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# REMARKS ON A PYTHAGOREAN APPROACH IN BANACH SPACES

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(*communicated by L. Maligranda*)

*Abstract.* We give relations between the parameters $E(X)$ and $f(X)$, introduced by the first author in [4], and other geometric constants. Main results in [4] and [5] are derived and strengthened. More precisely, a wider class of Banach spaces with uniform normal structure is obtained. A relation between $E(X)$ and $E(X^*)$ is also given.

## 1. Introduction

Let $X$ be a Banach space. If $\dim X \geqslant 2$, then we say that $X$ is nontrivial. Denote by $S_X$ and $B_X$ the unit sphere and the closed unit ball of $X$, respectively. Parameters

$$E(X) = \sup\{\|x+y\|^2 + \|x-y\|^2 : x, y \in S_X\}$$

and

$$f(X) = \inf\{\|x+y\|^2 + \|x-y\|^2 : x, y \in S_X\}$$

were introduced and studied by the first author [4]. The values of these parameters in the $l_p$ spaces and function spaces $L_p[0,1]$ are estimated. Among the other results, it was proved that (a) if $E(X) < 8$ or $f(X) > 2$, then $X$ is uniformly nonsquare; (b) if $E(X) < 5$ or $f(X) > \frac{32}{9}$, then $X$ has uniform normal structure.

In this short paper, we give relations between $E(X)$, $f(X)$ and other geometric constants. Consequently, main results in [4] and [5] are derived and strengthened. More precisely, we prove that a Banach space $X$ and its dual space $X^*$ have uniform normal structure if $E(X) < 3+\sqrt{5}$. Moreover, if $E(X) < 2+2\sqrt{3}$, then $X^*$ has uniform normal structure. The exact values of $E(L_p[0,1])$ and $E(\ell_p)$ when $1<p<2$ are also obtained.

Recall that a Banach space $X$ is said to have *uniform normal structure* if there exists $0<c<1$ such that for any closed bounded convex subset $K$ of $X$ that contains more than one point, there exists $x_0 \in K$ such that

$$\sup\{\|x_0 - y\| : y \in K\} < c \sup\{\|x-y\| : x, y \in K\}.$$

It follows from W. A. Kirk's result (see [8, 10]) that every Banach space $X$ with uniform normal structure has the *fixed point property*, that is for every nonexpansive mapping $T$ from each bounded closed convex subset $C \subset X$ into itself always has a fixed point.

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In a recent paper, M. Kato, L. Maligranda and Y. Takahashi [9] gave a sufficient condition for uniform normal structure in terms of the *von Neumann–Jordan constant* $C_{\mathrm{NJ}}(X)$, which was defined in 1937 by J. A. Clarkson [2] as

$$C_{\mathrm{NJ}}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X \text{ and } \|x\| + \|y\| \neq 0 \right\}.$$

This result has been recently improved by the second author in [13], where among other things, it was proved that $X$ and $X^*$ have uniform normal structure whenever $C_{\mathrm{NJ}}(X) < \frac{1+\sqrt{3}}{2}$.

Another constants which were used to give sufficient conditions for uniform normal structure are $J(X)$ and $g(X)$ defined as

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in S_X\},$$
$$g(X) = \inf\{\max\{\|x+y\|, \|x-y\|\} : x, y \in S_X\}.$$

J. Gao and K.-S. Lau proved in [7] that a Banach space $X$ has normal structure if $J(X) < \frac{3}{2}$ (equivalently, $g(X) > \frac{4}{3}$), and again this result has been recently improved by S. Dhompongsa, A. Kaewkhao and S. Tasena, who proved in [3, Theorem 2.1] that the constant $\frac{3}{2}$ can be replaced by $\frac{1+\sqrt{5}}{2}$.

