Mathematical
Inequalities
& Applications
Volume 11, Number 1 (2007), 173–188

AN EXPLICIT BOUND FOR THE ERROR TERM OF THE DEVELOPMENT AT $s = 1$ OF A SET OF LACUNARY SERIES

GIUSEPPE MOLTENI

(*communicated by A. Laforgia*)

Abstract. An explicit bound for the error term of the expansion of $F_n(x) := \sum_{k=0}^{\infty} k^n x^{2^k}$ as $x \to 1^-$ is given.

Notation. The symbol $\delta_{something}$ assumes the value 1 when *something* holds, 0 otherwise; $B_n(\cdot)$ is the nth Bernoulli polynomial and B_n is the nth Bernoulli number; $\lfloor x \rfloor$, $\{x\}$ and $\operatorname{Im} x$ denote the integer, fractional and imaginary parts of x, respectively; $\log_2(x)$ denotes the base 2 logarithm; at last, the symbol $x \to 1^-$ will be used to mean that $x \in \mathbb{R}$, $x < 1$ and $x \to 1$.

1. Introduction

In a couple of recent papers [1, 2] where the expected length of certain instantaneous integer codes for the compression of web graphs (graphs having web pages as nodes and hyperlinks as arcs) is estimated, the authors needed some results about the behavior of $F_0(x)$ and $F_1(x)$ when x approaches 1, where F_n is the function that for $x \in \mathbb{C}$ with $|x| < 1$, is defined by

$$F_n(x) := \sum_{k=0}^{\infty} k^n x^{2^k}.$$

Such functions are probably the simplest examples of lacunary series whose essential properties are well known in literature; for example, the asymptotic behavior in a neighborhood of almost every point of the boundary of their natural domain is easily described by the iterated logarithm law proved by Erdös and Gál. On the other hand, the Euler-Mc Laurin summation formula gives almost immediately that

$$F_n(x) = \frac{1}{n+1}(-\log_2(1-x))^{n+1} + O((-\log_2(1-x))^n) \quad \text{as } x \to 1^-.$$

The aim of the present paper is to improve this result providing an explicit bound for the constant appearing into the O-term. Actually we prove that

Mathematics subject classification (2000): 33E20.
Key words and phrases: Lacunary series.

Paper MIA-11-12

THEOREM. *Let* $L(x) := -\log_2(1-x)$. *If* $x \to 1^-$, *then for every* $n \geqslant 0$

$$|F_n(x) - \big(\tfrac{1}{n+1}L^{n+1}(x) - \tfrac{\gamma}{\log 2}L^n(x) + \tfrac{1}{2}\delta_{n=0}\big)| \leqslant (4.49 * 10^{-6} + o_n(1))L^n(x),$$

where γ *is the Euler-Mascheroni constant.*

In order to appreciate this result, in Section 3. some numerical tests and conjectures are given.

Next section is devoted to the proof of the theorem.

2. Proof of the theorem

Henceforth we take $x \to 1^-$. It is convenient to define $\epsilon := -\log x \sim (1-x)$ and

$$w_n(k,\epsilon) := k^n e^{-2^k\epsilon}.$$

Using the Euler-Mc Laurin summation formula (see Ch. I.0 of [9]) we have

$$F_n(x) = \sum_{k=0}^{\infty} w_n(k,\epsilon) = \hat{F}_n(x) + \check{F}_n(x),$$

where

$$\hat{F}_n(x) := \tfrac{1}{2}e^{-\epsilon}\delta_{n=0} + \int_0^{\infty} w_n(k,\epsilon)dk, \qquad \check{F}_n(x) := \int_0^{+\infty} w_n'(k,\epsilon)B_1(\{k\})\,dk \qquad (1)$$

and $B_1(x) = x - 1/2$ is the first Bernoulli polynomial. The behavior of $\hat{F}_n(x)$ when $x \to 1^-$ is well understood since with simple manipulations it is possible to find the complete asymptotic expansion of $\hat{F}_n$; however, the following lemma gives explicitly only the first two terms, since for our purposes these ones are enough.

LEMMA 1. *For every* $n \geqslant 0$

$$\hat{F}_n(x) = \frac{(\log_2(1/\epsilon))^{n+1}}{n+1} + \frac{\delta_{n=0}}{2} + \left(-\frac{\gamma}{\log 2} + o(1)\right)(\log_2(1/\epsilon))^n.$$

Proof. In fact,

$$\hat{F}_n(x) = \tfrac{1}{2}e^{-\epsilon}\delta_{n=0} + \int_0^{\log_2(1/\epsilon)} k^n e^{-2^k\epsilon}\,dk + \int_{\log_2(1/\epsilon)}^{+\infty} k^n e^{-2^k\epsilon}\,dk.$$

A translation in the second integral: $k \to \log_2(1/\epsilon) + k$ gives

$$= \tfrac{1}{2}e^{-\epsilon}\delta_{n=0} + \int_0^{\log_2(1/\epsilon)} k^n e^{-2^k\epsilon}\,dk + \int_0^{+\infty} (\log_2(1/\epsilon)+k)^n e^{-2^k}\,dk.$$

The introduction of a new variable $v = 2^k\epsilon$ in the first integrale and $v = 1/2^k$ in the second one gives

$$= \tfrac{1}{2}e^{-\epsilon}\delta_{n=0} + \frac{1}{\log 2}\int_\epsilon^1 (\log_2(v/\epsilon))^n e^{-v}\,\frac{dv}{v} + \frac{1}{\log 2}\int_0^1 (\log_2(1/\epsilon)+\log_2(1/v))^n e^{-1/v}\,\frac{dv}{v}.$$

The binomial formular for $(\log_2(v/\epsilon))^n = (\log_2 v + \log_2(1/\epsilon))^n$ gives

$$= \tfrac{1}{2}e^{-\epsilon}\delta_{n=0} + \sum_{m=0}^{n}\binom{n}{m}\frac{\log_2(1/\epsilon)^m}{\log 2}\Big[\int_\epsilon^1 (\log_2 v)^{n-m}e^{-v}\frac{dv}{v} + \int_0^1 (\log_2(1/v))^{n-m}e^{-1/v}\frac{dv}{v}\Big]$$

and separating the contribute of the diverging terms near $\epsilon = 0$ we get

$$= \tfrac{1}{2}e^{-\epsilon}\delta_{n=0} + \sum_{m=0}^{n}\binom{n}{m}\frac{\log_2(1/\epsilon)^m}{\log 2}\Big[\int_0^1 (\log_2 v)^{n-m}\frac{e^{-v}-1}{v}\,dv$$

$$- \int_0^\epsilon (\log_2 v)^{n-m}\frac{e^{-v}-1}{v}\,dv + \int_\epsilon^1 (\log_2 v)^{n-m}\,\frac{dv}{v} + \int_0^1 (\log_2(1/v))^{n-m}e^{-1/v}\,\frac{dv}{v}\Big]$$

$$= \tfrac{1}{2}e^{-\epsilon}\delta_{n=0} + \sum_{m=0}^{n}\binom{n}{m}\frac{\log_2(1/\epsilon)^m}{\log 2}\Big[\int_0^1 (\log_2 v)^{n-m}\frac{e^{-v}-1}{v}\,dv + O(\epsilon(\log\epsilon)^{n-m})$$

