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# CLASSIFIED CONSTRUCTION OF GENERALIZED FURUTA TYPE OPERATOR FUNCTIONS

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*Abstract.* A kind of construction of generalized Furuta type operator functions $F_{p,t,q}(r,s) = A^{-r/2}(A^{r/2}(A^{t/2}B^pA^{t/2})^sA^{r/2})^{(q+r)/((p+t)s+r)}A^{-r/2}$ is introduced. It is based on the classification of the functions $F_{p,t,q}(r,s)$ according to the existence of Furuta inequality. It is showed that all such functions before can be generated by this construction and some of them are sharpened. Also, characterizations of operator order $A \geqslant B \geqslant 0$ and chaotic order $\log A \geqslant \log B$ are obtained which extend the related results before.

## 1. Introduction

A capital letter (such as $T$) means a bounded linear operator on a Hilbert space. $T \geqslant 0$ and $T > 0$ mean a positive operator and an invertible positive operator respectively.

As an essential extension of the celebrated Löwner-Heinz inequality: $A \geqslant B \geqslant 0$ ensures $\Rightarrow A^\alpha \geqslant B^\alpha$ for each $\alpha \in [0,1]$, Furuta [14] showed the following operator inequality.

THEOREM 1.1. (Furuta inequality, [14]) *If $A \geqslant B \geqslant 0$, then for each $r \geqslant 0$,*

$$\left(B^{r/2}A^pB^{r/2}\right)^{1/q} \geqslant \left(B^{r/2}B^pB^{r/2}\right)^{1/q}, \tag{1.1}$$

$$\left(A^{r/2}A^pA^{r/2}\right)^{1/q} \geqslant \left(A^{r/2}B^pA^{r/2}\right)^{1/q} \tag{1.2}$$

*as long as real numbers $p$, $r$, $q$ satisfy*

$$p \geqslant 0, \quad q \geqslant 1 \text{ with } (1+r)q \geqslant p+r. \tag{1.3}$$

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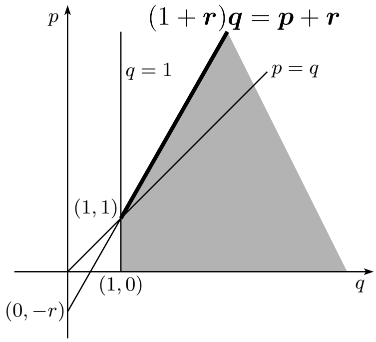


*Domain of Furuta inequality.*

[15] provided an element proof of Theorem 1.1. Kamei [32] gave the first improvement of Furuta inequality. Tanahashi [36] showed that, for each $r \geqslant 0$, condition (1.3) is optimal for the validity of Furuta inequality. It's well known that Furuta inequality has many applications. See [1], [2], [5], [18], [29], [47] and [48].

Yoshino [45] initiated an attempt to extend the domain in which Furuta inequality holds. Tanahashi [37] obtained Furuta inequality with negative powers stated below.

THEOREM 1.2. ([6, 27, 37, 45]) *For each $r < 0$, $A \geqslant B \geqslant 0$ with $A > 0$ implies each of the following inequalities.*

(1) $A^{1+r} \geqslant \left(A^{r/2}B^pA^{r/2}\right)^{(1+r)/(p+r)}$ *when* $1 \geqslant p > -r \geqslant 0$ *and* $p \geqslant 1/2$.

(2) $A^r \geqslant \left(A^{r/2}B^pA^{r/2}\right)^{r/(p+r)}$ *when* $1 \geqslant -r > p \geqslant 0$ *and* $1/2 \geqslant p$.

(3) $A^{2p+r} \geqslant \left(A^{r/2}B^pA^{r/2}\right)^{(2p+r)/(p+r)}$ *when* $1/2 \geqslant p > -r \geqslant 0$.

(4) $A^{2p-1+r} \geqslant \left(A^{r/2}B^pA^{r/2}\right)^{(2p-1+r)/(p+r)}$ *when* $1 \geqslant -r > p \geqslant 1/2$.

(5) $A^{p+r} \geqslant A^{r/2}B^pA^{r/2}$ *when* $-1 > r, 1 \geqslant p > 0$.

(6) $A^{(p+r)/q} \geqslant \left(A^{r/2}B^pA^{r/2}\right)^{1/q}$ *when* $1 \geqslant -r > 0, p+r = 0$ *and* $q > 0$.

Extensions of Theorem 1.2 are shown in [7, 8, 20, 30]. Afterwards, the domain given by [14] was enlarged to (1) by Fujii-Kamei-Furuta [6]. [33] gave simplified proofs of (1) and (3). [37] showed that the outer exponents of (1)–(2), (4)–(6) are the best possible. But it has not been proved yet whether the outer exponent of (3) is the best possible or not. [27] proved equivalence relations among these inequalities and gave a concrete counter example of case $1+r$ of the outer exponent of (3). Still, the domain of (3) is called *mysterious delta zone* (in $rp$-plane).

As further development of Furuta inequality, many researchers discuss the following Furuta type operator functions with monotone property. They play an important role in the theory of $p$-hyponormal operators. We cite [39, 42]. They have been generalized to the following.

THEOREM 1.3. ([3, 5, 16, 17]) *Let $A, B \geqslant 0$ such that $A^t \geqslant B^t$ for $t \geqslant 0$ ($A > 0$, $B > 0$ and $\log A \geqslant \log B$ if $t = 0$). Then for each $r > 0$ and $s > -r$ the following hold.*

(1) $f_{r,s}(p) = \left(A^{r/2}B^pA^{r/2}\right)^{(s+r)/(p+r)}$ *is decreasing for* $p \geqslant \max\{s,0\}$.

(2) $g_{r,s}(p) = \left(B^{r/2}A^pB^{r/2}\right)^{(s+r)/(p+r)}$ *is increasing for* $p \geqslant \max\{s,0\}$.

