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REMARKS ON EMBEDDING RESULTS OF SINE SERIES

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(*communicated by L. Leindler*)

Abstract. We consider some embedding relations among many important functional classes, such as $S_p(\lambda)$, $H^{\omega}_{S,\beta}$, H^{ω}_{β} etc..

1. Introduction

Recently, many new kinds of sequences were introduced for extending a lot of classical results in Fourier analysis. Among them, Leindler (see for example, [2]) defined the *class of sequences of rest bounded variation*, and denoted by $RBVS$. The definition of $RBVS$ can be read as follows: A nonnegative sequence $\mathbf{C} := \{c_n\}$ is of rest bounded variation, or $\mathbf{C} \in RBVS$, if $c_n \to 0$ and for any $m \in N$ it holds that

$$\sum_{n=m}^{\infty} |\Delta c_n| \leqslant K(\mathbf{C})c_m,$$

where $\Delta c_n = c_n - c_{n+1}$, and $K(\mathbf{C})$ denotes a constant only depending on $\mathbf{C}$.

It has been proved that the class[1] $CQMS$ and $RBVS$ are not comparable (see [2]). Very recently, Le and Zhou [1] suggested the following new class of sequences to include both $RBVS$ and $CQMS$: Let $\mathbf{C} := \{c_n\}$ be a nonnegative sequence tending to zero, if

$$\sum_{n=m}^{2m} |\Delta c_n| \leqslant K(\mathbf{C})c_m$$

holds for all $m = 1, 2, \cdots$, then we say $\mathbf{C} \in GBVS$.

As a further generalization of $GBVS$, we [7] defined the following class of sequences named as $NBVS$, which can be stated as follows: Let $\mathbf{C} := \{c_n\}$ be a nonnegative sequence tending to zero, if

$$\sum_{n=m}^{2m} |\Delta c_n| \leqslant K(\mathbf{C})\left(c_m + c_{2m}\right)$$

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[1] $\mathbf{C} = \{c_n\} \in CQMS$ means that there is an $\alpha \geqslant 0$ such that c_n/n^{α} is decreasing.

Paper MIA-11-10

holds for all $m = 1, 2, \cdots$, then we say $\mathbf{C} \in NBVS$. By the definition of $NBVS$, we see that for any $\mathbf{C} := \{c_n\} \in NBVS$, it holds that

$$c_n \leqslant \sum_{k=n}^{2n} |\Delta c_k| + c_{2n} \leqslant K(\mathbf{C})\,(c_n + c_{2n})\,.$$

Generally speaking, the term c_{2n} can not be canceled, so $NBVS$ essentially extending monotonicity from “one sided" to “two sided" in some sense.

Very recently, Leindler [4] further extended the definition of $RBVS$, by introducing the so-called $\gamma RBVS$, that is,

DEFINITION 1. Let $\gamma := \{\gamma_n\}$ be a positive sequence. If a null-sequence $\mathbf{C} := \{c_n\}$ of real numbers has the property

$$\sum_{k=m}^{\infty} |\Delta c_k| \leqslant K(\mathbf{C})\gamma_m$$

for all $m \in N$, then we call the sequence $\mathbf{C}$ a $\gamma RBVS$, briefly denoted by $\mathbf{C} \in \gamma RBVS.$

If $\gamma \equiv \mathbf{C}$, then $\mathbf{C}RBVS \equiv RBVS.$ Furthermore, a sequence of $\gamma RBVS$ may have infinitely many zeros and negative terms.

A nondecreasing continuous function $\omega(\delta)$ defined on the interval $[0, 2\pi]$ is called a *a modulus of continuity*, if it satisfies the properties:

$$\omega(0) = 0,\ \omega(\delta_1 + \delta_2) \leqslant \omega(\delta_1) + \omega(\delta_2) \text{ for any } 0 \leqslant \delta_1 \leqslant \delta_2 \leqslant \delta_1 + \delta_2 \leqslant 2\pi.$$

Let $f(x)$ be a continuous and $2\pi-$ periodic function, and let

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx \tag{1}$$

be its Fourier series. Denote by $s_n = s_n(f, x)$ the nth partial sum of (1), by $\|\cdot\|$ the usual supremum norm, and by $E_n = E_n(f)$ the best approximation of f by trigonometric polynomials of order at most n. Define the modulus of smoothness of order $\beta(> 0)$ of the f by

$$\omega_\beta(f, t) = \sup_{|h| \leqslant t} \left\| \sum_{k=0}^{\infty} (-1)^k \begin{pmatrix} \beta \\ k \end{pmatrix} f(x + (\beta - k)h) \right\|,$$

where

$$\begin{pmatrix} \beta \\ k \end{pmatrix} = \begin{cases} \frac{\beta(\beta-1)\cdots(\beta-k+1)}{k!}, & k \geqslant 1, \\ 1, & k = 0. \end{cases}$$

