpha-n}e^{i|y|}f(x-y)\,dy$.

By taking the Fourier transform of $I_jf(x)$, we have $\widehat{I_jf}(\zeta)=m_j(\zeta)\hat{f}(\zeta)$, where $m_j(\zeta)=\int_{|y|\cong 2^j}\Omega(y')|y|^{\alpha-n}e^{i|y|}e^{-i\zeta\cdot y}dy$. Let

$$F(s)=(1-s^2)^{(n-3)/2}\chi_{(-1,1)}(s)\int_{S^{n-2}}\Omega(s,(1-s^2)^{1/2}\tilde{y})\,d\sigma(\tilde{y}),$$

and note that $\int_{\mathbb{R}}|F(s)|\,ds\leqslant ||\Omega||_{L^1(S^{n-1})}$. We then have $m_j(\zeta)=\int_{\mathbb{R}}F(s)N_j(s|\zeta|)\,ds$, where $N_j(u)=2^{j\alpha}\int_1^2 t^{\alpha-1}e^{i2^jt(1-u)}dt$. If $|\zeta|\leqslant 1/2$, then $|u|=|s\zeta|\leqslant 1/2$ since $|s|\leqslant 1$. By van der Corput's lemma, $|N_j(u)|\leqslant C2^{j(\alpha-1)}$. Therefore,

$$|m_j(\zeta)|\leqslant C2^{j(\alpha-1)}\int_{\mathbb{R}}|F(s)|\,ds\leqslant C2^{j(\alpha-1)}||\Omega||_{L^1(S^{n-1})}\leqslant C2^{j(\alpha-1)},|\zeta|\leqslant 1/2. \tag{33}$$

If $|\zeta|>1/2$, we write

$$m_j(\zeta)=\int_{2^j}^{2^{j+1}}t^{\alpha-1}e^{it}\left(\int_{\mathbb{R}}F(s)e^{-it|\zeta|s}ds\right)dt=|\zeta|^{-\alpha}\int_{2^j|\zeta|}^{2^{j+1}|\zeta|}t^{\alpha-1}e^{i|\zeta|^{-1}t}\hat{F}(t)\,dt.$$

Thus by Hölder's inequality and by Hausdorff-Young's inequality, we obtain

$$\begin{aligned}|m_j(\zeta)|&\leqslant C|\zeta|^{-\alpha}(2^j|\zeta|)^{\alpha-1/r'}||\hat{F}||_{r'}\leqslant C|\zeta|^{-1/r'}2^{j(\alpha-1/r')}||\Omega||_{L^r(S^{n-1})}\\&\leqslant C2^{j(\alpha-1/r')},\quad |\zeta|>1/2.\end{aligned} \tag{34}$$

Combining inequalities (33)and (34) yields

$$|m_j(\zeta)| \leqslant C2^{j(\alpha-1/r')} \quad \text{for all } \zeta \in \mathbb{R}^n. \tag{35}$$

By Plancherel's Theorem, we have

$$||I_j f||_2 \leqslant C2^{j(\alpha-1/r')}||f||_2. \tag{36}$$

It is obvious that

$$||I_j f||_1 \leqslant C2^{j\alpha}||\Omega||_{L^1(S^{n-1})}||f||_1 \tag{37}$$

$$||I_j f||_\infty \leqslant C2^{j\alpha}||\Omega||_{L^1(S^{n-1})}||f||_\infty. \tag{38}$$

Interpolating (36)-(37) and (36)-(38) yields for $2/(2-\alpha r') < p < 2/(\alpha r')$,

$$||I_j f||_p \leqslant C2^{j(\alpha-\frac{2}{r'\tilde{p}})}||f||_p, \text{ where } 0<\alpha<2/(r'\tilde{p}) \text{ and } \tilde{p} = \max\{p,p'\}. \tag{39}$$

$$\text{Therefore } ||J_2 f||_p \leqslant \sum_{j=0}^{\infty} ||I_j f||_p \leqslant C||f||_p \text{ for } 2/(2-\alpha r') < p < 2/(\alpha r'). \tag{40}$$