$$- \frac{(\log_2\epsilon)^{n-m+1}}{n-m+1}\cdot\log 2 + \int_0^1 (\log_2(1/v))^{n-m}e^{-1/v}\,\frac{dv}{v}\Big]$$

$$= \log_2(1/\epsilon)^{n+1}\sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^{n-m}}{n-m+1} + \sum_{m=0}^{n}\binom{n}{m}\frac{\log_2(1/\epsilon)^m}{\log 2}\times$$

$$\times\Big[\int_0^1 (\log_2 v)^{n-m}\frac{e^{-v}-1+(-1)^{n-m}e^{-1/v}}{v}\,dv\Big] + \tfrac{1}{2}\delta_{n=0} + O(\epsilon(\log\epsilon)^n)$$

$$= \frac{\log_2(1/\epsilon)^{n+1}}{n+1} + \sum_{m=0}^{n}\binom{n}{m}\frac{\log_2(1/\epsilon)^m}{\log 2}\Big[\int_0^1 (\log_2 v)^{n-m}\frac{e^{-v}-1+(-1)^{n-m}e^{-1/v}}{v}\,dv\Big]$$

$$+ \tfrac{1}{2}\delta_{n=0} + O(\epsilon(\log\epsilon)^n).$$

The proof is concluded by noting that $\int_0^1 \frac{e^{-v}-1+e^{-1/v}}{v}\,dv = -\gamma$ (see [3], Art. 178.)

□

In order to get good upper-bounds for $\check{F}_n$ we have to take advantage of the variations in sign of the oscillating term $B_1(\{k\})$ appearing in (1). The following simple lemma provides the main tool.

LEMMA 2. *Let* $g:\mathbb{R}\to\mathbb{R}$ *be integrable, bounded and* 1*-periodic with* $\int_0^1 g(u)du = 0$. *Let* $M := \sup_{x,y\in[0,1]}\int_x^y g(u)du$. *Let* $a < b \in \mathbb{R}$ *and let* $f : [a,b]\to\mathbb{R}$ *be monotonic. Then,*

$$\Big|\int_a^b f(x)g(x)\,dx\Big| \leqslant M(|f(a)| + |f(b)|).$$

Proof. By the second mean value formula (see Ch. 0.1 of [9]), we have

$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^\xi g(x)dx + f(b)\int_\xi^b g(x)dx$$

for some $\xi \in [a,b]$. By hypothesis g is periodic and its mean value is zero, hence

$$\Big|\int_\xi^b g(x)dx\Big|,\ \ \Big|\int_a^\xi g(x)dx\Big| \leqslant M,$$

so that the desired inequality follows. □

We do not apply this lemma directly to the integral defining $\check{F}_n$, but to the integral appearing in the following formula (Euler-Mc Laurin formula with arbitrary index)

$$\check{F}_n(x) = \sum_{l=1}^{m-1} \frac{(-1)^{l+1}B_{l+1}}{(l+1)!}\, w_n^{(l)}(k,\epsilon)\Big|_0^{+\infty} - \frac{(-1)^m}{m!}\int_0^{+\infty} w_n^{(m)}(k,\epsilon)B_m(\{k\})\,dk, \quad (2)$$

where $B_m(\{k\})$ is the mth Bernoulli polynomial, $B_m := B_m(0)$ is the mth Bernoulli number and $w_n^{(m)}(k,\epsilon)$ is the mth partial derivative of $w_n(k,\epsilon)$ with respect to k. When $m = 1$ such formula actually is the definition of $\check{F}_n$; as we will see, the theorem will follow from a suitable choice of the parameter m.

As first task we have to get a convenient upper-bound for $M_m := \sup_{x,y} \int_x^y B_m(\{k\})\,dk$, where $B_m(\{k\})$ is the mth Bernoulli polynomial. This is the simplest step since the recursive definition $B_m(x) = m\int_0^x B_{m-1}(v)dv$ implies that

$$(m+1)M_m = \sup_{x,y\in[0,1]} (B_{m+1}(x) - B_{m+1}(y)),$$

so that M_m can be computed using the bounds for $B_{m+1}(x)$ that have been proved by Lehmer in [6]. Actually, the following non-optimal but simply proved bounds for M_m are sufficient for our purposes.

LEMMA 3. *For every $m > 0$, we have*

$$M_m \leqslant \begin{cases} \frac{1}{2}|B_m| = \frac{m!\zeta(m)}{(2\pi)^m}, & \text{if } m \text{ is even,} \\ \frac{2}{m+1}|B_{m+1}| = \frac{2\cdot m!\cdot\zeta(m+1)}{\pi(2\pi)^m}, & \text{if } m \text{ is odd,} \end{cases}$$

where ζ is the Riemann zeta function.

Proof. We use some simple and well known results. When m is even, $m \geqslant 2$, $B_m(\{x\})$ has only two roots into $[0,1]$ which are symmetric with respect to $1/2$; furthermore $|B_m(\{x\})| \leqslant |B_m|$ for every x, so that the first inequality follows.

When m is odd, the unique roots of $B_m(\{x\})$ in $[0,1]$ are 0, $\frac{1}{2}$ and 1 (only $\frac{1}{2}$ if $m = 1$). Since every root is simple, we have

$$\begin{aligned} M_m &= \Big|\int_0^{1/2} B_m(\{u\})\,du\Big| = \tfrac{1}{m+1}|B_{m+1}(\tfrac{1}{2}) - B_{m+1}| \\ &= \tfrac{1}{m+1}|(2^{-m}-1)B_{m+1} - B_{m+1}| = \tfrac{2}{m+1}(1-2^{-1-m})|B_{m+1}|, \end{aligned}$$

proving the second inequality. □

REMARK 1. Lehmer [6] (see also Delange [4, 5]) proved that for m even the unique zero of B_m belonging to $(0,1/2)$ tends to $1/4$ as $m\to\infty$; as a consequence the upper-bound $M_m \leqslant (1/4+o(1))|B_m|$ holds as $m\to\infty$; this weak improvement is not important here since we will not consider M_m for arbitrarily large m.

Now we have to find the intervals where $w_n^{(m)}(k,\epsilon)$ is monotonic as a function of k, at least for ϵ small enough. Evidently, these intervals will be known when the roots of $w_n^{(m+1)}(k,\epsilon)$ will be known, therefore an explicit formula fo $w_n^{(m)}(k,\epsilon)$ is needed. The following lemma provides such description in terms of the *exponential polynomials* $\phi_m(x)$ (see [8]), i.e., in terms of polynomials recursively defined by

$$\begin{cases} \phi_m(x) = 0 & \text{if } m<0,\\ \phi_0(x)=1, & \\ \phi_{m+1}(x) = x(\phi_m'(x)+\phi_m(x)) & \text{if } m\geqslant 0. \end{cases}$$

LEMMA 4. *Let $y := 2^k\epsilon$. Then for every $m\geqslant 0$,*

$$w_n^{(m)}(k,\epsilon) = e^{-y}\sum_{u=0}^{n}\binom{n}{u} m_{(u)} k^{n-u}\phi_{m-u}(-y)\log^{m-u} 2,$$

with

$$\begin{cases} m_{(0)} = 1 & \textit{for every } m,\\ m_{(u)} = m(m-1)\cdots(m-u+1) & \textit{if } u>0. \end{cases}$$