Set

$$H_\beta^\omega := \{f : \omega_\beta(f, \delta) = O(\omega(\delta))\},$$

$$S_p(\lambda) := \left\{ f : \left\| \sum_{n=1}^{\infty} \lambda_n |s_n - f|^p \right\| < \infty \right\},$$

where $\lambda := \{\lambda_n\}$ is monotone (nondecreasing or increasing) sequence of positive numbers and $0 < p < \infty$,

$$H^{\omega}_{S,\beta} := \left\{ f : f = \sum_{n=1}^{\infty} b_n \sin nx,\ \{b_n\} \in RBVS,\ \text{and}\ \omega_\beta\left(f, \frac{1}{n}\right) = O\left(\omega\left(\frac{1}{n}\right)\right) \right\},$$

$$H_{S,\Omega} := \left\{ f : f = \sum_{n=1}^{\infty} b_n \sin nx \text{ and } \{b_n\} \in \Omega RBVS, \Omega := \{n^{-1}\omega(n^{-1})\} \right\},$$

$$H^{\omega,*}_{S,\beta} := \left\{ f : f = \sum_{n=1}^{\infty} b_n \sin nx,\ \{b_n\} \in NBVS,\ \text{and}\ \omega_\beta\left(f, \frac{1}{n}\right) = O\left(\omega\left(\frac{1}{n}\right)\right) \right\},$$

A sequence $\eta := \{\eta_n\}$ of positive numbers is *quasi* $\beta-$*power-monotone increasing* (*decreasing*) if there exists a constant $K := K(\beta, \eta) \geqslant 1$ such that

$$Kn^\beta \eta_n \geqslant m^\beta \eta_m \quad (n^\beta \eta_n \leqslant K m^\beta \eta_m)$$

holds for any $n \geqslant m$. If $K = 1$, then we neglect the attribute “quasi".

2. Main Results

The main results of the present paper are the follows:

THEOREM 1. *It holds the following embedding results*

$$H^{\omega}_{S,\beta} \subset H^{\omega,*}_{S,\beta} \subset H_{S,\Omega} \cap H^{\omega}_{\beta}. \tag{2}$$

THEOREM 2. *Let $p \geqslant 1$ and ω be a modulus of continuity. If $\lambda := \{\lambda_n\}$ is a positive monotone sequence such that λ is quasi $\beta-$power-monotone increasing with some $\beta < 1$, then the condition*

$$\omega\left(\frac{1}{n}\right) = O\left((n\lambda_n)^{-1/p}\right)$$

implies that

$$H^{\omega}_{S,\beta} \subset H^{\omega,*}_{S,\beta} \subset H_{S,\Omega} \subset S_p(\lambda). \tag{3}$$

If there exists a positive nondecreasing sequence $\rho := \{\rho_n\}$ tending to infinity such that the sequences $\{\lambda_n \rho_n^{-p}\}$ is simultaneously $\gamma-$power-monotone increasing with some $\gamma < 1$ and quasi $\alpha-$power-monotone decreasing with some $\alpha > 1-\min(1,\beta)p$, and

$$\omega\left(\frac{1}{n}\right) \geqslant K \frac{\rho_n}{(n\lambda_n)^{1/p}}, \tag{4}$$

then (3) does not hold; namely, there exists a function f_0 such that

$$f_0 \in H^{\omega}_{S,\beta} \text{ but } f_0 \notin S_p(\lambda).$$

THEOREM 3. *Let $0 < p \leqslant 1$ and ω be a modulus of continuity. If $\lambda := \{\lambda_n\}$ is a positive monotone sequence such that λ is a quasi $\beta-$power-monotone increasing with some $\beta < 1-p$, then the condition*

$$\omega\left(\frac{1}{n}\right) = O\left((n(n^2\lambda_{n^2})^{-1/p}\right)$$

implies that

$$H^{\omega}_{S,\beta} \subset H^{\omega,*}_{S,\beta} \subset H_{S,\Omega} \subset S_p(\lambda). \tag{5}$$

If there exists a positive nondecreasing sequence $\rho := \{\rho_n\}$ tending to infinity such that the sequences $\{\lambda_{n^2}\rho_n^{-p}\}$ is simultaneously $\gamma-$power-monotone increasing with some $\gamma < 2-p$ and quasi $\alpha-$power-monotone decreasing with some $\alpha > 2(1-\min(1,\beta)p)$, and

$$\omega\left(\frac{1}{n}\right) \geqslant K\rho_n n(n^2\lambda_{n^2})^{-1/p},$$

then (3) does not hold; namely, there exists a function f_0 such that

$$f_0 \in H^{\omega}_{S,\beta} \text{ but } f_0 \notin S_p(\lambda).$$

Theorem 2 and Theorem 3 are more complete than the relevant results of Leindler [3] and [4].