See [46] for further properties of such functions. As a highlight of such discussion, [19] showed the following generalized Furuta type operator functions which interpolates the Furuta inequalities (as extremal case $t = 0$ in (1.4)) and Ando- Hiai inequality [4] (as extremal case $t = -1$ and $r = s$ in (1.4)).

THEOREM 1.4. (grand Furuta inequality, [10, 19, 23]) *If $A \geqslant B \geqslant 0$ with $A > 0$, then for each $p \geqslant 1$ and $t \in [-1, 0]$, the function*

$$F_{p,t,1+t}(r,s) = A^{-r/2}\left(A^{r/2}(A^{t/2}B^pA^{t/2})^sA^{r/2}\right)^{(1+t+r)/((p+t)s+r)}A^{-r/2}$$

*is decreasing for both $r \geqslant -t$ and $s \geqslant 1$. In particular, the inequality*

$$A^{1+t+r} \geqslant \left(A^{r/2}(A^{t/2}B^pA^{t/2})^sA^{r/2}\right)^{(1+t+r)/((p+t)s+r)} \tag{1.4}$$

*holds for $r \geqslant -t$ and $s \geqslant 1$*

It is interesting that the outer exponent $1+t+r$ of the inequality (1.4) obtained in Theorem 1.4 is also the best possible similar to that of Furuta inequality in Theorem 1.1 (see [13, 38, 40]).

Another useful and interesting form of such functions (stated below) appeared in [21]. They are important to research several problems associated with operator functions, for instance, [22].

THEOREM 1.5. ([21]) *If $A \geqslant B \geqslant 0$ with $A > 0$, then for each $p \geqslant 1$ and $t \geqslant 0$, the function*

$$F_{p,t,1+t}(r,s) = A^{-r/2}\left(A^{r/2}(A^{t/2}B^pA^{t/2})^sA^{r/2}\right)^{(1+t+r)/((p+t)s+r)}A^{-r/2}$$

*is increasing for $s$ such that $1 \geqslant s \geqslant \frac{1+t}{p+t}$ and decreasing for $r$ such that $0 \geqslant r \geqslant -t$.*

The original authors took hard work to discuss the form of such functions similar to Theorem 1.4 and Theorem 1.5 because they are varied and complex.

In this paper, we will introduce a kind of classified construction (or a method) to obtain such operator functions. It is showed that all such functions before can be generated by our construction and some of them can be improved greatly.

For instance, it is showed that, for each $-(1+t) < r \leqslant 0$, $F_{p,t,1+t}(r,s)$ in Theorem 1.5 is increasing for $s$ such that $\infty > s \geqslant \frac{1+t}{p+t}$. See Theorem 4.2 (1) for detail.

Also, characterizations of operator order $A \geqslant B \geqslant 0$ and chaotic order $\log A \geqslant \log B$ are obtained which extend the related results before ([11, 21, 22]).

For convenience, the $\alpha$-power mean $\sharp_\alpha$, a typical operator mean in the Kubo-Ando theory [35], is used:

$$A\sharp_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}$$

for $0 \leqslant \alpha \leqslant 1$, $A > 0$ and $B \geqslant 0$. $\natural_s$ for $s \in R$ is defined by

$$A\natural_s B = A^{1/2}(A^{-1/2}BA^{-1/2})^s A^{1/2}$$

for $A > 0$ and $B \geqslant 0$, see [19].

## 2. Main idea and steps of classified construction

Main idea

- Classification. According to the existence of Furuta inequality (see [37]), all functions $F_{p,t,q}(r,s)$ can be divided into class (F) and class (NF), that is,
  - (F) call a function $F_{p,t,q}(r,s)$ belongs to class (F) if Furuta inequality $(A^{t/2}B^pA^{t/2})^{1/\alpha} \leqslant A^{\frac{p+t}{\alpha}}$ is valid for some $\alpha \neq 0$, it is only need to consider the case $t<0$, $1 \geqslant p > 0$ and the case $t \geqslant 0$, $p \geqslant 0$,
  - (NF) call a function $F_{p,t,q}(r,s)$ belongs to class (NF) if Furuta inequality $(A^{t/2}B^pA^{t/2})^{1/\alpha} \leqslant A^{\frac{p+t}{\alpha}}$ is not valid for any $\alpha \neq 0$, it is only need to consider the case $t<0$, $p>1$;
- Simplification. According to the classification of $F_{p,t,q}(r,s)$, $F_{p,t,q}(r,s)$ can be regarded as Furuta type functions (such as $f_{r,s}(p)$ in Theorem 1.3) by selecting a proper part in $F_{p,t,q}(r,s)$ as a unit to deal with. In general, we select $(A^{t/2}B^pA^{t/2})^{1/(p+t)}$ as a unit if $F_{p,t,q}(r,s)$ belongs to class (F) and $\big(A^{-t/2}(A^{t/2}B^pA^{t/2})^sA^{-t/2}\big)^{1/((p+t)s-t)}$ as a unit if $F_{p,t,q}(r,s)$ belongs to class (NF).

By using means of operators, the function

$$F_{p,t,q}(r,s) = A^{-r/2}\big(A^{r/2}(A^{t/2}B^pA^{t/2})^sA^{r/2}\big)^{(q+r)/((p+t)s+r)}A^{-r/2}$$

can be rewritten as

$$\begin{aligned} F_{p,t,q}(r,s) &= A^{-r}\sharp_{(q+r)/((p+t)s+r)}(A^{t/2}B^pA^{t/2})^s \\ &= A^{t/2}\big(A^{-(r+t)}\sharp_{(q+r)/((p+t)s+r)}(A^{-t}\natural_s B^p)\big)A^{t/2}. \end{aligned} \tag{2.1}$$

Construction of class (F)

(I) Denote $A_1 = A$, $B_1 = (A^{t/2}B^pA^{t/2})^{1/(p+t)}$, then there exist a real number $\alpha \neq 0$ so that $A_1^{\alpha} \geqslant B_1^{\alpha}$.