Its proof by induction on m is quite simple and we leave it to the reader. Three examples of this identity are:

$$\begin{aligned} w_0^{(m)}(k,\epsilon) &= e^{-y}\phi_m(-y)\log^m 2,\\ w_1^{(m)}(k,\epsilon) &= e^{-y}\big(k\phi_m(-y)\log^m 2 + m\phi_{m-1}(-y)\log^{m-1} 2\big),\\ w_2^{(m)}(k,\epsilon) &= e^{-y}\big(k^2\phi_m(-y)\log^m 2 + 2mk\phi_{m-1}(-y)\log^{m-1} 2\\ &\qquad + m(m-1)\phi_{m-2}(-y)\log^{m-2} 2\big). \end{aligned}$$

Previous lemma shows that the roots of $w_0^{(m)}(k,\epsilon)$ can be calculated immediately if the roots of ϕ_m are known. We will see that a similar fact holds also for the roots of every $w_n^{(m)}(k,\epsilon)$, at least when ϵ is small enough: as a consequence the next step is to study the roots of ϕ_m. Such polynomials appear frequently in combinatorial analysis (a fact which is not surprising since $\phi_m(x) = \sum_{k=1}^m S(m,k)x^k$ where $S(m,k)$ is the Stirling number of second kind) but we have not been able to locate in letterature an explicit reference for the following result.

LEMMA 5. *For every $m\geqslant 1$, 0 is a simple zero of ϕ_m and the $m-1$ non-zero roots of ϕ_m are simple, real, negative and interlaced to those ones of ϕ_{m-1}, i.e., if $\eta_{m-1}<\cdots<\eta_2<\eta_1<\eta_0=0$ are the roots of ϕ_m and $\zeta_{m-2}<\cdots<\zeta_2<\zeta_1<\zeta_0=0$ those ones of ϕ_{m-1}, we have*

$$\eta_{m-1}<\zeta_{m-2}<\eta_{m-2}<\cdots<\eta_2<\zeta_1<\eta_1<0.$$

Proof. It is convenient to introduce $\Phi_m(x) := \phi_m(x)e^x$, so that $\Phi_0(x) = e^x$ and $\Phi_{m+1}(x) = x\Phi_m'(x)$ by the recursion formula for ϕ_m. If we prove that Φ_m admits m simple real roots, interlaced to those ones of $\Phi_{m-1}(x)$, then the claim about the roots of $\phi_m(x)$ follows since the roots of Φ_m and those ones of ϕ_m are the same, multiplicity included. As a preliminary fact we note that the degree of ϕ_m is m for every m and that when $m > 0$ the recursion formula implies that 0 is a simple root of Φ_m. In particular, the claim holds when $m = 0$.

By induction, Φ_m has $m-1$ simple and negative roots, besides 0. Every couple of consecutive roots of Φ_m delimits an interval containing an extremal point for Φ_m which is a root of Φ_m'. There are $m-1$ such intervals, therefore there are $m-1$ distinct roots of Φ_m'. Such points are extremal points for Φ_m, therefore their multiplicity as root of Φ_m' is odd: if any of them has a multiplicity greater than 1 then the roots we have found for Φ_m' would be $\geqslant (m-2)+3 = m+1$, when the multiplicity is considered, but this is impossible since every such root is also a negative root of Φ_{m+1} which admits m negative roots, at most. This fact proves that the roots of Φ_m' we have found are simple roots of Φ_{m+1} and are interlaced to those ones of Φ_m. Also 0 is a simple root of Φ_{m+1} so that we actually have found m simple roots of Φ_{m+1}. Let γ_{m+1} be the smallest root of Φ_{m+1} we have found up to now. We note that

$$\lim_{x\to+\infty} \Phi_{m+1}(x) = +\infty, \quad \lim_{x\to-\infty} \Phi_{m+1}(x) = 0^{(-1)^{m+1}},$$

and that the sign of $\Phi_{m+1}(x)$ changes when x crosses a root (since every roots is simple), therefore another root of Φ_{m+1}, lower than γ_{m+1}, exists. Also such root is simple, otherwise Φ_{m+1} would have more than $m+1$ roots. □

It is convenient to denote by $\zeta_{m,0} = 0 < \zeta_{m,1} < \zeta_{m,2} < \cdots < \zeta_{m,m-1}$ the *opposite* of the roots of ϕ_m. The following two lemmas describe the roots of $w_n^{(m)}(k,\epsilon)$ in the limit $\epsilon \to 0$.

LEMMA 6. *For every $m \geqslant 2$ and $n \geqslant 0$, there exists a positive constant $\bar\epsilon$ depending on m and n such that for $\epsilon < \bar\epsilon$ the equation $w_n^{(m)}(k,\epsilon) = 0$ under the restriction $2^k\epsilon \geqslant \zeta_{m,1}$ has $m-1$ simple roots: $\tilde k_{m,n,j}(\epsilon)$ with $j = 1,\cdots,m-1$, say. Furthermore, $\tilde k_{m,n,j}(\epsilon) = \log_2(1/\epsilon) + \log_2 \zeta_{m,j} + o_{m,n}(1)$ for every $j = 1,\cdots,m-1$, as $\epsilon \to 0$.*

Proof. Let $y := 2^k\epsilon$. By Lemma 4, $w_n^{(m)}(k,\epsilon) = 0$ if and only if $G_{m,n,\epsilon}(y) = 0$, where

$$G_{m,n,\epsilon}(y) := \sum_{u=0}^{n} \binom{n}{u} m_{(u)} (\log(y/\epsilon))^{n-u} \phi_{m-u}(-y).$$

We take y as new independent variable. By hypothesis $y \geqslant \zeta_{m,1}$. The claim is evident if $n = 0$, hence we can suppose that $n \geqslant 1$. We prove the claim by several steps.

i) For $\epsilon \to 0$ and uniformly on $y > \zeta_{m,1}$ we have

$$G_{m,n,\epsilon}(y) = (\log(y/\epsilon))^n \phi_m(-y) + O(y^{m-1}(\log(y/\epsilon))^{n-1}),$$

hence, if for infinitely many ϵ_i converging to 0 there exists a point y_i such that $G_{m,n,\epsilon_i}(y_i) = 0$ and $0 < a < y_i < b$ for some constants a, b independent of i, then $\phi_m(-y_i) \to 0$.

ii) By Lemma 5 the roots of ϕ_m and ϕ_{m-1} are simple and interlaced. Furthermore, $\phi_m(x) > 0$ when $x > 0$, so that

$$\begin{cases} \phi_m(-\zeta_{m,j}) = 0 & \Longrightarrow\ (-1)^j \phi_{m-1}(-\zeta_{m,j}) > 0 \\ \phi_{m-1}(-\zeta_{m-1,j}) = 0 & \Longrightarrow\ (-1)^j \phi_m(-\zeta_{m-1,j}) < 0, \end{cases}$$

for every m and j.