## 2. Results

PROPOSITION 1. *For any nontrivial Banach space $X$,*

(*i*) $f(X) \leqslant 2(g(X))^2$;

(*ii*) $E(X) = \sup\{4(1-\delta_X(\varepsilon))^2 + \varepsilon^2 : \varepsilon \in [0,2]\}$, *where* $\delta_X(\varepsilon) = \inf\{1 - \frac{1}{2}\|x+y\| : x, y \in S_X, \|x-y\| = \varepsilon\}$ *is the modulus of convexity of $X$ (see* [1]*).*

*Proof.* (*i*) Let $x, y \in S_X$, then $\|x+y\|^2 + \|x-y\|^2 \leqslant 2\max\{\|x+y\|^2, \|x-y\|^2\}$. This implies that $f(X) \leqslant 2(g(X))^2$.

(*ii*) Let $x, y \in S_X$. Then $\|x-y\| = \varepsilon$ for some $\varepsilon \in [0,2]$ and so $\|x+y\| \leqslant 2(1-\delta_X(\varepsilon))$. This implies that $\|x+y\|^2 + \|x-y\|^2 \leqslant 4(1-\delta_X(\varepsilon))^2 + \varepsilon^2$. On the other hand, let $\varepsilon \in [0,2]$, we choose sequences $\{x_n\}$ and $\{y_n\}$ in $S_X$ so that $\|x_n - y_n\| = \varepsilon$ for all $n$ and $\|x_n + y_n\| \to 2(1-\delta_X(\varepsilon))$ (see [11]). This gives $E(X) \geqslant \|x_n+y_n\|^2 + \|x_n-y_n\|^2 \to 4(1-\delta_X(\varepsilon))^2 + \varepsilon^2$. This completes the proof. □

As a consequence of the preceding proposition, we have the following.

COROLLARY 2. *Let $X$ be any nontrivial Banach space.*

(*i*) [4, Theorem 2.8] *If either $E(X) < 8$ or $f(X) > 2$, then $X$ is uniformly nonsquare;*

(*ii*) [4, Theorem 5.3] *If $E(X) < 5$ ($f(X) > \frac{32}{9}$ resp.), then $\delta_X(1) > 0$ ($g(X) > \frac{4}{3}$ resp.), which in turn implies that $X$ and $X^*$ have uniform normal structure.*

*Proof.* (*i*) It is well known that $X$ is uniformly nonsquare if and only if $g(X) > 1$; equivalently $\delta_X(\varepsilon) > 0$ for some $\varepsilon \in (0,2)$.

(*ii*) It is easy to see that if $E(X) < 5$ (or $f(X) > \frac{32}{9}$ resp.), then $\delta_X(1) > 0$ (or $g(X) > \frac{4}{3}$ resp.). Finally, we need to prove that $g(X) > \frac{4}{3}$ implies $\delta_X(\varepsilon) > \frac{\varepsilon - 1}{2}$

for some $\varepsilon \in [1,2)$ and the result follows from [12, Corollary 6]. It is known that $J(X) = \sup\{\varepsilon \in [0,2] : \varepsilon \leqslant 2 - 2\delta_X(\varepsilon)\}$ [6, Theorem 3.3] and $J(X)g(X) = 2$ [6, Theorem 2.5]. If $g(X) > \frac{4}{3}$, then $J(X) < \frac{3}{2}$. This implies that $\frac{3}{2} > 2 - 2\delta_X(\frac{3}{2})$, and so $\delta_X(\frac{3}{2}) > \frac{(3/2)-1}{2}$. □

PROPOSITION 3. *For any nontrivial Banach space $X$,*

$$\frac{(J(X))^2}{2} \leqslant \frac{E(X)}{4} \leqslant C_{\mathrm{NJ}}(X).$$

*Proof.* For $x, y \in S_X$, we have $\frac{1}{2}\min\{\|x+y\|^2, \|x-y\|^2\} \leqslant \frac{1}{4}(\|x+y\|^2 + \|x-y\|^2) \leqslant C_{\mathrm{NJ}}(X)$. This gives the assertion. □

PROPOSITION 4. *For any nontrivial Banach space $X$,*

$$\frac{(J(X^*))^2}{2} \leqslant \frac{E(X)}{4} \leqslant C_{\mathrm{NJ}}(X^*).$$