(II) By applying the related results of Furuta type functions to $A_1^{\alpha} \geqslant B_1^{\alpha}$, we obtain the monotonicity of the variables (at least one) in $F_{p,t,q}(r,s)$. Such as:
(1) if Theorem 2.1 (3) is used (stated below), construction is over.
(2) if Theorem 2.3 is used, only the monotonicity of $s$ (in $F_{p,t,q}(r,s)$) is obtained (and the next stop should be done).

(III) If the construction is not completed by (II). By applying Furuta's classical steps (p144, [26]), the construction is completed.

Construction of class (NF)

(I) Denote $B_1(s) = (A^{-t}\natural_s B^p)^{1/((p+t)s-t)}$, then try to find a fixed $s_0$ so that $A \geqslant B_1(s_0)$.

(II) By applying Theorem 2.6 to $A \geqslant B_1(s_0)$, we obtain the monotonicity of $B_1(s)$ for $s \geqslant s_0$ (or $s \leqslant s_0$), thus $A \geqslant B_1(s)$ when $s \geqslant s_0$ (or $s \leqslant s_0$).

(III) By applying the related results of Furuta type functions to $A \geqslant B_1(s)$, we obtain the monotonicity of r in $F_{p,t,q}(r,s)$. Warning: the monotonicity of s in $F_{p,t,q}(r,s)$ can not be obtained because $B_1(s)$ is regarded as a fixed operator now.

(IV) By applying Furuta's classical steps (p144, [26]), the monotonicity of $s$ is obtained. Construction is over.

In section 3–5, we will give some examples of the construction. Special attention is paid to the form of functions similar to Theorem 1.4 and 1.5. For convenience, we write down the following results which will be used in section 3–5.

Theorem 1.3 (1) can be improved into the following form. It will be used in section 3 and section 5.

THEOREM 2.1. *Let* $A \geqslant 0$, $B \geqslant 0$ *such that* $A^t \geqslant B^t$ *for* $t \geqslant 0$ *(* $A > 0$, $B > 0$ *and* $\log A \geqslant \log B$ *if* $t = 0$ *). Then the following hold.*

(1) *For each* $r > 0$ *and* $s > -r$, $f_{r,s}(p) = \left(A^{r/2}B^pA^{r/2}\right)^{(s+r)/(p+r)}$ *is decreasing for* $p \geqslant \max\{s, 0\}$.

(2) *For each* $p > 0$ *and* $s < p$, $f_{p,s}(r) = A^{-r}\sharp_{(s+r)/(p+r)}B^p$ *is decreasing for* $r \geqslant \max\{-s, 0\}$.

(3) *For each* $s \in R$, $f_s(p,r) = A^{-r}\sharp_{(s+r)/(p+r)}B^p$ *is decreasing for* $p \geqslant \max\{s, 0\}$ *and* $r \geqslant \max\{-s, 0\}$.

LEMMA 2.2. ([16, 19]) *Let* $\alpha \in R$ *and* $X$ *be invertible. Then* $(X^*X)^\alpha = X^*(XX^*)^{\alpha-1}X$, *especially if* $\alpha \geqslant 1$ *the equality holds without invertibility of* $X$.

THEOREM 2.3. ([44]) *Let* $A \geqslant B \geqslant 0$ *with* $A > 0$. *Then for each* $-1 \leqslant r < 0$,

(1) *if* $s > -r$, $f_{r,s}(p) = \left(A^{r/2}B^pA^{r/2}\right)^{(s+r)/(p+r)}$ *is increasing for* $p \in (-\infty, -2r - s] \cup [s, \infty)$.

(2) *if* $s < -r$, $f_{r,s}(p) = \left(A^{r/2}B^pA^{r/2}\right)^{(s+r)/(p+r)}$ *is decreasing for* $p \in (-\infty, s] \cup [-2r - s, \infty)$.

Theorem 2.3 will be used in section 4. By an observation, the following is equivalent to Theorem 2.3.

THEOREM 2.4. *Let* $A \geqslant B \geqslant 0$ *with* $A > 0$. *Then for each* $0 < r \leqslant 1$,

(1) *if* $s > -r$, $f_{r,s}(p)$ *is decreasing for* $p \in (-\infty, -2r - s] \cup [s, \infty)$.

(2) *if* $s < -r$, $f_{r,s}(p)$ *is increasing for* $p \in (-\infty, s] \cup [-2r - s, \infty)$.

THEOREM 2.5. ([29]) *Let* $A, B \geqslant 0$. *Then for each* $p, r \geqslant 0$, *the following assertions hold:*

(1) $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geqslant B^r \Rightarrow (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}} \leqslant A^p$.

(2) $(A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}} \leqslant A^p$ *and* $N(A) \subset N(B) \Rightarrow (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geqslant B^r$.

Theorem 2.5 will be used in section 5. The following result is a generalization of Theorem 1.3 because of Furuta inequality.