iii) We prove that for ϵ small enough there exists a solution of $G_{m,n,\epsilon}(y) = 0$ in the segment $[\zeta_{m,1}, \zeta_{m-1,1})$. In fact,

$$\begin{aligned} G_{m,n,\epsilon}(\zeta_{m,1}) &= (\log(\zeta_{m,1}/\epsilon))^n \phi_m(-\zeta_{m,1}) + mn(\log(\zeta_{m,1}/\epsilon))^{n-1} \phi_{m-1}(-\zeta_{m,1}) \\ &\quad + O_{m,n}((\log(1/\epsilon))^{n-2}) \\ &= mn(\log(\zeta_{m,1}/\epsilon))^{n-1} \phi_{m-1}(-\zeta_{m,1}) + O_{m,n}((\log(1/\epsilon))^{n-2}). \end{aligned}$$

Since $\phi_{m-1}(-\zeta_{m,1}) < 0$ by Step ii), we have that $G_{m,n,\epsilon}(\zeta_{m,1}) < 0$ when ϵ is small enough. Moreover,

$$G_{m,n,\epsilon}(\zeta_{m-1,1}) = (\log(\zeta_{m,1}/\epsilon))^n \phi_m(-\zeta_{m-1,1}) + O_{m,n}((\log(1/\epsilon))^{n-1})$$

so that $G_{m,n,\epsilon}(\zeta_{m-1,1}) > 0$ by Step ii), when ϵ is small enough.

iv) We prove that for ϵ small enough the solution of $G_{m,n,\epsilon}(y) = 0$ in $[\zeta_{m,1}, \zeta_{m-1,1})$ is unique and simple. In fact, let $y_1(\epsilon)$ be the solution whose existence is proved in previous step. Let $y_2(\epsilon) \geqslant y_1(\epsilon)$ be a second root belonging to the same interval $[\zeta_{m,1}, \zeta_{m-1,1})$, with $y_2(\epsilon) = y_1(\epsilon)$ meaning that $y_1(\epsilon)$ is a non-simple root. Then, there exists a point $z(\epsilon)$ such that $y_1(\epsilon) \leqslant z(\epsilon) \leqslant y_2(\epsilon)$ and $G'_{m,n,\epsilon}(z(\epsilon)) = 0$. By Step i), $y_1(\epsilon)$ and $y_2(\epsilon)$ tend to $\zeta_{m,1}$, so that $z(\epsilon) \to \zeta_{m,1}$, too. Then

$$\begin{aligned} 0 &= G'_{m,n,\epsilon}(z(\epsilon)) \\ &= -(\log(z(\epsilon)/\epsilon))^n \phi'_m(-z(\epsilon)) + n(\log(z(\epsilon)/\epsilon))^{n-1} \frac{\phi_m(-z(\epsilon))}{z(\epsilon)} \\ &\quad + O_{m,n}((\log(1/\epsilon))^{n-1}) \\ &= (\log(1/\epsilon))^n(-\phi'_m(-\zeta_{m,1}) + o_{m,n}(1)) \end{aligned}$$

which is impossible for ϵ small enough since $\phi'_m(-\zeta_{m,1}) \neq 0$, being $-\zeta_{m,1}$ a simple root of ϕ_m by Lemma 5.

v) We prove that for ϵ small enough $G_{m,n,\epsilon}(y) \neq 0$ in the segment $[\zeta_{m-1,1}, \zeta_{m,2})$. In fact,

$$\begin{aligned} G_{m,n,\epsilon}(\zeta_{m,2}) &= (\log(\zeta_{m,1}/\epsilon))^n \phi_m(-\zeta_{m,2}) + mn(\log(\zeta_{m,2}/\epsilon))^{n-1} \phi_{m-1}(-\zeta_{m,2}) \\ &\quad + O_{m,n}((\log(1/\epsilon))^{n-2}) \\ &= mn(\log(\zeta_{m,2}/\epsilon))^{n-1} \phi_{m-1}(-\zeta_{m,2}) + O_{m,n}((\log(1/\epsilon))^{n-2}), \end{aligned}$$

which by Step ii) is positive when ϵ is small enough. Since we know that $G_{m,n,\epsilon}(\zeta_{m-1,1}) > 0$, it follows that if $G_{m,n,\epsilon}(y)$ has a root in $[\zeta_{m-1,1}, \zeta_{m,1})$, then in this segment there are two roots, at least. Let $y_1(\epsilon) \leqslant y_2(\epsilon)$ be a couple of such roots. Then there exists an intermediate point $z(\epsilon)$ such that $y_1(\epsilon) \leqslant z(\epsilon) \leqslant y_2(\epsilon)$ and $G'_{m,n,\epsilon}(z(\epsilon)) = 0$. By Step i), $y_1(\epsilon)$ and $y_2(\epsilon)$ tend to $\zeta_{m,2}$, so that $z(\epsilon) \to \zeta_{m,2}$, too. Then

$$0 = G'_{m,n,\epsilon}(z(\epsilon)) = (\log(1/\epsilon))^n(-\phi'_m(-\zeta_{m,2}) + o_{m,n}(1))$$

which is impossible for ϵ small enough since $\phi'_m(-\zeta_{m,2}) \neq 0$ by Lemma 5.

vi) The argument we just employed can be repeated in every segment $[\zeta_{m,j}, \zeta_{m,j+1})$ when $j = 1, \dots, m-2$, proving the existence of a unique, simple root of $G_{m,n,\epsilon}(y) = 0$ in every subinterval $[\zeta_{m,j}, \zeta_{m-1,j})$ whenever ϵ is small enough. Furthermore, we proved that the root belonging to $[\zeta_{m,j}, \zeta_{m,j+1})$ tends to $\zeta_{m,j}$ as $\epsilon \to 0$. The argument can easily adapted to provide the same conclusions also for the last interval $[\zeta_{m,m-1}, +\infty)$ noting that $(-1)^m G_{m,n,\epsilon}(\zeta_{m,m-1}) < 0$ and that $(-1)^m G_{m,n,\epsilon}(+\infty) = +\infty$ when ϵ is small enough.

vii) The proof concludes noting that $2^k\epsilon = y = \zeta_{m,j} + o_{m,n}(1)$ implies

$$k = \log_2(1/\epsilon) + \log_2 \zeta_{m,j} + o_{m,n}(1).$$

□

LEMMA 7. *For every $m \geqslant 0$ and $n \geqslant 0$ the equation $w_n^{(m)}(k,\epsilon) = 0$ under the restriction $\epsilon < 2^k\epsilon < \zeta_{m,1}$ has no solutions when $m > n$. On the contrary, when $m \leqslant n$, there exists a positive constant $\bar\epsilon$ depending on m and n such that for $\epsilon < \bar\epsilon$ the equation has $M \leqslant m$ roots, at most: $\bar k_{m,n,j}(\epsilon)$ with $j = 1, \cdots, M$, say. Furthermore, every such root satisfies $2^{\bar k_{m,n,j}(\epsilon)}\epsilon \to 0$ as $\epsilon \to 0$.*

Proof. As in previous lemma we set $y := 2^k\epsilon$ and

$$G_{m,n,\epsilon}(y) := \sum_{u=0}^{n} \binom{n}{u} m_{(u)} (\log(y/\epsilon))^{n-u} \phi_{m-u}(-y)$$

so that $w_n^{(m)}(k,\epsilon) = 0$ if and only if $G_{m,n,\epsilon}(y) = 0$. We take y as new independent variable. By hypothesis $\epsilon < y < \zeta_{m,1}$. The claim is evident if $n = 0$, hence we can suppose that $n \geqslant 1$. We prove the claim by several steps.