Combining (32) and (40) yields

$$||I_{\alpha,\beta} f||_p \leqslant C||f||_p \text{ for } 2/(2-\alpha r') < p < 2/(\alpha r'). \tag{41}$$

Now suppose $0 < \alpha < 2/(r'^2)$ and $\omega \in A_{p/r'}$, $p > r'$. Note that

$$|I_j f(x)| \leqslant C2^{j\alpha}||\Omega||_{L^r(S^{n-1})}\left(M(f^{r'})(x)\right)^{1/r'},$$

where $Mf(x)$ is the Hardy-Littlewood maximal function. Thus we have for $p > r'$,

$$||I_j f||_{L^p(\omega dx)} \leqslant C2^{j\alpha}||M(f^{r'})||^{1/r'}_{L^{p/r'}(\omega dx)} \leqslant C2^{j\alpha}||f^{r'}||^{1/r'}_{L^{p/r'}(\omega dx)} = C2^{j\alpha}||f||_{L^p(\omega dx)}. \tag{42}$$

For $r' < p < 2/(\alpha r')$, we interpolate between (39) and (42) with the same p but change of measures (see [19]) to obtain $||I_j f||_{L^p(\omega^t dx)} \leqslant C2^{-\delta j}||f||_{L^p(\omega^t dx)}$, where $\delta = \frac{2(1-t)}{r'\tilde{p}} - \alpha = \frac{2(1-t)}{r'p} - \alpha > 0$, provided that $0 < t < 1 - \alpha r'p/2$. Thus

$$\begin{aligned} ||J_2 f||_{L^p(\omega^t dx)} \leqslant \sum_{j=0}^{\infty} ||I_j f||_{L^p(\omega^t dx)} \leqslant C||f||_{L^p(\omega^t dx)}, \\ 0 < t < 1-\alpha r'p/2, \quad r' < p < 2/(\alpha r'). \end{aligned} \tag{43}$$

Now write $J_1 f(x) = \sum_{j=-\infty}^{0} I_j f(x)$, where $I_j f(x) = \int_{|y|\cong 2^j} |y|^{\alpha-n}\Omega(y')e^{i|y|}f(x-y)\,dy$. Recall from inequality (42) that $||I_j f||_{L^p(\omega dx)} \leqslant C2^{j\alpha}||f||_{L^p(\omega dx)}$ for $p > r'$. Therefore $||J_1 f||_{L^p(\omega dx)} \leqslant \sum_{-\infty}^{0} ||I_j f||_{L^p(\omega dx)} \leqslant C||f||_{L^p(\omega dx)}$. In particular,

$$||J_1 f||_{L^p(\omega^t dx)} \leqslant C||f||_{L^p(\omega^t dx)} \text{ for } r' < p < 2/(\alpha r') \text{ and } 0 < t < 1-\alpha r'p/2. \tag{44}$$

Combining the inequalities (43) and (44) yields

$$||I_{\alpha,\beta} f||_{L^p(\omega^t dx)} \leqslant C||f||_{L^p(\omega^t dx)} \tag{45}$$

for $r' < p < 2/(\alpha r')$, $0 < t < 1-\alpha r'p/2$ and $\omega \in A_{p/r'}$. The proof of Theorem 5 is complete. $\square$

COROLLARY 5.1. *Let* $\Omega \in L^r(S^{n-1})$, $1 < r \leqslant 2$, $0 < \alpha < 2/r'^2$, *and* $\beta = 1$. *If* $\omega^{1/(1-p)} \in A_{p'/r'}$, $2/(2-\alpha r') < p < r$, *then* $||I_{\alpha,\beta} f||_{L^p(\omega^t dx)} \leqslant C||f||_{L^p(\omega^t dx)}$ *for* $0 < t < 1 - \alpha r' p'/2$.

