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A GENERALIZED REVERSE INEQUALITY OF THE CORDES INEQUALITY

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Abstract. The Cordes inequality was extended by using concave or convex functions. In this note, we give reverse inequalities of its extended ones for an increasing strictly concave submultiplicative function. As an application, we obtain a generalization of Bourin's inequality which gives an estimation of operator norm by spectral radius.

1. Introduction

Throughout this note, an operator means a bounded linear operator acting on a Hilbert space H. Let A and B be positive operators on H. The Cordes inequality [5] for the operator norm asserts

$$\|A^pB^p\| \quad \leqslant \quad \|AB\|^p \qquad \text{for all} \quad 0 \leqslant p \leqslant 1. \tag{1.1}$$

In [1], Araki showed a trace inequality which entailed the following inequality:

$$\|B^pA^pB^p\| \quad \leqslant \quad \|BAB\|^p \quad \text{for all} \quad 0 \leqslant p \leqslant 1. \tag{1.2}$$

The Araki inequality (1.2) is equivalent to the Cordes inequality (1.1) ([3], [8]). Furuta [10] proved that (1.1) is equivalent to the Löwner-Heinz inequality (e.g. [14])

$$A \geqslant B \geqslant 0 \quad \text{implies} \quad A^p \geqslant B^p \quad \text{for all } 0 \leqslant p \leqslant 1. \tag{1.3}$$

Let f be a continuous real valued function on $[0,\infty)$. Then f is semi-operator monotone, if $f(A^{\frac{1}{2}})^2 \geqslant f(B^{\frac{1}{2}})^2$ for $A \geqslant B \geqslant 0$, and f is submultiplicative (resp. supermultiplicative), if $f(ab) \leqslant f(a)f(b)$ (resp. $f(ab) \geqslant f(a)f(b)$) for all $a,b \geqslant 0$. The adjoint f^* of f is defined by $f^*(t) := f(t^{-1})^{-1}$ for $t > 0$ ([12]). J. I. Fujii and M. Fujii essentially gave the extension of (1.1) ([6], cf. [2, Theorem 2.6]).

THEOREM A. *If a nonnegative semi-operator monotone function f on $(0,\infty)$ is submultiplicative, then*

$$\|f(A)f^*(B)\| \quad \leqslant \quad f(\|AB\|) \tag{1.4}$$

for all positive operators A and B.

We have the following theorem which is a generalization of (1.2) and equivalent to (1.4). Moreover it is a refinement of [2, Theorem 2.9].

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THEOREM 1.1. *If a nonnegative operator monotone function f on $(0,\infty)$ is submultiplicative, then*

$$\|f^*(B^2)^{\frac{1}{2}}f(A^2)^{\frac{1}{2}}f^*(B^2)^{\frac{1}{2}}\| \leqslant \|f^*(B^2)^{\frac{1}{2}}f(A)f^*(B^2)^{\frac{1}{2}}\| \leqslant f(\|BAB\|) \tag{1.5}$$

for all positive operators A and B.

In this note, we give complementary inequalities of Theorems A and 1.1. As an application, we generalize the following Bourin's reverse inequality [4]: For a positive definite matrix A with $0 < m \leqslant A \leqslant M$ and a positive semidefinite matrix B

$$\|AB\| \leqslant \frac{M+m}{2\sqrt{Mm}}\, r(AB) \tag{1.6}$$

where $\mathrm{r}(\cdot)$ is the spectral radius.

2. Estimations of $f(\|BAB\|)$ by $\|f^*(B^2)^{\frac{1}{2}}f(A)f^*(B^2)^{\frac{1}{2}}\|$

Let f be a real valued continuous function on the interval $I(\supset [m,M])$ and

$$\alpha_f = \alpha_f(m,M) := \frac{f(M)-f(m)}{M-m} \quad \text{and} \quad \beta_f = \beta_f(m,M) := \frac{Mf(m)-mf(M)}{M-m}. \tag{2.1}$$

For an increasing strictly concave (resp. strictly convex) differentiable function f on $[m,M]$, we put the interval

$$I_f = I_{f,m,M} := \left[\frac{f'(M)}{\alpha_f}, \frac{f'(m)}{\alpha_f}\right] \quad \left(\text{resp. } I_f = I_{f,m,M} := \left[\frac{f'(m)}{\alpha_f}, \frac{f'(M)}{\alpha_f}\right]\right).$$

Here for each $\lambda \in I_f$ the equation $f'(\mu) = \lambda\alpha_f$ has a unique solution $\mu = \mu_\lambda \in [m,M]$. Furthermore we put

$$F(m,M,f;\lambda) := \begin{cases} (1-\lambda)f(c_1) & \text{if } 0<\lambda<\frac{f'(c_1)}{\alpha_f} \\ f(\mu_\lambda) - (\mu_\lambda\alpha_f+\beta_f)\lambda & \text{if } \lambda \in I_f \\ (1-\lambda)f(c_2) & \text{if } \lambda > \frac{f'(c_2)}{\alpha_f} \end{cases} \tag{2.2}$$

where $c_1 = M$ and $c_2 = m$ (resp. $c_1 = m$ and $c_2 = M$).

The function $F(m,M,p;\lambda)$ for $\lambda > 0$ is monotone decreasing and the equation $F(m,M,p;\lambda) = 0$ has a unique solution $\lambda = \lambda_f (\in I_f)$ ([13], [15]).