i) Suppose $m > n$. We prove that $G_{m,n,\epsilon}(y) < 0$ when $y \in (\epsilon, \zeta_{m,1})$ so that, in particular, equation $G_{m,n,\epsilon}(y) = 0$ has no solutions here. In fact, by Lemma 5 every ϕ_{m-u} with $m-u > 0$ is negative in $(-\zeta_{m-u,1}, 0)$ since 0 is a simple root of ϕ_{m-u} and $\phi_{m-u}(x)$ is positive when $x > 0$. Furthermore, $-\zeta_{m-u,1} \leqslant -\zeta_{m,1}$ since the roots are interlaced: these two facts show that every $\phi_{m-u}(-y)$ appearing in $G_{m,n,\epsilon}(y)$ is negative in $(0, \zeta_{m,1})$, being by hypothesis $m > n$. Since $\log(y/\epsilon) > 0$, the first claim follows.

ii) Suppose $m \leqslant n$. Then the terms with $m < u \leqslant n$ appearing in the definition of $G_{m,n,\epsilon}(y)$ do not contribute to the sum (because in this case $m_{(u)} = 0$). Besides, we can divide $G_{m,n,\epsilon}(y)$ by $(\log(y/\epsilon))^{n-m}$ which is not zero in range $\epsilon < y < \zeta_{m,1}$, getting in this way that $G_{m,n,\epsilon}(y) = 0$ if and only if $\tilde{G}_{m,n,\epsilon}(y) = 0$, where

$$\tilde{G}_{m,n,\epsilon}(y) := \sum_{u=0}^{m} \binom{n}{u} m_{(u)} (\log(y/\epsilon))^{m-u} \phi_{m-u}(-y).$$

iii) When $m \leqslant n$ one root at least in $(0, \zeta_{m,1})$ exists because $\tilde{G}_{m,n,\epsilon}(\epsilon) = \binom{n}{m} m_{(m)} > 0$ and

$$\tilde{G}_{m,n,\epsilon}(\zeta_{m,1}) = (\log(1/\epsilon))^{m-1}(mn\phi_{m-1}(-\zeta_{m,1}) + o(1))$$

which is negative when ϵ is small enough since $\phi_{m-1}(-\zeta_{m,1}) < 0$ (see Step ii) in proof of Lemma 6).

iv) Let us consider the equation

$$\sum_{u=0}^{m} \binom{n}{u} m_{(u)} \phi_{m-u}(-y) H^{m-u} = 0, \tag{3}$$

defining m algebraic functions $H_j = H_j(y)$, for $j = 1, \dots, m$. Let $j = 1, \dots, M$ be those ones which assume positive real values when $0 < y < \zeta_{m,1}$: one such function at least exists by previous step. Since $\phi_m(-y)$ is not zero in $(0, \zeta_{m,1})$, every H_j is analytical in this interval. We note that for suitable positive constants c_1 and c_2,

$$H_j(y) \sim \frac{c_1}{y^{1/m}}, \quad H_j'(y) \sim -\frac{c_2}{y^{1+1/m}}, \quad \text{as } y \to 0^+, \quad \forall l = 1, \dots, M. \tag{4}$$

In fact, the polynomials $\phi_{m-u}(-y)$ with $u < m$ and appearing in (3) are divisible by $-y$, while the polynomial with $u = m$ is constant and non-zero, therefore H_j diverges as $y \to 0^+$. As a consequence

$$H_j^m(\phi_m(-y) + O(H_j^{-1})) + \binom{n}{m} m_{(m)} = 0,$$

the first claim follows from this equality since $\phi_m(-y) \sim -\phi_m'(0)y$ as $y \to 0$ and $\phi_m'(0) > 0$. The second claim follows by the first one simply taking the derivative of (3) with respect to y.

v) We prove that

$$\limsup_{y \to \zeta_{m,1}^-} H_j(y) < +\infty, \quad \forall j = 1, \dots, M. \tag{5}$$

In fact, let use suppose that there exists a sequence $y_l < \zeta_{m,1}$ such that $y_l \to \zeta_{m,1}$ and a second diverging sequence H_l satisfying (3) with $y = y_l$. Dividing (3) by H_l^{m-1} we get

$$\phi_m(-y_l)H_l + mn\phi_{m-1}(-y_l) + O(H_l^{-1}) = 0.$$

Since by Lemma 5 both $\phi_{m-1}(-y)$ and $\phi_m(-y)$ are negative as $y \in (0, \zeta_{m,1})$ we conclude that $H_l \to -\infty$.

vi) For every $\epsilon > 0$, every solution $y = y(\epsilon)$ of $\tilde{G}_{m,n,\epsilon}(y) = 0$ must satisfy

$$\log(y/\epsilon) = H_j(y), \quad \text{i.e.,} \quad \epsilon = ye^{-H_j(y)}, \tag{6}$$

for some $j = 0, \dots, M$. R.H.S. of (6) is positive when $y > 0$ and tends to 0^+ as $y \to 0^+$ by first claim in (4). Moreover, its derivative is $(1 - yH_j')e^{-H_j}$ so that R.H.S. has only finitely many oscillations on every compact subset of $(0, \zeta_{m,1})$ (since H_j is analytical here). Actually, R.H.S. has only finitely many oscillations also in the larger interval $(0, \zeta_{m,1} - \eta)$ for every $\eta > 0$, since $(1 - yH_j')$ diverges to $+\infty$ as $y \to 0^+$ by (4). Besides, $\liminf_{y \to \zeta_{m,1}^-} ye^{-H_j(y)} > 0$ by Step v). As a consequence, for ϵ small enough, Equation (6) admits only one solution $y_j = y_j(\epsilon)$ and such solution tends to zero as $\epsilon \to 0$. Since there are $M \leqslant m$ distinct equations of type (6), also the last claim of Lemma 7 is proved.

□

We are now able to prove the following result.

LEMMA 8. *For every $m > 0$ let*

$$b_m := 2\big(\tfrac{2\zeta(m+1)}{\pi}\delta_{m \text{ odd}} + \zeta(m)\delta_{m \text{ even}}\big)\big(\tfrac{\log 2}{2\pi}\big)^m \sum_{j=1}^{m} |\phi_m(-\zeta_{m+1,j})| e^{-\zeta_{m+1,j}}.$$

Then, for every $m > 0$ and $n \geqslant 0$ we have

$$|\check{F}_n|(x) \leqslant (b_m + o_{m,n}(1))(\log_2(1/\epsilon))^n \quad \text{as } x \to 1^-.$$

Proof. By Lemma 4 we have $w_n^{(m)}(0,\epsilon) = m_{(n)}\phi_{m-n}(-\epsilon)\log^{m-n} 2 = \mathscr{O}_{m,n}(\epsilon)$ where

$$\mathscr{O}_{m,n}(\epsilon) := \begin{cases} O_m(\epsilon) & \text{if } n = 0, \\ O_{m,n}(1) & \text{if } n > 0, \end{cases} \quad \text{as } \epsilon \to 0.$$