Proof of Corollary 5.1. Denote $I^*_{\alpha,\beta}$ to be the dual of $I_{\alpha,\beta}$. For $f \in L^p(\omega^t dx)$, $g \in L^{p'}(\omega^{t/(1-p)} dx)$ with $||g||_{L^{p'}(\omega^{t/(1-p)}dx)} \leqslant 1$, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^n} I_{\alpha,\beta} f(x) g(x)\,dx\right| &= \left|\int_{\mathbb{R}^n} f(x) I^*_{\alpha,\beta} g(x)\,dx\right| \\ &\leqslant \left(\int_{\mathbb{R}^n} |f(x)|^p \omega^t(x) dx\right)^{1/p} \left(\int_{\mathbb{R}^n} |I^*_{\alpha,\beta} g(x)|^{p'} \omega^{-tp'/p}(x) dx\right)^{1/p'} \\ &\leqslant C||f||_{L^p(\omega^t dx)} ||g||_{L^{p'}(\omega^{t/(1-p)}dx)} \leqslant C||f||_{L^p(\omega^t dx)},\end{aligned}$$

where the first inequality follows from Hölder's inequality and the second inequality follows from Theorem 5. Now let g run over the unit ball of $L^{p'}(\omega^{t/(1-p)} dx)$, we obtain the result. Corollary 5.1 is proved □

COROLLARY 5.2. *Let* $\Omega \in L^r(S^{n-1})$, $1 < r \leqslant 2$, $0 < \alpha < 2/r'^2$, $\beta = 1$, $2/(2-\alpha r') < p_0 < r$, $r' < p_1 < 2/(\alpha r')$, *and* $\omega \in A_{p_1/r'}$. *Then*

$$||I_{\alpha,\beta} f||_{L^{p_s}(\omega^{r(s)}dx)} \leqslant C||f||_{L^{p_s}(\omega^{r(s)}dx)},$$

where $p_s = \dfrac{p_0 p_1}{(1-s)p_1 + p_0 s}$, $r(s) = \dfrac{t p_0 s}{(1-s)p_1 + p_0 s}$, $0 < s < 1$, *and* $t \in (0, 1 - \alpha r' p_1/2)$.

Proof of Corollary 5.2. By interpolating between (41) and (45) with change of measures (see [1, p.119]), we obtain the above result. □

COROLLARY 5.3. *Let* $\Omega \in L^r(S^{n-1})$, $r > n/(n-\alpha)$, $0 < \alpha < 1/r'$, $\beta = 1$, $r' < p_1 < n/\alpha$, $1/q_1 = 1/p_1 - \alpha/n$. *Suppose* $\omega(x) \geqslant 0$ *and* $v(x) = \omega^{r'}(x)$ *satisfies inequality (1.1) in [15], i.e.,*

$$\left(\frac{1}{|Q|}\int_Q v^q(x)dx\right)^{1/q} \left(\frac{1}{|Q|}\int_Q v^{-p'}(x)dx\right)^{1/p'} \leqslant C, \tag{46}$$

where C *is independent of the n-dimensional cube* Q, $1 < p < n/\gamma$, $0 < \gamma < n$, *and* $1/q = 1/p - \gamma/n$. *Then for* $2 < p_s < p_1$, *we have* $||I_{\alpha,\beta} f||_{L^{q_s}(\omega^{q_s s}dx)} \leqslant C||f||_{L^{p_s}(\omega^{p_s s}dx)}$, *where* $0 < s < 1$, $p_s = \dfrac{2p_1}{(1-s)p_1 + 2s}$, *and* $q_s = \dfrac{2q_1}{(1-s)q_1 + 2s}$.

Proof of Corollary 5.3. Write $I_{\alpha,\beta} f(x) = \sum_j I_j f(x)$, and by taking the Fourier transform, we obtain $\widehat{I_j f}(\zeta) = m_j(\zeta)\hat{f}(\zeta)$, with $m_j(\zeta) = \displaystyle\int_{|y|\cong 2^j} |y|^{\alpha-n}\Omega(y')e^{i|y|}e^{-i\zeta\cdot y}dy$. If $j \leqslant 0$, then it is clear that $|m_j(\zeta)| \leqslant C2^{j\alpha}$. If $j > 0$, then recall from inequality (35) that $|m_j(\zeta)| \leqslant C2^{j(\alpha-1/r')}$. Thus $|m_j(\zeta)| \leqslant C2^{-\delta|j|}$, where $\delta = \min\{\alpha, 1/r'-\alpha\} > 0$. It follows from Plancherel's Theorem that