THEOREM 2.6. ([41, 43]) *Let* $A, B \geqslant 0$; $\alpha_0$, $\beta_0 > 0$; $-\beta_0 < \delta_0 \leqslant \alpha_0$, $-\beta_0 \leqslant \overline{\delta}_0 < \alpha_0$. *Then the following assertions hold:*

(1) *If* $(B^{\frac{\beta_0}{2}}A^{\alpha_0}B^{\frac{\beta_0}{2}})^{\frac{\beta_0+\delta_0}{\beta_0+\alpha_0}} \geqslant B^{\beta_0+\delta_0}$, *then* $(B^{\frac{\beta}{2}}A^{\alpha_0}B^{\frac{\beta}{2}})^{\frac{\beta+\delta_0}{\beta+\alpha_0}} \geqslant B^{\beta+\delta_0}$ *for any* $\beta \geqslant \beta_0$. *Moreover, for each fixed* $\delta_0' \leqslant \alpha_0$, *the function*

$$f_{\alpha_0,\delta_0'}(\beta) = (A^{\frac{\alpha_0}{2}}B^\beta A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0-\delta_0'}{\alpha_0+\beta}}$$

*is a decreasing function for* $\beta \geqslant \max\{\beta_0, -\delta_0'\}$, *thus* $A^{\frac{\alpha_0}{2}}B^{\beta_1}A^{\frac{\alpha_0}{2}} \geqslant (A^{\frac{\alpha_0}{2}}B^{\beta_2}A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0+\beta_1}{\alpha_0+\beta_2}}$ *for* $\beta_2 \geqslant \beta_1 \geqslant \beta_0$.

(2) *If* $A^{\alpha_0-\overline{\delta}_0} \geqslant (A^{\frac{\alpha_0}{2}}B^{\beta_0}A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0-\overline{\delta}_0}{\alpha_0+\beta_0}}$, *then* $A^{\alpha-\overline{\delta}_0} \geqslant (A^{\frac{\alpha}{2}}B^{\beta_0}A^{\frac{\alpha}{2}})^{\frac{\alpha-\overline{\delta}_0}{\alpha+\beta_0}}$ *for any* $\alpha \geqslant \alpha_0$. *Moreover, for each fixed* $\overline{\delta}_0' \geqslant -\beta_0$, *the function*

$$g_{\beta_0,\overline{\delta}_0'}(\alpha) = (B^{\frac{\beta_0}{2}}A^{\alpha}B^{\frac{\beta_0}{2}})^{\frac{\beta_0+\overline{\delta}_0'}{\beta_0+\alpha}}$$

*is an increasing function for* $\alpha \geqslant \max\{\alpha_0, \overline{\delta}_0'\}$, *thus* $B^{\frac{\beta_0}{2}}A^{\alpha_1}B^{\frac{\beta_0}{2}} \leqslant (B^{\frac{\beta_0}{2}}A^{\alpha_2}B^{\frac{\beta_0}{2}})^{\frac{\beta_0+\alpha_1}{\beta_0+\alpha_2}}$ *for* $\alpha_2 \geqslant \alpha_1 \geqslant \alpha_0$.

Case $\delta_0 = \overline{\delta}_0 = 0$ of Theorem 2.6 appeared in [41], the complete form was obtained by [43]. It plays an important role in the discussion of powers of operators. See [41, 48]. Here this theorem will be used in section 5.

## 3. Examples of the construction of (F), I

This section is to show some functions $F_{p,t,q}(r,s)$ that are obtained by step (I)-(II) of the construction of (F) only.

THEOREM 3.1. *Let* $A \geqslant B \geqslant 0$ *with* $A > 0$ *and* $q \in R$, *then the following hold.*

(1) *If* $t \geqslant 0$ *and* $p > 0$, *the function*

$$F_{p,t,q}(r,s) = A^{-r}\natural_{(q+r)/((p+t)s+r)}(A^{t/2}B^pA^{t/2})^s$$

*is decreasing for both* $r \geqslant \max\{0,-q\}$ *and* $s \geqslant \max\{0, \frac{q}{p+t}\}$.

(2) *If* $t < 0$, $1 \geqslant p > 0$ *and* $p > -t$, *the function*

$$F_{p,t,q}(r,s) = A^{-r}\natural_{(q+r)/((p+t)s+r)}(A^{t/2}B^pA^{t/2})^s$$

*is decreasing for both* $r \geqslant \max\{0,-q\}$ *and* $s \geqslant \max\{0, \frac{q}{p+t}\}$.

(3) *If* $t < 0$, $1 \geqslant p > 0$ *and* $p < -t$, *the function*

$$F_{p,t,q}(r,s) = A^{-r}\natural_{(q+r)/((p+t)s+r)}(A^{t/2}B^pA^{t/2})^s$$

*is increasing for both* $r \geqslant \max\{0,-q\}$ *and* $s \leqslant \min\{0, \frac{q}{p+t}\}$.

[20] showed Theorem 3.1 (1)–(2) under some condition on $q \in R$, the complete Theorem 3.1 (1)–(2) appeared in [31] in form of means of operators. Here we use the construction of (F) to obtain them. Theorem 3.1 (3) do not seem to has been obtained before. It can be regarded as a parallel result and a natural complementary to Theorem 3.1 (1)–(2).

We remark that the roles of $r$ and $s$ in theorem 3.1 (1)–(2) are similar to that of $r$ and $p$ in theorem 2.1 (3) respectively.

*Proof of Theorem 3.1.* We prove (1)–(3) in order.

Proof of (1).

(I) Denote $A_1 = A$, $B_1 = (A^{t/2}B^pA^{t/2})^{1/(p+t)}$, then $A_1^{\alpha} \geqslant B_1^{\alpha}$ where $\alpha = \min\{p,1\} + t > 0$ by Theorem 1.1.

(II) By Theorem 2.1 (3), the function

$$F_{p,t,q}(r,s) = A^{-r}\natural_{(q+r)/((p+t)s+r)}(A^{t/2}B^pA^{t/2})^s = A_1^{-r}\natural_{\frac{q+r}{p_1+r}}B_1^{p_1}$$

where $p_1 = (p+t)s$ is decreasing for both $r \geqslant \max\{0,-q\}$ and $p_1 \geqslant \max\{0,q\}$, that is, $s \geqslant \max\{0,\frac{q}{p+t}\}$.

Proof of (2).

(I) Denote $A_1 = A$, $B_1 = (A^{t/2}B^pA^{t/2})^{1/(p+t)}$, then $A_1^{\alpha} \geqslant B_1^{\alpha}$ where $\alpha = p+t > 0$.