By this fact, from (2) we have

$$\check{F}_n(x) = \mathscr{O}_{m,n}(\epsilon) - \frac{(-1)^m}{m!}\int_0^{+\infty} w_n^{(m)}(k,\epsilon)B_m(\{k\})\,dk. \tag{7}$$

Let $\tilde{k}_{m,n,j}(\epsilon)$ for $j = 1, \dots, m-1$ be the sequence of points we found in Lemma 6. Define $\tilde{k}_{m+1,n,m}(\epsilon) := +\infty$. Let $\bar{k}_{m,n,j}(\epsilon)$ for $j = 1, \dots, M$ be the sequence of points we found in Lemma 7. Define $\bar{k}_{m+1,n,0}(\epsilon) := 0$. By (7) we have

$$\begin{aligned}
|\check{F}_n(x)| &\leqslant \mathscr{O}_{m,n}(\epsilon) + \tfrac{1}{m!}\Big|\int_0^{+\infty} w_m^{(n)}(k,\epsilon)B_m(\{k\})\,dk\Big| \\
&\leqslant \mathscr{O}_{m,n}(\epsilon) + \tfrac{1}{m!}\sum_{j=0}^{M-1}\Big|\int_{\bar{k}_{m+1,n,j}}^{\bar{k}_{m+1,n,j+1}} w_n^{(m)}(k,\epsilon)B_m(\{k\})\,dk\Big| \\
&\quad + \tfrac{1}{m!}\Big|\int_{\bar{k}_{m+1,n,M}}^{\tilde{k}_{m+1,n,1}} w_n^{(m)}(k,\epsilon)B_m(\{k\})\,dk\Big| + \tfrac{1}{m!}\sum_{j=1}^{m-1}\Big|\int_{\tilde{k}_{m+1,n,j}}^{\tilde{k}_{m+1,n,j+1}} w_n^{(m)}(k,\epsilon)B_m(\{k\})\,dk\Big|.
\end{aligned}$$

In Lemmas 6 and 7 we proved that $w_n^{(m)}(k,\epsilon)$ is monotonous in intervals $(\bar{k}_{m+1,n,j}, \bar{k}_{m+1,n,j+1})$, $(\tilde{k}_{m+1,n,j}, \tilde{k}_{m+1,n,j+1})$ and $(\bar{k}_{m+1,n,M}, \tilde{k}_{m+1,n,1})$, so that Lemma 2 can be used here, obtaining

$$\begin{aligned} |\check{F}_n(x)| &\leqslant \mathscr{O}_{m,n}(\epsilon) + \tfrac{M_m}{m!}\sum_{j=0}^{M-1}\Big(|w_n^{(m)}(\bar{k}_{m+1,n,j},\epsilon)| + |w_n^{(m)}(\bar{k}_{m+1,n,j+1},\epsilon)|\Big) \\ &\quad + \tfrac{M_m}{m!}\Big(|w_n^{(m)}(\tilde{k}_{m+1,n,1},\epsilon)| + |w_n^{(m)}(\bar{k}_{m+1,n,M},\epsilon)|\Big) \\ &\quad + \tfrac{M_m}{m!}\sum_{j=1}^{m-1}\Big(|w_n^{(m)}(\tilde{k}_{n+1,n,j},\epsilon)| + |w_n^{(m)}(\tilde{k}_{m+1,n,j+1},\epsilon)|\Big) \\ &= \mathscr{O}_{m,n}(\epsilon) + \tfrac{M_m}{m!}|w_n^{(m)}(\bar{k}_{m+1,n,0},\epsilon)| + \tfrac{2M_m}{m!}\sum_{j=1}^{M}|w_n^{(m)}(\bar{k}_{m+1,n,j},\epsilon)| \\ &\quad + \tfrac{2M_m}{m!}\sum_{j=1}^{m-1}|w_n^{(m)}(\tilde{k}_{m+1,n,j},\epsilon)| + \tfrac{M_m}{m!}|w_n^{(m)}(\tilde{k}_{m+1,n,m},\epsilon)|. \end{aligned}$$

Since $w_n^{(m)}(\bar{k}_{m+1,n,0},\epsilon) = w_n^{(m)}(0,\epsilon) = \mathscr{O}_{m,n}(\epsilon)$ and $w_n^{(m)}(\tilde{k}_{m+1,n,m},\epsilon) = w_n^{(m)}(+\infty,\epsilon) = 0$, previous inequality becomes

$$|\check{F}_n(x)| \leqslant \mathscr{O}_{m,n}(\epsilon) + \tfrac{2M_m}{m!}\sum_{j=1}^{M}|w_n^{(m)}(\bar{k}_{m+1,n,j},\epsilon)| + \tfrac{2M_m}{m!}\sum_{j=1}^{m-1}|w_n^{(m)}(\tilde{k}_{m+1,n,j},\epsilon)|. \tag{8}$$

By Lemma 7 we know that each $2^{\bar{k}_{m+1,n,j}}\epsilon$ tends to zero. As a consequence, $\bar{k}_{m+1,n,j}(\epsilon) < \log_2(1/\epsilon)$ when ϵ is small enough. Using the explicit formula for $w_n^{(m)}(k,\epsilon)$ we found in Lemma 4, we have that

$$\begin{aligned} w_n^{(m)}(\bar{k}_{m+1,n,j},\epsilon) &\ll e^{-2^{\bar{k}_{m+1,n,j}(\epsilon)}\epsilon}\big((\bar{k}_{m+1,n,j}(\epsilon))^n\phi_m(-2^{\bar{k}_{m+1,n,j}(\epsilon)}\epsilon) + (\bar{k}_{m+1,n,j}(\epsilon))^{n-1}\big) \\ &\ll (\log(1/\epsilon))^n\phi_m(-2^{\bar{k}_{m+1,n,j}(\epsilon)}\epsilon) + (\log(1/\epsilon))^{n-1}. \end{aligned}$$

We note that $\phi_m(x)$ is divisible by x when $m > 0$, and we know that $2^{\bar{k}_{m+1,n,j}(\epsilon)}\epsilon \to 0$, hence the previous inequality gives

$$w_n^{(m)}(\bar{k}_{m+1,n,j},\epsilon) = o_{m,n}((\log(1/\epsilon))^n),$$

so that (8) becomes

$$|\check{F}_n(x)| \leqslant \mathscr{O}_{m,n}(\epsilon) + o_{m,n}((\log(1/\epsilon))^n) + \tfrac{2M_m}{m!}\sum_{j=1}^{m-1}|w_n^{(m)}(\tilde{k}_{m+1,n,j},\epsilon)|.$$

In this estimate the term $\mathscr{O}_{m,n}(\epsilon)$ is always dominated by the term $o_{m,n}(\cdot)$, therefore, we proved that

$$|\check{F}_n(x)| \leqslant o_{m,n}((\log(1/\epsilon))^n) + \tfrac{2M_m}{m!}\sum_{j=1}^{m-1}|w_n^{(m)}(\tilde{k}_{m+1,n,j},\epsilon)|.$$