$$||I_jf||_2 \leqslant C2^{-\delta|j|}||f||_2 \tag{47}$$

Observe that

$$\begin{aligned} |I_jf(x)| &\leqslant 2^{j\alpha}\int_{|y|\cong 2^j}|y|^{-n}|\Omega(y')f(x-y)|\,dy \\ &\leqslant C2^{j\alpha}||\Omega||^{1/r}_{L^r(S^{n-1})}\left(\int_{|y|\cong 2^j}|y|^{-n}|f(x-y)|^{r'}dy\right)^{1/r'}. \end{aligned}$$

Thus

$$\begin{aligned} |I_jf(x)|^{r'} &\leqslant C\frac{1}{2^{jn(1-\alpha r'/n)}}\int_{|y|\cong 2^j}|f(x-y)|^{r'}dy \\ &\leqslant C(f^{r'})^*_\gamma \ (\gamma = \alpha r'), \end{aligned} \tag{48}$$

where $f^*_\gamma = \sup_{j\in\mathbb{Z}}\left\{\frac{1}{2^{jn(1-\alpha r'/n)}}\int_{|y|\cong 2^j}|f(x-y)|dy\right\}$. Let $\tilde{p} = p_1/r'$, $\tilde{q} = q_1/r'$. Then $1 < \tilde{p} < n/\gamma$ and $1/\tilde{q} = 1/\tilde{p} - \gamma/n$, where $\gamma = \alpha r'$. For $v(x) = \omega^{r'}(x)$ which satisfies inequality (46), we have

$$\begin{aligned} \left(\int_{\mathbb{R}^n}|I_jf(x)|^{q_1}\omega^{q_1}(x)dx\right)^{r'/q_1} &= \left\|(I_jf(x))^{r'}\right\|_{L^{\tilde{q}}(v^{\tilde{q}}dx)} \leqslant C\left\|(f^{r'})^*_\gamma\right\|_{L^{\tilde{q}}(v^{\tilde{q}}dx)} \\ &\leqslant C||f^{r'}||_{L^{\tilde{p}}(v^{\tilde{p}}dx)} = C\left(\int_{\mathbb{R}^n}|f(x)|^{p_1}\omega^{p_1}(x)dx\right)^{r'/p_1}, \end{aligned}$$

where the first inequality follows from (48) and the second inequality follows from [15, Theorem 3]. It follows that

$$||I_jf||_{L^{q_1}(w^{q_1}dx)} \leqslant C||f|||_{L^{p_1}(w^{p_1}dx)}. \tag{49}$$

Interpolating between (47) and (49) with change of measures (see [1, p.119]), we get $||I_jf(x)||_{L^{q_s}(\omega^{q_s s}dx)} \leqslant C2^{-\delta(1-s)|j|}||f||_{L^{p_s}(\omega^{p_s s}dx)}$, where $0 < s < 1$, $p_s = \frac{2p_1}{(1-s)p_1+2s}$, $q_s = \frac{2q_1}{(1-s)q_1+2s}$, $1/q_1 = 1/p_1 - \alpha/n$, and $r' < p_1 < n/\alpha$. Consequently,

$$||I_{\alpha,\beta}f||_{L^{q_s}(\omega^{q_s s}dx)} \leqslant \sum_j ||I_jf(x)||_{L^{q_s}(\omega^{q_s s}dx)} \leqslant C||f||_{L^{p_s}(\omega^{p_s s}dx)}.$$

Corollary 5.3 is proved. □

By interpolating between (41) and the results of [9, Theorem 1] and [9, Theorem 2] respectively, we obtain the following corollaries.