*Proof.* The second inequality holds since $C_{\mathrm{NJ}}(X^*) = C_{\mathrm{NJ}}(X)$. To prove the first inequality, let $\varepsilon > 0$. We choose $u, v \in S_{X^*}$ so that $J(X^*) - \varepsilon \leqslant \min\{\|u+v\|, \|u-v\|\}$. Now, let $x, y \in S_X$ so that

$$\|u+v\| - \varepsilon \leqslant (u+v)(x),$$
$$\|u-v\| - \varepsilon \leqslant (u-v)(y).$$

Hence

$$\begin{aligned}
J(X^*) - 2\varepsilon &\leqslant \min\{\|u+v\|, \|u-v\|\} - \varepsilon \\
&\leqslant \frac{\|u+v\| + \|u-v\|}{2} - \varepsilon \\
&\leqslant \frac{(u+v)(x) + (u-v)(y)}{2} \\
&= \frac{u(x+y) + v(x-y)}{2} \\
&\leqslant \frac{\|x+y\| + \|x-y\|}{2} \\
&\leqslant \left(\frac{\|x+y\|^2 + \|x-y\|^2}{2}\right)^{1/2} \\
&\leqslant \left(\frac{E(X)}{2}\right)^{1/2}.
\end{aligned}$$

Letting $\varepsilon \to 0$ yields the assertion. □

The next corollary strengthens results in [4] and [5].

COROLLARY 5. *Let $X$ be any nontrivial Banach space.*

(*i*) [5, Theorem 8] *If $E(X) < 3 + \sqrt{5}$, then both $X$ and its dual $X^*$ have uniform normal structure.*

(*ii*) [5, Theorem 9] *If* $f(X) > 12 - 4\sqrt{5}$, *then* $X$ *has uniform normal structure.*
(*iii*) [4, Theorem 4.2, Theorem 4.3] *If* $1 \leqslant p \leqslant \infty$, *then* $E(L_p[0,1]) = E(l_p) = 2^{2/r+1}$ *where* $r = \min\{p, p'\}$ *and* $p'$ *is the conjugate exponent of* $p$, *that is* $1/p + 1/p' = 1$.

*Proof.* (*i*) If $E(X) < 3 + \sqrt{5}$, then $J(X) < \frac{1+\sqrt{5}}{2}$ and $J(X^*) < \frac{1+\sqrt{5}}{2}$. By [3, Theorem 2.1], $X$ and $X^*$ have uniform normal structure.

(*ii*) If $f(X) > 12 - 4\sqrt{5}$, then $g(X) > \sqrt{5} - 1$ and so $J(X) < \frac{1+\sqrt{5}}{2}$. This implies $X$ has uniform normal structure.

(*iii*) It follows from the fact that $J(L_p[0,1]) = J(l_p) = 2^{1/r}$ [6, Theorem 3.2] and $C_{\mathrm{NJ}}(L_p[0,1]) = C_{\mathrm{NJ}}(l_p) = 2^{2/r-1}$ [2]. □

REMARK 6. There exists a two-dimensional space $X$ for which $E(X) \neq E(X^*)$. Let $X = \mathbb{R}^2$ with the $\ell_2$-$\ell_1$ norm given by

$$|||(x_1, x_2)||| = \begin{cases} (|x_1|^2 + |x_2|^2)^{1/2}, & \text{if } x_1 x_2 \geqslant 0; \\ |x_1| + |x_2|, & \text{if } x_1 x_2 \leqslant 0. \end{cases}$$

Then (see [8, Example 5.8, page 60])

$$\delta_X(\varepsilon) = \begin{cases} 0, & \text{if } 0 \leqslant \varepsilon \leqslant \sqrt{2}; \\ 1 - \sqrt{2 - \frac{\varepsilon^2}{2}}, & \text{if } \sqrt{2} \leqslant \varepsilon \leqslant \sqrt{\frac{8}{3}}; \\ 1 - \sqrt{1 - \frac{\varepsilon^2}{8}}, & \text{if } \sqrt{\frac{8}{3}} \leqslant \varepsilon \leqslant 2. \end{cases}$$

It follows from Proposition 1 (*ii*) that $E(X) = 6$. The dual space $X^*$ of $X$ is $\mathbb{R}^2$ equipped with the $\ell_2$-$\ell_\infty$ norm given by

$$\|(x_1, x_2)\| = \begin{cases} (|x_1|^2 + |x_2|^2)^{1/2}, & \text{if } x_1 x_2 \geqslant 0; \\ \max\{|x_1|, |x_2|\}, & \text{if } x_1 x_2 \leqslant 0. \end{cases}$$

PROPOSITION 7. $E(\ell_2\text{-}\ell_\infty) = 3 + 2\sqrt{2}$.