By Lemma 6 we known that $\tilde{k}_{m+1,n,j} = \log_2(1/\epsilon) + \log_2 \zeta_{m+1,j} + o_{m,n}(1)$, so that using again the explicit formula for $w_n^{(m)}(k,\epsilon)$ we get

$$|w_n^{(m)}(\tilde{k}_{m+1,n,j},\epsilon)| \sim e^{-\zeta_{m+1,j}}|\phi_m(-\zeta_{m+1,j})| \log^m 2(\log_2(1/\epsilon))^n,$$

hence

$$|\check{F}_n(x)| \leqslant o_{m,n}((\log(1/\epsilon))^n) + \tfrac{2M_m \log^m 2}{m!} \sum_{j=1}^{m-1} e^{-\zeta_{m+1,j}}|\phi_m(-\zeta_{m+1,j})|(\log_2(1/\epsilon))^n.$$

The proof concludes introducing the upper-bounds of Lemma 3 for M_m in previous estimate. □

Previous lemma shows that the best bound for $|\check{F}_n|$ will be attaint by choosing m giving the smallest b_m. We can prove that such minimum exists. In fact, for every m there exists $\bar{\jmath}$ such that

$$|\phi_m(-\zeta_{m+1,\bar{\jmath}})|e^{-\zeta_{m+1,\bar{\jmath}}} = \max_{x\in(-\infty,0)} |\phi_m(x)|e^x,$$

therefore,

$$b_m \gg \left(\tfrac{\log 2}{2\pi}\right)^m \max_{(-\infty,0)} |\phi_m(x)|e^x \gg \left(\tfrac{\log 2}{2\pi}\right)^m |\phi_m(-1)|.$$

It is known that the growth of the Bell numbers $\phi_m(1)$ is over-exponential (see for example [7]). Modifying a little bit the argument in [7] it is possible to prove that also $\phi_m(-1)$ grows in over-exponential way so that $b_m \to +\infty$ as m diverges. Actually, a computation we performed by the `PARI` program shows that b_m decreases for $m \leqslant 13$ with $b_{13} \leqslant 4.49 * 10^{-6}$, then suddenly increases (Figure 1). As a consequence, it seams that the best choice we can make is $m = 13$. Choosing $m = 13$ in Lemma 8 the theorem follows.

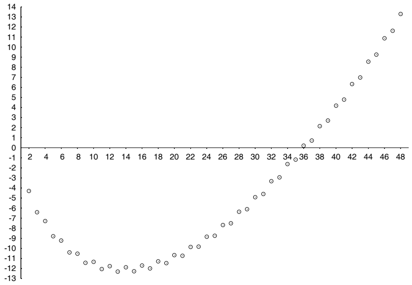

Figure 1. Values of $\log(b_m)$ *for* $1 < m < 50$.

A further improvement can be attaint renouncing to estimate M_m by Lemma 3 and using its exact value. We do not present the details of this computation since the formula we get in this way is much more involuted and the upper-bound we found in this way is only very weakly better than the previous one: $4.41 * 10^{-6}$ against previous $4.49 * 10^{-6}$.

3. What is the true behavior of $\check{F}_n$?

About $\check{F}_0$

Computations we performed using the arbitrary precision arithmetic of PARI show that $\check{F}_0(x)$ oscillates with an amplitude which is practically constant over the range $10^{-57} < 1 - x < 10^{-9}$ (Figure 2, Left). The amplitude of these oscillations is $1.5 * 10^{-6}$, i.e., exactly of the order of the upper-bound we proved in previous section. Figure 2 (Left) suggests that data will be probably better displayed if a logarithmic scale will be used for the abscissa. In fact, when the abscissa is $L(x)$ the graphs appear almost sinusoidal (Figure 2, Right): the Fourier transform of these numerical data shows a very high peak close to the frequency 1 (Figure 3). Roughly speaking, therefore, Figures 2 and 3 suggest that

$$\alpha \sin(2\pi L(x) + \beta), \quad \text{with} \quad \alpha \approx 1.58 * 10^{-6}, \quad \beta \text{ suitable},$$

provide an excellent fit for $\check{F}_0(x)$ as $x \to 1^-$.

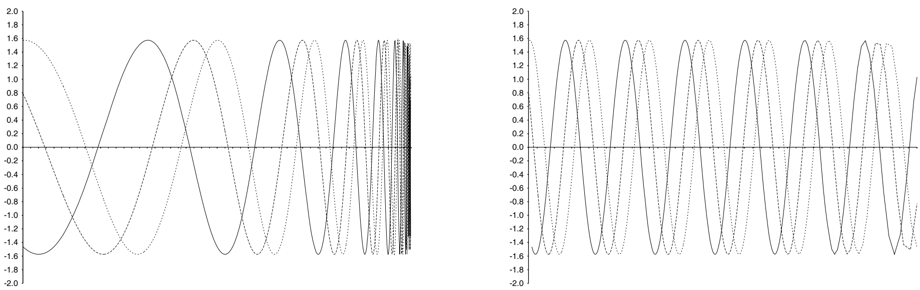

Figure 2. (Left) The overlapped graphes of $\check{F}_0(x)$ in different ranges: (continue line) $10^{-15} < 1 - x < 10^{-18}$, (dashed line) $10^{-39} < 1 - x < 10^{-42}$, (small dashes line) $10^{-54} < 1 - x < 10^{-57}$. The ordinata has been magnified by a factor 10^6. (Right) Same data, but logarithmic abscissa.

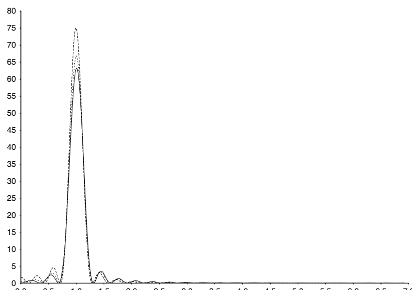

Figure 3. Numerical Fourier transform of $\check{F}_0(x)$ in different ranges: (continue line) $10^{-15} < 1 - x < 10^{-18}$, (dashed line) $10^{-39} < 1 - x < 10^{-42}$, (small dashes line) $10^{-54} < 1 - x < 10^{-57}$.

About $\check{F}_1$

The computations reveal two interesting and unexpected aspects. The first one is that $\check{F}_1(x)/L(x)$ is still of order 10^{-3} when $1-x \approx 10^{-230}$ (see Figure 4). This fact means that such range cannot yet be considered as 'close enough' to 1: by Lemma 8, in fact, we know that $\check{F}_1(x)/L(x)$ must be lower than $4.49 * 10^{-6}$ when $1-x$ is small enough.

The second one is that oscillations are present also for $1-x \approx 10^{-130}$, but being of order $\approx 10^{-6}$ they are completely hidden by the main term which is still of order $\approx 10^{-3}$, as we told. The oscillations reveal them-self if we remove from the data their linear baseline, see Figure 5.

At last, we note that the oscillating graphs we produced appear again as simple sinusoids: in fact, the Fourier transform of the oscillating part of $\check{F}_1(x)/L(x)$ shows a strong peak around the frequency ≈ 1 (see Figure 6): the same frequency we already noted for $\check{F}_0(x)$. As we will see in next paragraph, this fact can be heuristically explicated as a consequence of the functional equation $F_0(x) = F_1(x) - F_1(x^2) + x$.