COROLLARY 5.4. *Let* $\Omega \in L^r(S^{n-1})$, $r > n/(n-\alpha)$, $0 < \alpha < 1/r'$, $\beta = 1$, $2/(2-\alpha r') < p_0 < 2/(\alpha r')$, $r' < p_1 < n/\alpha$, $1/q_1 = 1/p_1 - \alpha/n$, *and* $w^{r'} \in A_{(p_1/r', q_1/r')}$. *Then*

$$||I_{\alpha,\beta} f||_{L^{q_s}(\omega^{q_s s}dx)} \leqslant C||f||_{L^{p_s}(\omega^{p_s s}dx)},$$

where $0 < s < 1$, $p_s = \dfrac{p_0 p_1}{(1-s)p_1 + p_0 s}$, *and* $q_s = \dfrac{p_0 q_1}{(1-s)q_1 + p_0 s}$.

COROLLARY 5.5. *Let* $\Omega \in L^r(S^{n-1})$, $r > 1$, $0 < \alpha < 1/r'$, $\beta = 1$, $2/(2-\alpha r') < p_0 < 2/(\alpha r')$, $1 < p_1 < nr/(n+\alpha r)$, $1/q_1 = 1/p_1 - \alpha/n$, *and* $w^{-r'} \in A_{(q_1'/r', p_1'/r')}$. *Then*

$$||I_{\alpha,\beta} f||_{L^{q_s}(\omega^{q_s s}dx)} \leqslant C||f||_{L^{p_s}(\omega^{p_s s}dx)},$$

where $0 < s < 1$, $p_s = \dfrac{p_0 p_1}{(1-s)p_1 + p_0 s}$, *and* $q_s = \dfrac{p_0 q_1}{(1-s)q_1 + p_0 s}$.

We now consider the case $\beta > 0$, $\beta \neq 1$.

THEOREM 6. *Suppose* $\Omega \in L^1(S^{n-1})$. *If* $\beta > 2\alpha > 0$, $\beta \neq 1$, *then*

$$||I_{\alpha,\beta} f||_p \leqslant C||f||_p \text{ for } \beta/(\beta-\alpha) < p < \beta/\alpha.$$

Moreover, if $\Omega \in L^r(S^{n-1})$, $\beta/(\beta-\alpha) < r \leqslant 2$, *and* $\omega \in A_{p/r'}$, $p > r'$, *then*

$$||I_{\alpha,\beta} f||_{L^p(\omega^t dx)} \leqslant C||f||_{L^p(\omega^t dx)} \text{ for } r' < p < \beta/\alpha, \text{ and } 0 < t < 1 - \alpha p/\beta.$$

Proof of Theorem 6. The proof of this theorem is similar to the proof of Theorem 5. We write $I_{\alpha,\beta} f(x) = \int_{|y|\leqslant 1} ...dy + \int_{|y|>1} ...dy \equiv J_1 f(x) + J_2 f(x)$. Then $||J_1 f||_p \leqslant C||\Omega||_{L^1(S^{n-1})}||f||_p$ for $1 < p < \infty$. Now write $J_2 f(x) = \sum_{j=0}^{\infty} I_j f(x)$, with $\widehat{I_j f}(\zeta) = m_j(\zeta)\hat{f}(\zeta)$. Note that

$$m_j(\zeta) = \int_{|y|\cong 2^j} \Omega(y')|y|^{\alpha-n} e^{i|y|^\beta} e^{-i\zeta\cdot y} dy \equiv \int_{\mathbb{R}} F(s) N_j(s|\zeta|)\, ds,$$

with $N_j(u) = 2^{j\alpha}\int_1^2 t^{\alpha-1} e^{i2^j(t^\beta - tu)} dt$. By van de Corput's lemma, we obtain $|N_j(u)| \leqslant C2^{-j(\beta/2-\alpha)}$, and thus $|m_j(\zeta)| \leqslant C||\Omega||_{L^1(S^{n-1})} 2^{-j(\beta/2-\alpha)}$. By Plancherel's Theorem, we have

$$||I_j f||_2 \leqslant C2^{-j(\beta/2-\alpha)}||f||_2. \tag{50}$$

Also it is clear that

$$||I_j f||_1 \leqslant C2^{j\alpha}||f||_1, \text{ and} \tag{51}$$

$$||I_j f||_\infty \leqslant C2^{j\alpha}||f||_\infty. \tag{52}$$

Interpolating between (50)-(51) and (50)-(52), we obtain

$$||I_j f||_p \leqslant C2^{j(\alpha-\beta/\tilde{p})}||f||_p, \tag{53}$$

where $\tilde{p} = \max\{p, p'\}$, and $\alpha < \beta/\tilde{p}$, if $\beta/(\beta-\alpha) < p < \beta/\alpha$.