(II) By Theorem 2.1 (3), the function

$$F_{p,t,q}(r,s) = A_1^{-r}\natural_{\frac{q+r}{p_1+r}}B_1^{p_1}$$

where $p_1 = (p+t)s$ is decreasing for both $r \geqslant \max\{0,-q\}$ and $p_1 \geqslant \max\{0,q\}$, that is, $s \geqslant \max\{0,\frac{q}{p+t}\}$.

Proof of (3).

(I) Denote $A_1 = A$, $B_1 = (A^{t/2}B^pA^{t/2})^{1/(p+t)}$, then $A_1^{\alpha} \geqslant B_1^{\alpha}$ where $\alpha = p+t < 0$.

(II) By Theorem 2.1 (3), the function

$$F_{p,t,q}(r,s) = A_1^{-r}\natural_{\frac{q+r}{p_1+r}}B_1^{p_1}$$

where $p_1 = (p+t)s$ is increasing for both $r \geqslant \max\{0,-q\}$ and $p_1 \geqslant \max\{0,q\}$, that is, $s \leqslant \min\{0,\frac{q}{p+t}\}$ for $p+t<0$. □

It is obvious that Theorem 3.1 (1) holds if the operators $A$ and $B$ satisfy the condition in Theorem 2.1.

Fujii et al. [11] showed the following characterizations of chaotic order $\log A \geqslant \log B$ and operator order $A \geqslant B \geqslant 0$ via generalized Furuta inequalities.

THEOREM 3.2. ([11]) *For $A,\ B>0$, $\log A \geqslant \log B$ if and only if*

$$A^{r+t} \geqslant \left(A^{r/2}(A^{t/2}B^pA^{t/2})^sA^{r/2}\right)^{(r+t)/((p+t)s+r)}$$

*holds for $p \geqslant 0$, $t \geqslant 0$, $r \geqslant 0$ and $s \in [1,2]$.*

THEOREM 3.3. ([11]) *For $A,\ B>0$, $A \geqslant B$ if and only if*

$$A^{1+r+t} \geqslant \left(A^{r/2}(A^{t/2}B^pA^{t/2})^sA^{r/2}\right)^{(1+r+t)/((p+t)s+r)}$$

*holds for $p \geqslant 1$, $t \geqslant 0$, $r \geqslant 0$ and $s \in [1,2]$.*

See [9, 12, 25] for related topics. Although the proof of Theorem 3.1 is short and not difficult, it can be utilized to extend the results above greatly.

THEOREM 3.4. *For $A,\ B>0$, $\log A \geqslant \log B$ if and only if*

$$A^{r+t} \geqslant \left(A^{r/2}(A^{t/2}B^pA^{t/2})^sA^{r/2}\right)^{(r+t)/((p+t)s+r)} \tag{3.1}$$

*holds for $p>0$, $t \geqslant 0$, $r \geqslant 0$ and $s \geqslant \frac{t}{p+t}$.*

THEOREM 3.5. *For $A>0$, $B\geqslant 0$, $A\geqslant B$ if and only if*

$$A^{1+t+r}\geqslant \left(A^{r/2}(A^{t/2}B^pA^{t/2})^sA^{r/2}\right)^{(1+t+r)/((p+t)s+r)} \tag{3.2}$$

*holds for $p\geqslant 1$, $t\geqslant 0$, $r\geqslant 0$ and $s\geqslant \frac{1+t}{p+t}$.*

We remark that Theorem 3.4 can be regarded as a parallel result to Theorem 3.5 by contrasting (3.1) with (3.2).

*Proof of Theorem 3.4.* Obviously, the part "only if" is essential and we only prove this part. By Theorem 3.1 (1) and Furuta inequality under chaotic order [5],

$$F_{p,t,t}(r,s)\leqslant F_{p,t,t}(0,\frac{t}{p+t})=(A^{t/2}B^pA^{t/2})^{\frac{t}{p+t}}\leqslant A^t$$

holds for $p>0$, $t\geqslant 0$, $r\geqslant 0$ and $s\geqslant \frac{t}{p+t}$. Hence, (3.1) follows. □

*Proof of Theorem 3.5.* The proof is similar to Proof of Theorem 3.4. By Theorem 3.1 (1) and Furuta inequality,

$$F_{p,t,1+t}(r,s)\leqslant F_{p,t,1+t}(0,\frac{1+t}{p+t})=(A^{t/2}B^pA^{t/2})^{\frac{1+t}{p+t}}\leqslant A^{1+t}$$

holds for $p\geqslant 1$, $t\geqslant 0$, $r\geqslant 0$ and $s\geqslant \frac{1+t}{p+t}$. Hence, (3.2) follows. □

## 4. Examples of the construction of (F), II

This section is to show some functions $F_{p,t,q}(r,s)$ that are obtained by step (I)–(III) of the construction of (F).

As a chaotic order version of Theorem 1.5, Furuta [22] showed the following generalized operator function.

THEOREM 4.1. ([22]) *If $A,B>0$ with $\log A\geqslant \log B$, then for each $t>0$ and $p>0$, the function*

$$F_{p,t,t}(r,s)=A^{-r}\natural_{(t+r)/((p+t)s+r)}(A^{t/2}B^pA^{t/2})^s$$

*is decreasing for $0\geqslant r\geqslant -t$ and increasing for $\frac{t}{p+t}\leqslant s\leqslant 1$.*

Here we show the following Theorem 4.2 and 4.3 which are extensions of Theorem 1.5 and Theorem 4.1 respectively by construction of (F).