*Proof.* In order to calculate $E(\ell_2\text{-}\ell_\infty)$, it suffices to take only elements $x$ and $y$ from the upper half of the unit sphere of $\ell_2$-$\ell_\infty$. We suppose for a moment that $x$ is in the interior of the second quadrant.

We observe that if $x = (u, 1)$ where $-1 < u < 0$, then $x = (1 - \lambda)(-1, 1) + \lambda(0, 1)$ for some $0 < \lambda < 1$. It follows that

$$\begin{aligned} &\|x + y\|^2 + \|x - y\|^2 \\ &\leqslant (1 - \lambda)(\|(-1, 1) + y\|^2 + \|(-1, 1) - y\|^2) + \lambda(\|(0, 1) + y\|^2 + \|(0, 1) - y\|^2) \\ &\leqslant \max\{\|(-1, 1) + y\|^2 + \|(-1, 1) - y\|^2, \|(0, 1) + y\|^2 + \|(0, 1) - y\|^2\} \end{aligned}$$

Similarly, if $x = (-1, v)$ where $0 < v < 1$, then

$$\begin{aligned} &\|x + y\|^2 + \|x - y\|^2 \\ &\leqslant \max\{\|(-1, 1) + y\|^2 + \|(-1, 1) - y\|^2, \|(-1, 0) + y\|^2 + \|(-1, 0) - y\|^2\} \\ &= \max\{\|(-1, 1) + y\|^2 + \|(-1, 1) - y\|^2, \|(1, 0) - y\|^2 + \|(1, 0) + y\|^2\}. \end{aligned}$$

This implies that $E(\ell_2\text{-}\ell_\infty)$ is essentially determined by elements $x$ and $y$ being in the first quadrant of $\mathbb{R}^2$ or $(-1,1)$. We now consider the following cases.

*Case 1:* $x$ and $y$ are in the first quadrant of $\mathbb{R}^2$. We write $x=(u,v)$ and $y=(w,z)$. It follows that

$$\begin{aligned}\|x+y\|^2+\|x-y\|^2 &= (u+w)^2+(v+z)^2+\max\{|u-w|^2,|v-z|^2\}\\ &\leqslant (u+w)^2+(v+z)^2+(u-w)^2+(v-z)^2\\ &= 4.\end{aligned}$$

*Case 2:* $x$ is in the first quadrant of $\mathbb{R}^2$ and $y=(-1,1)$. In this case, we have

$$\begin{aligned}\|x+y\|^2+\|x-y\|^2 &= \max\{|u-1|^2,|v+1|^2\}+\max\{|u+1|^2,|v-1|^2\}\\ &= |v+1|^2+|u+1|^2\\ &= 3+2(u+v)\\ &\leqslant 3+4\left(\frac{u^2+v^2}{2}\right)^{1/2}=3+2\sqrt{2}.\end{aligned}$$

Consequently, $E(\ell_2\text{-}\ell_\infty)\leqslant 3+2\sqrt{2}$. Moreover, equality is attained for $x=(1/\sqrt{2},1/\sqrt{2})$ and $y=(-1,1)$ and this completes the proof. □

We now present a relationship between $E(X)$ and $E(X^*)$.

PROPOSITION 8. *Let $X$ be a Banach space. Then*
(*i*) $E(X)\leqslant\sqrt{8E(X^*)}$ *and*
(*ii*) $E(X^*)\leqslant\sqrt{8E(X)}$.