Thus $||J_2 f||_p \leqslant \sum_{j=0}^{\infty} ||I_j f||_p \leqslant C||f||_p$ for $\beta/(\beta-\alpha) < p < \beta/\alpha$. Therefore,

$$||I_{\alpha,\beta} f||_p \leqslant ||J_1 f||_p + ||J_2 f||_p \leqslant C||f||_p, \quad \beta/(\beta-\alpha) < p < \beta/\alpha.$$

Now suppose $\Omega \in L^r(S^{n-1})$, $\beta/(\beta-\alpha) < r \leqslant 2$, $\beta > 2\alpha > 0$, and $\beta \neq 1$. From the proof of Theorem 5, we have for $\omega \in A_{p/r'}$, $(p > r')$,

$$||I_j f||_{L^p(\omega dx)} \leqslant C 2^{j\alpha} ||f||_{L^p(\omega dx)}, \quad j \in \mathbb{Z}. \tag{54}$$

$$\text{Thus } ||J_1 f||_{L^p(\omega dx)} \leqslant \sum_{j=-\infty}^{0} ||I_j f||_{L^p(\omega dx)} \leqslant C||f||_{L^p(\omega dx)} \quad \text{for } p > r'. \tag{55}$$

By interpolating between (53) and (54) with the same p but with change of measures, we obtain for $r' < p < \beta/\alpha$,

$$||I_j f||_{L^p(\omega^t dx)} \leqslant C 2^{-j\delta} ||f||_{L^p(\omega^t dx)}, \tag{56}$$

where $\delta = (1-t)\beta/p - \alpha > 0$ if $0 < t < 1 - \alpha p/\beta$. Hence, $||J_2 f||_{L^p(\omega^t dx)} \leqslant \sum_{j=0}^{\infty} ||I_j f||_{L^p(\omega^t dx)} \leqslant C||f||_{L^p(\omega^t dx)}$ for $r' < p < \beta/\alpha$, $0 < t < 1 - \alpha p/\beta$.
Recall from (55) that $||J_1 f||_{L^p(\omega dx)} \leqslant C||f||_{L^p(\omega dx)}$ for $p > r'$.
In particular, $||J_1 f||_{L^p(\omega^t dx)} \leqslant C||f||_{L^p(\omega^t dx)}$ for $r' < p < \beta/\alpha$, $0 < t < 1 - \alpha p/\beta$.

Consequently, $||I_{\alpha,\beta} f||_{L^p(\omega^t dx)} \leqslant C||f||_{L^p(\omega^t dx)}$ for $r' < p < \beta/\alpha$, $0 < t < 1 - \alpha p/\beta$. The proof of Theorem 6 is complete. □

COROLLARY 6.1. *Let $\Omega \in L^r(S^{n-1})$, $\beta/(\beta-\alpha) < r \leqslant 2$, $\beta > 2\alpha > 0$, $\beta \neq 1$. If $\omega^{1/(1-p)} \in A_{p'/r'}$, $\beta/(\beta-\alpha) < p < r$, then*

$$||I_{\alpha,\beta} f||_{L^p(\omega^t dx)} \leqslant C||f||_{L^p(\omega^t dx)} \ \textit{for} \ 0 < t < 1 - \alpha p'/\beta.$$

COROLLARY 6.2. *Let $\Omega \in L^r(S^{n-1})$, $\beta/(\beta-\alpha) < r \leqslant 2$, $\beta > 2\alpha > 0$, $\beta \neq 1$, $\beta/(\beta-\alpha) < p_0 < r$, $r' < p_1 < \beta/\alpha$, and $\omega \in A_{p_1/r'}$. Then*

$$||I_{\alpha,\beta} f||_{L^{p_s}(\omega^{r(s)} dx)} \leqslant C||f||_{L^{p_s}(\omega^{r(s)} dx)},$$

where $p_s = \dfrac{p_0 p_1}{(1-s)p_1 + p_0 s}$, $r(s) = \dfrac{t p_0 s}{(1-s)p_1 + p_0 s}$, $0 < s < 1$, and $t \in (0, 1 - \alpha p_1/\beta)$.