THEOREM 4.2. *Let $A\geqslant B\geqslant 0$ with $A>0$, $t\geqslant 0$ and $p\geqslant 1$, then the following hold.*

(1) *For each $-(1+t)\leqslant r\leqslant 0$ and $q+r>0$, the function*

$$F_{p,t,q}(r,s)=A^{-r}\natural_{(q+r)/((p+t)s+r)}(A^{t/2}B^pA^{t/2})^s$$

*is increasing for $s\geqslant \frac{q}{p+t}$.*

(2) *For each* $p+t \geqslant q \geqslant t$ *and* $\frac{q}{p+t} \leqslant s \leqslant 1$, *the function*

$$F_{p,t,q}(r,s) = A^{-r} \natural_{(q+r)/((p+t)s+r)} (A^{t/2}B^pA^{t/2})^s$$

*is decreasing for* $0 \geqslant r \geqslant -t$.

(3) *For each* $p+t \geqslant q \geqslant t$, *the function*

$$F_{p,t,q}(r,s) = A^{-r} \natural_{(q+r)/((p+t)s+r)} (A^{t/2}B^pA^{t/2})^s$$

*is decreasing for* $0 \geqslant r \geqslant -t$ *and increasing for* $\frac{q}{p+t} \leqslant s \leqslant 1$.

Theorem 4.2 (3) implies Theorem 1.5 (a generalization of Kamei [34]) by putting $q = 1+t$. Theorem 4.1 (1) implies that the monotone interval $[(1+t)/(p+t), 1]$ of the variable $s$ in Theorem 1.5 can be extended to $[(1+t)/(p+t), \infty)$ for each $r$ such that $-(1+t) < r \leqslant 0$. Meanwhile, Theorem 4.1 (1) is a sharpen extension of Kamei's result ([34]) as follows:

Let $A \geqslant B \geqslant 0$ with $A > 0$, $t \geqslant 0$ and $p \geqslant 1$. Then

$$B_1(s) = (A^{-t} \natural_s B^p)^{1/((p+t)s-t)}$$

is increasing for $1 \geqslant s \geqslant \frac{1+t}{p+t}$.

THEOREM 4.3. *Let* $A, B > 0$ *with* $\log A \geqslant \log B$, $t > 0$ *and* $p > 0$, *then the following hold.*

(1) *For each* $-t \leqslant r \leqslant 0$ *and* $q+r > 0$, *the function*

$$F_{p,t,q}(r,s) = A^{-r} \natural_{(q+r)/((p+t)s+r)} (A^{t/2}B^pA^{t/2})^s$$

*is increasing for* $s \geqslant \frac{q}{p+t}$.

(2) *For each* $p+t \geqslant q \geqslant t$ *and* $\frac{q}{p+t} \leqslant s \leqslant 1$, *the function*

$$F_{p,t,q}(r,s) = A^{-r} \natural_{(q+r)/((p+t)s+r)} (A^{t/2}B^pA^{t/2})^s$$

*is decreasing for* $0 \geqslant r \geqslant -t$.

(3) *For each* $p+t \geqslant q \geqslant t$, *the function*

$$F_{p,t,q}(r,s) = A^{-r} \natural_{(q+r)/((p+t)s+r)} (A^{t/2}B^pA^{t/2})^s$$

*is decreasing for* $0 \geqslant r \geqslant -t$ *and increasing for* $\frac{q}{p+t} \leqslant s \leqslant 1$.

The relation between Theorem 4.3 and Theorem 4.1 is similar to that of Theorem 4.2 and Theorem 1.5.

*Proof of Theorem 4.2.* We only proof (1)-(2) for (3) follows them immediately.

(I) Denote $A_1 = A$, $B_1 = (A^{t/2}B^pA^{t/2})^{1/(p+t)}$, then $A_1^\alpha \geqslant B_1^\alpha$ where $\alpha = 1+t > 0$ by Theorem 1.1.

(II) Denote $q_1 = \frac{q}{1+t}$, $r_1 = \frac{r}{1+t}$, $p_1 = \frac{(p+t)s}{1+t}$, $A_2 = A_1^\alpha$ and $B_2 = B_1^\alpha$, then $A_2 \geqslant B_2$. By Theorem 2.3 (1), for each $-1 \leqslant r_1 < 0$ and $q_1 > -r_1$, the function

$$F_{p,t,q}(r,s) = A^{-r} \natural_{(q+r)/((p+t)s+r)} (A^{t/2}B^pA^{t/2})^s = A_2^{-r_1} \natural_{\frac{q_1+r_1}{p_1+r_1}} B_2^{p_1}$$

is increasing for $p_1 \geqslant q_1$. That is, for each $-(1+t) \leqslant r \leqslant 0$ and $q > -r$, $F_{p,t,q}(r,s)$ is increasing for $s \geqslant \frac{q}{p+t}$.

(III) Let $p+t \geqslant q \geqslant t$, $\frac{q}{p+t} \leqslant s \leqslant 1$ and $0 \geqslant r \geqslant -t$. By (II) and Theorem 1.1,

$$\begin{aligned} A^{1+t+r} &\geqslant \left(A^{r/2}A^{t/2}B^pA^{t/2}A^{r/2}\right)^{(1+t+r)/(p+t+r)} \\ &\geqslant \left(A^{r/2}(A^{t/2}B^pA^{t/2})^sA^{r/2}\right)^{(1+t+r)/((p+t)s+r)}. \end{aligned} \tag{4.1}$$

Put $D = A^{t/2}B^pA^{t/2}$, then for $0 < u \leqslant 1+t+r$,

$$\begin{aligned} F_{p,t,q}(r,s) &= A^{-r}\sharp_{(q+r)/((p+t)s+r)}D^s \\ &= D^{s/2}\left(D^{s/2}A^rD^{s/2}\right)^{(q-(p+t)s)/((p+t)s+r)}D^{s/2} \quad \text{by Lemma 2.2} \\ &= D^{s/2}\{\left(D^{s/2}A^rD^{s/2}\right)^{((p+t)s+r+u)/((p+t)s+r)}\}^{\frac{q-(p+t)s}{(p+t)s+r+u}}D^{s/2} \\ &= D^{s/2}\{D^{s/2}A^{r/2}\left(A^{r/2}D^sA^{r/2}\right)^{u/((p+t)s+r)}A^{r/2}D^{s/2}\}^{\frac{q-(p+t)s}{(p+t)s+r+u}}D^{s/2} \\ &\geqslant D^{s/2}\{D^{s/2}A^{r/2}A^uA^{r/2}D^{s/2}\}^{(q-(p+t)s)/((p+t)s+r+u)}D^{s/2} \quad \text{by (4.1)} \\ &= D^{s/2}\{D^{s/2}A^{r+u}D^{s/2}\}^{(q-(p+t)s)/((p+t)s+r+u)}D^{s/2} \\ &= F_{p,t,q}(r+u,s). \qquad \square \end{aligned} \tag{4.2}$$

*Proof of Theorem 4.3.* The proof is similar to Proof of Theorem 4.2.