*Proof.* (*i*) First we observe that

$$E(X)=\sup\{\|x+y\|^2+\|x-y\|^2 : x,y\in B_X\}.$$

Let $x,y\in S_X$. We choose $u,v\in S_{X^*}$ so that

$$u(x+y)=\|x+y\| \quad\text{and}\quad v(x-y)=\|x-y\|.$$

Take $u'=\frac{\|x+y\|}{2}u$ and $v'=\frac{\|x-y\|}{2}v$. It follows that $\|u'\|\leqslant 1$ and $\|v'\|\leqslant 1$. Therefore,

$$\begin{aligned}\|x+y\|^2+\|x-y\|^2 &= 2(u'(x+y)+v'(x-y))\\ &= 2((u'+v')(x)+(u'-v')(y))\\ &\leqslant 2(\|u'+v'\|\|x\|+\|u'-v'\|\|y\|)\\ &= 2(\|u'+v'\|+\|u'-v'\|)\\ &\leqslant 2\sqrt{2}\left(\|u'+v'\|^2+\|u'-v'\|^2\right)^{1/2}\\ &\leqslant\sqrt{8E(X^*)}.\end{aligned}$$

(*ii*) If $X$ is not reflexive, then $X$ is not uniformly nonsquare and so by Corollary 2, $E(X^*)=E(X)=8$ and the inequality holds immediately. On the other hand, if $X$ is reflexive, then $E(X^{**})=E(X)$. It follows from (*i*) that

$$E(X^*)\leqslant\sqrt{8E(X^{**})}=\sqrt{8E(X)}.$$

□

REMARK 9. If $X$ is not uniformly nonsquare (for example $X = l_1, l_\infty$ or $c_0$), then $E(X) = \sqrt{8E(X^*)}$ and $E(X^*) = \sqrt{8E(X)}$.

REMARK 10. The idea of the proofs of Propositions 4 and 8 is from [9].

We do not know whether or not $J(X) < (1+\sqrt{5})/2$ implies that the dual space $X^*$ has uniform normal structure. We now show the following improvement on a sufficient condition for uniform normal structure of the dual space.

THEOREM 11. *If $E(X) < 2+2\sqrt{3}$, then $X^*$ has uniform normal structure.*

*Proof.* We follow the idea of [14]. Here and hereafter $\widetilde{X}$ and $\widetilde{X}^*$ denote the Banach space ultrapower of $X$ and the dual space $X^*$ of $X$, respectively. For more details on the ultrapower construction, we refer the reader to [15]. We first show that if $E(\widetilde{X}^*) < 2+2\sqrt{3}$, then $X$ has normal structure. We note here that $E(\widetilde{X}^*) < 2+2\sqrt{3}$ implies that $\widetilde{X}$ is reflexive and hence $X$ is superreflexive. If $X$ does not have normal structure, then by Lemma 2 of [14], there are vectors $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3 \in S_{\widetilde{X}}$ and $\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3 \in S_{\widetilde{X}^*}$ such that

(a) $\|\widetilde{x}_i - \widetilde{x}_j\| = 1$ and $\widetilde{f}_i(\widetilde{x}_j) = 0$ for all $i \neq j$,
(b) $\widetilde{f}_i(\widetilde{x}_i) = 1$ for $i = 1,2,3$, and
(c) $\|\widetilde{x}_3 - (\widetilde{x}_2 + \widetilde{x}_1)\| \geqslant \|\widetilde{x}_2 + \widetilde{x}_1\|$.

It now follows that

$$\begin{aligned} E(\widetilde{X}^*) &\geqslant \|\widetilde{f}_2 + \widetilde{f}_1\|^2 + \|\widetilde{f}_2 - \widetilde{f}_1\|^2 \\ &\geqslant \Big((\widetilde{f}_2 + \widetilde{f}_1)\Big(\frac{\widetilde{x}_2 + \widetilde{x}_1}{\|\widetilde{x}_2 + \widetilde{x}_1\|}\Big)\Big)^2 + \Big((\widetilde{f}_2 - \widetilde{f}_1)(\widetilde{x}_2 - \widetilde{x}_1)\Big)^2 \\ &= \frac{4}{\|\widetilde{x}_2 + \widetilde{x}_1\|^2} + 4, \end{aligned}$$