Since $\omega\left(f,\frac{1}{n}\right) = O\left(\omega\left(\frac{1}{n}\right)\right)$ is not anymore an assumption in $H_{S,\Omega}$, then $H_{S,\Omega}$ is not necessary a subclass of H^{ω}_{β}. However, if we add some more conditions on $\omega\left(\frac{1}{n}\right)$, then $H_{S,\Omega} \subset H^{\omega}$ can be hold.

THEOREM 4. *If the sequence $\{\omega(n^{-1})\}$ is quasi $(1-\varepsilon)-$power-monotone increasing with some $1-\min(1,\beta) < \varepsilon < 1$, then*

$$H_{S,\Omega} \subset H^{\omega}_{\beta}.$$

holds.

Theorem 4 improves the proposition of [4], where $\{\omega(n^{-1})\}$ should be simultaneously quasi $\varepsilon-$power-monotone decreasing with some $0 < \varepsilon < 1$ and quasi $(1-\varepsilon)-$power-monotone increasing, and only established for the case $\beta = 1$.

3. Proofs

We need the following lemmas.

LEMMA 1. ([5]) *Let $\beta > 0$, and $f(x)$ be a continuous function, Then*

$$E_n(f) \leqslant K\omega_\beta\left(f,\frac{1}{n}\right) \leqslant Kn^{-\beta}\sum_{k=1}^{n} k^{\beta-1}E_k(f). \tag{6}$$

$$\omega_{\alpha+\beta}(f,\delta) \leqslant K\omega_\beta(f,\delta), \text{ for } \alpha \geqslant 0. \tag{7}$$

LEMMA 2. ([6]) *Let $f(x)$ be a continuous function, and $f(x)$ has a Fourier series of the form*

$$\sum_{n=1}^{\infty} b_n \sin nx,\ b_n \geqslant 0,$$

then

$$n^{-\beta}\sum_{k=1}^{n} k^{\beta} b_k \leqslant K\omega_{\beta}\left(f, \frac{1}{n}\right),\ \beta \neq 2l,\ l = 1, 2, \cdots ..$$

Proof of Theorem 1. By the definitions of $H^{\omega}_{S,\beta}$ and $H^{\omega,*}_{S,\beta}$, it is clear that

$$H^{\omega}_{S,\beta} \subset H^{\omega,*}_{S,\beta} \subset H^{\omega}_{\beta}.$$

So we only need to verify that

$$H^{\omega,*}_{S,\beta} \subset H_{S,\Omega}. \tag{8}$$

Let $f(x) = \sum\limits_{n=1}^{\infty} b_n \sin nx \in H^{\omega,*}_{S,\beta}$. By the definition of $NBVS$, we have

$$b_n \leqslant \sum_{i=k}^{n-1} |\Delta b_i| + b_k \leqslant K(b_k + b_{2k}),\ [n/2] + 1 \leqslant k \leqslant n - 1,$$

hence, by Lemma 2, we deduce that for $\beta \neq 2l,\ l = 1, 2, \cdots$,

$$\begin{aligned}
b_n + b_{2n} &\leqslant Kn^{-1}\left(\sum_{k=[n/2]+1}^{n-1}(b_k + b_{2k}) + \sum_{k=n+1}^{2n-1}(b_k + b_{2k})\right)\\
&\leqslant Kn^{-1}\sum_{k=[n/2]+1}^{2n}(b_k + b_{2k}) \leqslant Kn^{-1}\sum_{k=[n/2]+1}^{4n} b_k\\
&\leqslant Kn^{-1-\beta}\sum_{k=[n/2]+1}^{4n} k^{\beta} b_k \leqslant Kn^{-1}\omega_{\beta}\left(f, \frac{1}{n}\right) \leqslant Kn^{-1}\omega\left(\frac{1}{n}\right).
\end{aligned}$$