(I) Denote $A_1 = A$, $B_1 = (A^{t/2}B^pA^{t/2})^{1/(p+t)}$, then $A_1^\alpha \geqslant B_1^\alpha$ where $\alpha = t > 0$ by Furuta inequality under chaotic order [5].

(II) Denote $q_1 = \frac{q}{t}$, $r_1 = \frac{r}{t}$, $p_1 = \frac{(p+t)s}{t}$, $A_2 = A_1^\alpha$ and $B_2 = B_1^\alpha$, then $A_2 \geqslant B_2$. By Theorem 2.3 (1), for each $-1 \leqslant r_1 < 0$ and $q_1 > -r_1$, the function

$$F_{p,t,q}(r,s) = A_2^{-r_1}\sharp_{\frac{q_1+r_1}{p_1+r_1}}B_2^{p_1}$$

is increasing for $p_1 \geqslant q_1$. That is, for each $-t \leqslant r \leqslant 0$ and $q > -r$, $F_{p,t,q}(r,s)$ is increasing for $s \geqslant \frac{q}{p+t}$.

(III) Let $p+t \geqslant q \geqslant t$, $\frac{q}{p+t} \leqslant s \leqslant 1$ and $0 \geqslant r \geqslant -t$. By (II) and Furuta inequality under chaotic order,

$$\begin{aligned} A^{t+r} &\geqslant \left(A^{r/2}A^{t/2}B^pA^{t/2}A^{r/2}\right)^{(t+r)/(p+t+r)} \\ &\geqslant \left(A^{r/2}(A^{t/2}B^pA^{t/2})^sA^{r/2}\right)^{(t+r)/((p+t)s+r)}. \end{aligned} \tag{4.3}$$

The following is quite similar to the related part of Proof of Theorem 4.2, so we omit it here. $\square$

For convenience, we write down the following Theorem 4.4 and 4.5 by Furuta [21, 22] which are similar to Theorem 3.4 and 3.5 respectively.

THEOREM 4.4. ([22]) *For $A > 0$, $B > 0$, $\log A \geqslant \log B$ if and only if*

$$A^{t+r} \geqslant \left(A^{r/2}(A^{t/2}B^pA^{t/2})^sA^{r/2}\right)^{(t+r)/((p+t)s+r)} \tag{4.4}$$

*holds for $p > 0$, $t \geqslant 0$, $0 \geqslant r \geqslant -t$ and $1 \geqslant s \geqslant \frac{t}{p+t}$.*

THEOREM 4.5. ([21, 22]) *For $A > 0$, $B \geqslant 0$, $A \geqslant B$ if and only if*

$$A^{1+t+r} \geqslant \left(A^{r/2}(A^{t/2}B^pA^{t/2})^sA^{r/2}\right)^{(1+t+r)/((p+t)s+r)} \tag{4.5}$$

*holds for $p \geqslant 1$, $t \geqslant 0$, $0 \geqslant r \geqslant -t$ and $1 \geqslant s \geqslant \frac{1+t}{p+t}$.*

## 5. Examples of the construction of (NF)

The purpose of this section is to show the construction of (NF) via examples.

We shall show the following Theorem 5.1 which implies the well known grand Furuta inequality Theorem 1.4 easily. Theorem 1.4 is an important and interesting operator function because it interpolates Furuta inequality and Ando–Hiai inequality.

THEOREM 5.1. *If $A \geqslant B \geqslant 0$ with $A > 0$, $p > 1$ and $t \in [-1,0]$. Then the function*

$$F_{p,t,q}(r,s) = A^{t/2}\left(A^{-(r+t)}\sharp_{(q+r)/((p+t)s+r)}(A^{-t}\natural_s B^p)\right)A^{t/2}$$

*is decreasing for both $r \geqslant \max\{-t,-q\}$ and $s \geqslant \max\{1, \frac{q}{p+t}\}$.*

Theorem 3.1 and Theorem 5.1 imply the following result by Furuta et al. [28].

THEOREM 5.2. ([28]) *Let $A \geqslant B \geqslant 0$ with $A > 0$, $t \in [-1,0]$ and $p > -t$. Then the following (1) and (2) hold for a fixed real number $q$.*

(1) *If $q \geqslant t$, then the function*

$$F_{p,t,q}(r,s) = A^{-r}\sharp_{(q+r)/((p+t)s+r)}(A^{t/2}B^pA^{t/2})^s$$

*is decreasing for both $r \geqslant -t$ and $s \geqslant \max\{1, \frac{q}{p+t}\}$.*

(2) *If $p \geqslant q - t$, then the function*

$$F_{p,t,q}(r,s) = A^{-r}\sharp_{(q+r)/((p+t)s+r)}(A^{t/2}B^pA^{t/2})^s$$

*is decreasing for both $r \geqslant \max\{-t,-q\}$ and $s \geqslant 1$.*

After Furuta's original paper [19], Fujii–Kamei [10] provided a mean theoretic approach of Theorem 1.4. The complete Theorem 5.1 appeared in Jiang et al. [31] which showed a very short proof (less than half page), but the monotonicity of the variable $s$ of the proof is somewhat ambiguous. Here we show a new and clear proof by construction of (NF).