COROLLARY 6.3. *Let $\Omega \in L^r(S^{n-1})$, $r > n/(n-\alpha)$, $0 < \alpha < n$, $\beta > 2\alpha > 0$, $\beta \neq 1$, $r' < p_1 < n/\alpha$, and $1/q_1 = 1/p_1 - \alpha/n$. Suppose $\omega(x) \geqslant 0$ and $v(x) = \omega^{r'}(x)$ satisfies inequality (1.1) in [15], i.e.,*

$$\left(\frac{1}{|Q|}\int_Q v^q(x)dx\right)^{1/q} \left(\frac{1}{|Q|}\int_Q v^{-p'}(x)dx\right)^{1/p'} \leqslant C, \tag{57}$$

where C is independent of the n-dimensional cube Q, $1 < p < n/\gamma$, $0 < \gamma < n$, and $1/q = 1/p - \gamma/n$. Then for $2 < p_s < p_1$, we have

$$||I_{\alpha,\beta} f||_{L^{q_s}(\omega^{q_s s} dx)} \leqslant C||f||_{L^{p_s}(\omega^{p_s s} dx)},$$

where $0 < s < 1$, $p_s = \dfrac{2p_1}{(1-s)p_1 + 2s}$, and $q_s = \dfrac{2q_1}{(1-s)q_1 + 2s}$.

COROLLARY 6.4. *Let* $\Omega \in L^r(S^{n-1})$, $r > n/(n-\alpha)$, $0 < \alpha < n$, $\beta > 2\alpha > 0$, $\beta \neq 1$, $2/(2-\alpha r') < p_0 < 2/(\alpha r')$, $r' < p_1 < n/\alpha$, $1/q_1 = 1/p_1 - \alpha/n$, *and* $w^{r'} \in A_{(p_1/r', q_1/r')}$. *Then*

$$||I_{\alpha,\beta} f||_{L^{q_s}(\omega^{q_s s}dx)} \leqslant C||f||_{L^{p_s}(\omega^{p_s s}dx)},$$

where $0 < s < 1$, $p_s = \dfrac{p_0 p_1}{(1-s)p_1 + p_0 s}$, *and* $q_s = \dfrac{p_0 q_1}{(1-s)q_1 + p_0 s}$.

COROLLARY 6.5. *Let* $\Omega \in L^r(S^{n-1})$, $r > 1$, $0 < \alpha < n$, $\beta > 2\alpha > 0$, $\beta \neq 1$, $2/(2-\alpha r') < p_0 < 2/(\alpha r')$, $1 < p_1 < nr/(n+\alpha r)$, $1/q_1 = 1/p_1 - \alpha/n$, *and* $w^{-r'} \in A_{(q_1'/r', p_1'/r')}$. *Then*

$$||I_{\alpha,\beta} f||_{L^{q_s}(\omega^{q_s s}dx)} \leqslant C||f||_{L^{p_s}(\omega^{p_s s}dx)},$$

where $0 < s < 1$, $p_s = \dfrac{p_0 p_1}{(1-s)p_1 + p_0 s}$, *and* $q_s = \dfrac{p_0 q_1}{(1-s)q_1 + p_0 s}$.

Since the proof of Corollaries (6.1)-(6.5) is similar to the proof of Corollaries (5.1)-(5.5) respectively, we omit the proof of these corollaries.

REMARK 3. Consider the hypersingular integral

$$T_{\alpha,\beta} f(x) = \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \Omega(y')|y|^{n+\alpha} e^{i|y|^{-\beta}} f(x-y)\, dy.$$

If $\beta > 2\alpha > 0$, $\beta \neq 1$, then all the results in Theorem 6 and Corollaries 6.1-6.2 also apply to this integral $T_{\alpha,\beta} f(x)$.

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