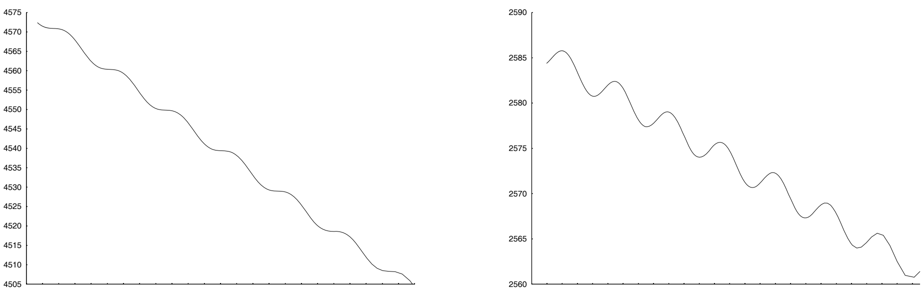

Figure 4. Graph of $\check{F}_1(x)/L(x)$ when $10^{-133} < 1-x < 10^{-130}$ (left) and $10^{-233} < 1-x < 10^{-230}$ (right). Logarithmic abscissa. The ordinata has been magnified by a factor 10^6.

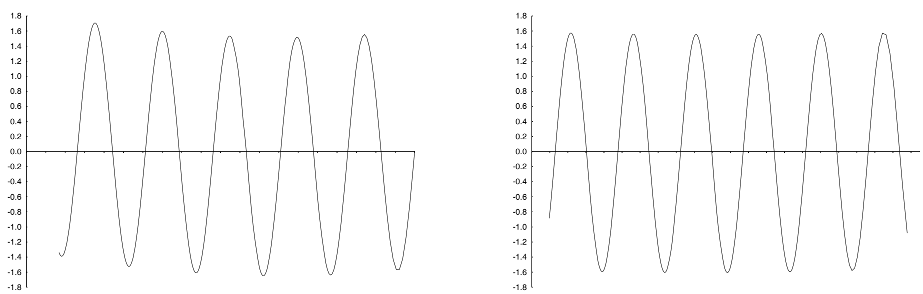

Figure 5. Oscillating part of $\check{F}_1(x)/L(x)$ when $10^{-133} < 1-x < 10^{-130}$ (left) and $10^{-233} < 1-x < 10^{-230}$ (right). Logarithmic abscissa. The ordinata has been magnified by a factor 10^6.

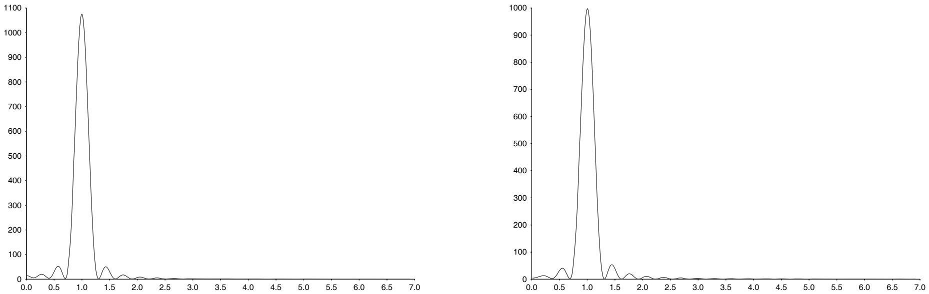

Figure 6. Fourier transform of the oscillating part of $\check{F}_1(x)/L(x)$ *in different ranges:* $10^{-133} < 1-x < 10^{-130}$ *(left) and* $10^{-233} < 1-x < 10^{-230}$ *(right).*

About $\check{F}_n$, $n > 1$.

We do not have done any numerical test for $\check{F}_n(x)$ with $n > 1$, but we can formulate some conjectures. In fact, functions F_n satisfy the relation

$$\sum_{m=0}^{n}(-1)^m\binom{n}{m}F_n(x^{2^m}) = n!F_0(x) - \sum_{m=1}^{n}\sum_{k=0}^{m-1}(-1)^m\binom{n}{m}(k-m)^n x^{2^k}$$

relating F_n directly with F_0. As $x \to 1^-$ this identity gives

$$\sum_{m=0}^{n}(-1)^m\binom{n}{m}\check{F}_n(x^{2^m}) = n!\check{F}_0(x) + o(1), \tag{9}$$

thus, it is quite natural to expect that $\check{F}_n(x)/L^n(x)$ oscillates with the same mean frequency of $\check{F}_0(x)$. Formula (9) provides also an interesting, heuristic argument explicating why this frequency is so near an integer. Let us assume that (in some heuristic meaning)

$$\check{F}_n(x) \approx \alpha L^n(x)\sin(2\pi\nu L(x) + \beta)$$

for some constants α, ν, β, as $x \to 1^-$. Then

$$\check{F}_n(x^{2^m}) \approx \alpha L^n(x)\sin(2\pi\nu L(x) + 2\pi m\nu + \beta + o(1)),$$

so that from (9) we get

$$\begin{aligned} n!\check{F}_0(x) + o(1) &\approx \alpha L^n(x)\Big(\sum_{m=0}^{n}(-1)^m\binom{n}{m}\sin(2\pi\nu L(x) + 2\pi m\nu + \beta + o(1))\Big) \\ &\approx \alpha L^n(x)\cdot \operatorname{Im}\Big[e^{2\pi i\nu L(x)+i\beta+o(1)}(1-e^{2\pi i\nu})^n\Big]. \end{aligned}$$

By this formula we see that ν must be an integer, otherwise R.H.S. has oscillations of order $L^n(x)$ whereas we know that L.H.S. is bounded.

Acknowledgement. I wish to thank prof. Vigna who pointed to my attention this problem.

REFERENCES

[1] P. BOLDI AND S. VIGNA, *Codes for The Word-Wide Web*, Internet Math. **2** (2005), no. 4, 407–429.
[2] ———, *The Webgraph Framework I: Compression Techniques*, Proc. of the Thirteenth International World Wide Web Conference (Manhattan, USA), ACM Press, 2004, pp. 595–601.
[3] T. J. BROMWICH, *An introduction to the theory of infinite series, 3rd ed.*, Chelsea, New York, 1991.
[4] H. DELANGE, *Sur les zéros réels des polynômes de Bernoulli*, C. R. Acad. Sci. Paris Sér. I Math. **303** (1986), no. 12, 539–542.
[5] ———, *Sur les zéros réels des polynômes de Bernoulli*, Ann. Inst. Fourier (Grenoble) **41** (1991), no. 2, 267–309.
[6] D. H. LEHMER, *On the maxima and minima of Bernoulli polynomials*, Amer. Math. Monthly **47** (1940), 533–538.
[7] L. LOVÁSZ, *Combinatorial problems and exercises*, second ed., North-Holland Publishing Co., Amsterdam, 1993.
[8] S. ROMAN, *The umbral calculus*, Pure and Applied Mathematics, vol. 111, Academic Press Inc., New York, 1984.
[9] G. TENENBAUM, *Introduction to analytic and probabilistic number theory*, Cambridge Studies in Advanced Mathematics, vol. 46, Cambridge University Press, Cambridge, 1995, Translated from the second French edition (1995) by C. B. Thomas.

(Received March 31, 2007)

G. Molteni
Dipartimento di Matematica
Università di Milano
Via Saldini 50
I-20133 Milano
Italy
e-mail: giuseppe.molteni@mat.unimi.it