If $\beta = 2l,\ l = 1, 2, \cdots$, then by (7), we have

$$\begin{aligned}
b_n + b_{2n} &\leqslant Kn^{-1-\beta-1}\sum_{k=[n/2]+1}^{4n} k^{\beta+1} b_k \leqslant Kn^{-1}\omega_{\beta+1}\left(f, \frac{1}{n}\right)\\
&\leqslant K\omega_{\beta}\left(f, \frac{1}{n}\right) \leqslant Kn^{-1}\omega\left(\frac{1}{n}\right)
\end{aligned}$$

Therefore, by the definition of $NBVS$ again, we see that

$$\sum_{k=n}^{2n} |\Delta b_k| \leqslant K(b_n + b_{2n}) \leqslant Kn^{-1}\omega\left(\frac{1}{n}\right).$$

Thus, for $2^j \leqslant n < 2^{j+1}$, $j = 1, 2, \cdots$,

$$\begin{aligned}\sum_{k=n}^{\infty} |\Delta b_k| &= \sum_{k=n}^{2n-1} |\Delta b_k| + \sum_{k=j}^{\infty} \sum_{s=2^{k+1}}^{2^{k+2}-1} |\Delta b_k| \\ &\leqslant K\left(b_n + b_{2n} + \sum_{k=j}^{\infty} (b_{2^{k+1}} + b_{2^{k+2}})\right) \\ &\leqslant K\left(n^{-1}\omega\left(\frac{1}{n}\right) + \sum_{k=j}^{\infty} 2^{-k-1}\omega\left(2^{-k-1}\right)\right) \\ &\leqslant Kn^{-1}\omega\left(\frac{1}{n}\right),\end{aligned}$$

which implies that $\{b_n\} \in \Omega RBVS$, and thus we finish (8). □

Proof of Theorem 2. First, (3) can be deduced by combining (2) and Theorem 1 of [4].

Set

$$f_0(x) := \sum_{n=1}^{\infty} \frac{1}{n} \frac{\rho_n}{(n\lambda_n)^{1/p}} \sin nx.$$

It was proved by Leindler [3] that, under the condition of Theorem 2, $f_0 \notin S_p(\lambda)$, and

$$E_n(f_0) \leqslant K \frac{\rho_n}{(n\lambda_n)^{1/p}}. \tag{9}$$

We verify that

$$\sum_{k=1}^{n} \frac{k^{\beta-1}\rho_k}{(k\lambda_k)^{1/p}} \leqslant K \frac{n^{\beta}\rho_n}{(n\lambda_n)^{1/p}}, \quad \beta > 0. \tag{10}$$

In fact, it can be deduced directly from (see [3])

$$\sum_{k=1}^{n} \frac{\rho_k}{(k\lambda_k)^{1/p}} \leqslant K \frac{n\rho_n}{(n\lambda_n)^{1/p}}$$

for $\beta \geqslant 1$. If $0 < \beta < 1$, by noting that the sequence $\{\lambda_n \rho_n^{-p}\}$ is quasi $\alpha-$power-monotone decreasing with some $\alpha > 1 - \min(1, \beta)p = 1 - \beta p$, we have

$$\sum_{k=1}^{n} \frac{k^{\beta-1}\rho_k}{(k\lambda_k)^{1/p}} = \sum_{k=1}^{n} \frac{\rho_k}{(k^{\alpha}\lambda_k)^{1/p}} \frac{1}{k^{1/p-\alpha/p+1-\beta}} \leqslant K \frac{\rho_n}{(n^{\alpha}\lambda_n)^{1/p}} \sum_{k=1}^{n} k^{\alpha/p-1/p-1+\beta}$$

$$\leqslant K \frac{\rho_n}{(n^{\alpha}\lambda_n)^{1/p}} n^{\alpha/p-1/p+\beta} \leqslant K \frac{n^{\beta}\rho_n}{(n\lambda_n)^{1/p}}.$$

Thus, by (4), (6), (9) and (10),

$$\omega_\beta\left(f_0,\frac{1}{n}\right) \leqslant Kn^{-\beta}\sum_{k=1}^{n} k^{\beta-1}E_k(f_0) \leqslant Kn^{-\beta}\sum_{k=1}^{n} k^{\beta-1}\frac{\rho_k}{(k\lambda_k)^{1/p}}$$

$$\leqslant K\frac{\rho_n}{(n\lambda_n)^{1/p}} \leqslant K\omega\left(\frac{1}{n}\right),$$

which means that $f_0 \in H^\omega_\beta$.