*Proof of Theorem 5.1.* When $q \geqslant t$, we only need to show $F_{p,t,q}(r,s)$ is decreasing for both $r \geqslant -t$ and $s \geqslant \max\{1, \frac{q}{p+t}\}$.

(I) Put $B_1(s) = (A^{-t}\natural_s B^p)^{1/((p+t)s-t)}$. Obviously, $A \geqslant B = B_1(1)$.

(II) Put $B_2 = (A^{t/2}B^pA^{t/2})^{1/(p+t)}$. By (I),

$$(A^{-t/2}B_2^{p+t}A^{-t/2})^{(-t)/(p+t-t)} = \left(B_1(1)\right)^{-t} \leqslant A^{-t}$$

for $-t \in [0,1]$. Thus, by Theorem 2.5 (2),

$$(B_2^{(p+t)/2}A^{-t}B_2^{(p+t)/2})^{(p+t)/(p+t-t)} \geqslant B_2^{p+t}.$$

Therefore, by Theorem 2.6 (1), $B_1(s) = (A^{-t/2}B_2^{(p+t)s}A^{-t/2})^{(1+t-t)/((p+t)s-t)}$ is decreasing for $s \geqslant 1$, so that $B_1(s) \leqslant B_1(1) = B \leqslant A$.

(III) Put $r_1 = r+t \geqslant 0$, $q_1 = q-t \geqslant 0$. By Theorem 2.1 (2) or (3), for each $q_1 \geqslant 0$ and $p_1 = (p+t)s - t \geqslant \max\{p, q_1\} > 0$, the function

$$F_{p,t,q}(r,s) = A^{-r_1} \sharp_{\frac{q_1+r_1}{p_1+r_1}} (B_1(s))^{p_1}$$

is decreasing for $r_1 \geqslant \max\{0, -q_1\}$, that is, $r \geqslant -t$.

(IV) Let $r \geqslant -t$, $s \geqslant 1$ and $D = A^{t/2}B^pA^{t/2}$, then

$$\begin{aligned} & A^{1+r+t} \geqslant \left(A^{r/2}DA^{r/2}\right)^{(1+r+t)/(p+t+r)} \quad \text{by Theorem 1.1 for } p > 1 \\ \Rightarrow\ & A^r \geqslant \left(A^{r/2}DA^{r/2}\right)^{r/(p+t+r)} \\ \Rightarrow\ & D \leqslant \left(D^{1/2}A^rD^{1/2}\right)^{(p+t)/(p+t+r)} \quad \text{by Theorem 2.5 (2)} \\ \Rightarrow\ & D^s \leqslant \left(D^{s/2}A^rD^{s/2}\right)^{(p+t)s/((p+t)s+r)} \quad \text{by Theorem 2.6 (1).} \end{aligned} \tag{5.1}$$

Then for $0 < w \leqslant s$,

$$\begin{aligned} F_{p,t,q}(r,s) &= A^{-r} \sharp_{(q+r)/((p+t)s+r)} D^s \\ &= A^{-r/2}\left\{\left(A^{r/2}D^sA^{r/2}\right)^{\frac{(p+t)(s+w)+r}{(p+t)s+r}}\right\}^{\frac{q+r}{(p+t)(s+w)+r}} A^{-r/2} \\ &= A^{-r/2}\left\{A^{r/2}D^{s/2}\left(D^{s/2}A^rD^{s/2}\right)^{\frac{(p+t)w}{(p+t)s+r}} D^{s/2}A^{r/2}\right\}^{\frac{q+r}{(p+t)(s+w)+r}} A^{-r/2} \\ &\geqslant A^{-r/2}\left\{A^{r/2}D^{s/2}D^wD^{s/2}A^{r/2}\right\}^{(q+r)/((p+t)(s+w)+r)} A^{-r/2} \quad \text{by (5.1)} \\ &= A^{-r/2}\left\{A^{r/2}D^{s+w}A^{r/2}\right\}^{(q+r)/((p+t)(s+w)+r)} A^{-r/2} \\ &= F_{p,t,q}(r, s+w). \end{aligned} \tag{5.2}$$

Proof of case $q < t$ is similar to the proof of case $q \geqslant t$, so we omit it here. □

*Proof of Theorem 5.2.* It is obvious that case $1 \geqslant p > -t$ of Theorem 5.2 belongs to class (F) and follows Theorem 3.1 (2) easily. Meanwhile, case $p > 1$ of Theorem 5.2 belongs to class (NF) and is clear by Theorem 5.1. □

We remark that the inequality (1.4) can be regarded as a parallel result to Theorem 3.5 in section 3 and Theorem 4.5 in section 4. Associated with (1.4), Furuta [24] posed the following question.

For $A > 0$, $B > 0$, $\log A \geqslant \log B$ if and only if

$$A^{t+r} \geqslant \left(A^{r/2}(A^{t/2}B^pA^{t/2})^sA^{r/2}\right)^{(t+r)/((p+t)s+r)} \tag{5.3}$$

holds for $p \geqslant 1$, $t \in [-1, 0]$, $r \geqslant -t$ and $s \geqslant 1$.

Fujii et al. [11] showed the following result which implies that the assertion above holds if $\log A \geqslant \log B$ is replaced with $A \geqslant B$ by using Kantorovich type operator inequality.

THEOREM 5.3. ([11]) *For each $\alpha \geqslant 0$, $A > 0$ and $B > 0$, $A^\alpha \geqslant A^\alpha$ (case $\alpha = 0$ means $\log A \geqslant \log B$) if and only if (5.3) holds for all $p \geqslant \alpha$, $t \in [-\alpha, 0]$, $r \geqslant -t$ and $s \geqslant 1$.*

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