and

$$\begin{aligned} E(\widetilde{X}^*) &\geqslant \|\widetilde{f}_3 + \widetilde{f}_1\|^2 + \|\widetilde{f}_3 - \widetilde{f}_1\|^2 \\ &\geqslant \Big((\widetilde{f}_3 + \widetilde{f}_1)\Big(\frac{\widetilde{x}_3 - \widetilde{x}_2 + \widetilde{x}_1}{\|\widetilde{x}_3 - \widetilde{x}_2 + \widetilde{x}_1\|}\Big)\Big)^2 + \Big((\widetilde{f}_3 - \widetilde{f}_1)(\widetilde{x}_3 - \widetilde{x}_1)\Big)^2 \\ &= \frac{4}{\|\widetilde{x}_3 - \widetilde{x}_2 + \widetilde{x}_1\|^2} + 4. \end{aligned}$$

Next, we choose $\widetilde{g}_1, \widetilde{g}_2 \in S_{\widetilde{X}^*}$ so that

$$\begin{aligned} \widetilde{g}_1(\widetilde{x}_3 - \widetilde{x}_2 + \widetilde{x}_1) &= \|\widetilde{x}_3 - \widetilde{x}_2 + \widetilde{x}_1\|, \\ \widetilde{g}_2(\widetilde{x}_3 - \widetilde{x}_2 - \widetilde{x}_1) &= \|\widetilde{x}_3 - \widetilde{x}_2 - \widetilde{x}_1\|. \end{aligned}$$

Then

$$\begin{aligned} \|\widetilde{g}_1 + \widetilde{g}_2\| + \|\widetilde{g}_1 - \widetilde{g}_2\| &\geqslant \|\widetilde{g}_1 + \widetilde{g}_2\|\|\widetilde{x}_3 - \widetilde{x}_2\| + \|\widetilde{g}_1 - \widetilde{g}_2\|\|\widetilde{x}_1\| \\ &\geqslant (\widetilde{g}_1 + \widetilde{g}_2)(\widetilde{x}_3 - \widetilde{x}_2) + (\widetilde{g}_1 - \widetilde{g}_2)(\widetilde{x}_1) \\ &= \widetilde{g}_1(\widetilde{x}_3 - \widetilde{x}_2 + \widetilde{x}_1) + \widetilde{g}_2(\widetilde{x}_3 - \widetilde{x}_2 - \widetilde{x}_1) \\ &= \|\widetilde{x}_3 - \widetilde{x}_2 + \widetilde{x}_1\| + \|\widetilde{x}_3 - \widetilde{x}_2 - \widetilde{x}_1\|. \end{aligned}$$

Hence,

$$
\begin{aligned}
E(\widetilde{X}^*) &\geqslant 2\left(\frac{\|\widetilde{g_1}+\widetilde{g_2}\|+\|\widetilde{g_1}-\widetilde{g_2}\|}{2}\right)^2\\
&\geqslant 2\left(\frac{\|\widetilde{x_3}-\widetilde{x_2}+\widetilde{x_1}\|+\|\widetilde{x_3}-\widetilde{x_2}-\widetilde{x_1}\|}{2}\right)^2\\
&\geqslant 2\left(\frac{\|\widetilde{x_3}-\widetilde{x_2}+\widetilde{x_1}\|+\|\widetilde{x_2}+\widetilde{x_1}\|}{2}\right)^2\\
&\geqslant \frac{8}{E(\widetilde{X}^*)-4},
\end{aligned}
$$

or equivalently $E(\widetilde{X}^*) \geqslant 2+2\sqrt{3}$ which is a contradiction.

Finally, as proved above, we have if $E(X^{**}) = E(X) < 2+2\sqrt{3}$, then $X^*$ has normal structure. To conclude $X^*$ has uniform normal structure, we just invoke the fact that $E(X) = E(\widetilde{X})$. This completes the proof. □

COROLLARY 12. ([13, Theorem 2]) *If* $C_{\mathrm{NJ}}(X) < (1+\sqrt{3})/2$, *then* $X$ *and* $X^*$ *have uniform normal structure.*

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