On the other hand, since the coefficients of f_0 are monotone decreasing (see [3]), then they belong to $RBVS$. Hence, $f_0 \in H^\omega_{S,\beta}$. We have completed Theorem 2. □

Proof of Theorem 3. (5) can be derived by (2) and Theorem 2 of [4].

Set

$$f_0(x) := \sum_{n=1}^{\infty} \rho_n(n^2\lambda_{n^2})^{-1/p}\sin nx.$$

Then, under the condition of Theorem 3(see [3] or [4]), $f_0 \notin S_p(\lambda)$, $\{\rho_n(n^2\lambda_{n^2})^{-1/p}\}$ is decreasing, and

$$E_n(f_0) \leqslant K\frac{n\rho_n}{(n^2\lambda_{N^2})^{1/p}}.$$

Noting that (see [3])

$$\sum_{k=1}^{n}\frac{k\rho_k}{(k^2\lambda_{k^2})^{1/p}} \leqslant K\frac{n^2\rho_n}{(n^2\lambda_{n^2})^{1/p}}$$

and $\{\rho_n(n^2\lambda_{n^2})^{-1/p}\}$ is $\alpha-$ power-monotone decreasing with some $\alpha > 2\,(1-\min(1,\beta)p)$, we can derive that

$$\sum_{k=1}^{n}\frac{k^\beta\rho_k}{(k^2\lambda_{k^2})^{1/p}} \leqslant K\frac{n^{\beta+1}\rho_n}{(n^2\lambda_{n^2})^{1/p}}$$

by a simialr way as that of (10). Thus, by a similar discussion to the proof of Theorem 2, we have $f_0 \in H^\omega_{S,\beta}$. □

Proof of Theorem 4. We consider the error

$$\Delta(x) := |f(x) - S_n(f,x)| = \left|\sum_{k=n+1}^{\infty} b_k\sin kx\right|.$$

By noting that $\Delta(0) = \Delta(\pi) = 0$, we only need to consider the case $x \in (0,\pi)$. Set $N := [1/x]$, and[2]

$$\Delta(x) \leqslant \left|\sum_{k=n+1}^{N-1} b_k\sin kx\right| + \left|\sum_{k=N}^{\infty} b_k\sin kx\right| := J_1(x) + J_2(x). \tag{11}$$

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[2]If $N \leqslant n+1$, then a similar discussion can be made directly to $\sum_{k=n+1}^{\infty} b_k\sin kx$.

Let

$$D_n(x)=\sum_{k=1}^{n}\sin kx.$$

It is well known that

$$|D_n(x)|\leqslant\frac{\pi}{x}.$$

If $f\in H_{S,\Omega}$, by the definition of $\Omega RBVS$, we have

$$|b_n|\leqslant\sum_{k=n}^{\infty}|\Delta b_k|\leqslant Kn^{-1}\omega\left(\frac{1}{n}\right), \tag{12}$$

hence

$$|J_1(x)|\leqslant Kx\sum_{k=n}^{N-1}k|b_k|\leqslant K\omega\left(\frac{1}{n}\right)x(N-1)\leqslant K\omega\left(\frac{1}{n}\right). \tag{13}$$

By Abel's transformation and (12), we have

$$\begin{aligned}|J_2(x)| &\leqslant \sum_{k=N}^{\infty}|\Delta b_k||D_k(x)|+|b_N||D_{N-1}(x)|\\ &\leqslant Kx^{-1}\left(\sum_{k=N}^{\infty}|\Delta b_k|+|b_N|\right)\\ &\leqslant KN|b_N|\leqslant K\omega\left(\frac{1}{N}\right)\leqslant K\omega\left(\frac{1}{n}\right).\end{aligned} \tag{14}$$

Altogether (11), (13), (14), we obtain that

$$E_n(f)\leqslant\|f-S_n(f)\|\leqslant K\omega\left(\frac{1}{n}\right). \tag{15}$$

Applying (6) and (15), we get

$$\omega_\beta\left(f,\frac{1}{n}\right)\leqslant Kn^{-\beta}\sum_{k=1}^{n}k^{\beta-1}E_k(f)\leqslant Kn^{-\beta}\sum_{k=1}^{n}k^{\beta-1}\omega\left(\frac{1}{k}\right)$$

$$\leqslant Kn^{-\beta}\sum_{k=1}^{n}k^{\beta+\varepsilon-2}\omega\left(\frac{1}{k}\right)k^{1-\varepsilon}\leqslant K\omega\left(\frac{1}{n}\right).$$

Hence, $f\in H_\beta^\omega$. The proof is over. □

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