In our previous note [13], we have the following result (cf. [11]):

THEOREM B. *Let A be a positive operator on a Hilbert space H such that $m \leqslant A \leqslant M$ for some scalars $0 < m < M$. Let f be a real valued continuous strictly concave (resp. strictly convex) differentiable function on $[m,M]$ with $f(m) \neq f(M)$. Then for each $\lambda > 0$*

$$f(\langle Ax,x\rangle) - \lambda\langle f(A)x,x\rangle \leqslant F(m,M,f;\lambda) \tag{2.3}$$
$$\left(\textit{resp. } f(\langle Ax,x\rangle) - \lambda\langle f(A)x,x\rangle \geqslant F(m,M,f;\lambda)\right)$$

holds for all unit vectors $x \in H$.

As our main theorem we give complementary inequalities of Theorem 1.1.

THEOREM 2.1. *Let A and B be positive operators on a Hilbert space H such that $m_1 \leqslant A \leqslant M_1$ and $m_2 \leqslant B \leqslant M_2$ for some scalars $0 < m_i < M_i$ $(i = 1, 2)$. Let f and g be nonnegative real valued differentiable functions on $(0, \infty)$. Then the following assertions (i) and (ii) hold and they are equivalent:*

(i) *Suppose that f is increasing strictly concave submultiplicative and λ_f is a unique solution of $F(m_1, M_1, f; \lambda) = 0$. Then for each $\lambda \in (0, \lambda_f]$*

$$f(\|BAB\|) \leqslant \lambda \sup_{t\in[m_2,M_2]} f(t^2) f\left(\frac{1}{t^2}\right) \|f^*(B^2)^{\frac{1}{2}} f(A) f^*(B^2)^{\frac{1}{2}}\| + F(m_1, M_1, f; \lambda) f(M_2^2). \tag{2.4}$$

(ii) *Suppose that g is increasing strictly convex supermultiplicative and λ_g is a unique solution of $F(g(m_1), g(M_1), g^{-1}; \lambda) = 0$. Then for each $\lambda \in (0, \lambda_g]$*

$$g^{-1}\left(\|g^*(B^2)^{\frac{1}{2}} g(A) g^*(B^2)^{\frac{1}{2}}\|\right) \leqslant \lambda \sup_{t\in[m_2,M_2]} g^{-1}(g^*(t^2)) t^{-2} \|BAB\| + F\left(g(m_1), g(M_1), g^{-1}, \lambda\right) g^{-1}(g^*(M_2^2)). \tag{2.5}$$

Proof. Firstly we prove the case (i). For each $\lambda > 0$ and unit vector $x \in H$

$$\begin{aligned}
f(\langle BABx, x\rangle) &= f\left(\left\langle A\frac{Bx}{\|Bx\|}, \frac{Bx}{\|Bx\|}\right\rangle \|Bx\|^2\right) \\
&\leqslant f\left(\left\langle A\frac{Bx}{\|Bx\|}, \frac{Bx}{\|Bx\|}\right\rangle\right) f(\|Bx\|^2) \\
&\leqslant \left\{\lambda \left\langle f(A)\frac{Bx}{\|Bx\|}, \frac{Bx}{\|Bx\|}\right\rangle + F(m_1, M_1, f; \lambda)\right\} f(\|Bx\|^2) \quad \text{(by (2.3))} \\
&= \lambda \left\langle f(B^{-2})^{-\frac{1}{2}} f(A) f(B^{-2})^{-\frac{1}{2}} \cdot \frac{f(B^{-2})^{\frac{1}{2}} Bx}{\|f(B^{-2})^{\frac{1}{2}} Bx\|}, \frac{f(B^{-2})^{\frac{1}{2}} Bx}{\|f(B^{-2})^{\frac{1}{2}} Bx\|}\right\rangle \\
&\quad \times \frac{f(\|Bx\|^2)\|f(B^{-2})^{\frac{1}{2}} Bx\|^2}{\|Bx\|^2} + F(m_1, M_1, f; \lambda) f(\|Bx\|^2) \\
&\leqslant \lambda \|f^*(B^2)^{\frac{1}{2}} f(A) f^*(B^2)^{\frac{1}{2}}\| \cdot \frac{f(\|Bx\|^2)\|f(B^{-2})^{\frac{1}{2}} Bx\|^2}{\|Bx\|^2} \\
&\quad + F(m_1, M_1, f; \lambda) f(\|Bx\|^2).
\end{aligned}$$

Here, we have

$$\begin{aligned}
f(\|Bx\|^2)\|f(B^{-2})^{\frac{1}{2}} \frac{Bx}{\|Bx\|}\|^2 &= f(\|Bx\|^2) \left\langle f(B^{-2})\frac{Bx}{\|Bx\|}, \frac{Bx}{\|Bx\|}\right\rangle \\
&\leqslant f(\|Bx\|^2) f\left(\left\langle \frac{x}{\|Bx\|}, \frac{x}{\|Bx\|}\right\rangle\right) \\
&= f(\|Bx\|^2) f\left(\frac{1}{\|Bx\|^2}\right) \\
&\leqslant \sup_{t\in[m_2,M_2]} f(t^2) f\left(\frac{1}{t^2}\right).
\end{aligned} \tag{2.6}$$

Moreover since $0 < f(\|Bx\|^2) \leqslant f(M_2^2)$ and $F(m_1, M_1, f; \lambda) \geqslant 0$ for $\lambda \in (0, \lambda_f]$, we have $0 < F(m_1, M_1, f; \lambda) f(\|Bx\|^2) \leqslant F(m_1, M_1, f; \lambda) f(M_2^2)$. So the desired inequality (2.4) holds.

Next we show (2.4) $\Longrightarrow$ (2.5). We replace A, B and f by $g(A)$, $g^*(B^2)^{\frac{1}{2}}$ and g^{-1}, respectively in (2.4). Since $(g^{-1})^*(g^*(X)) = X$ for all positive operator X and g^* is also increasing, the inequality (2.4) ensures the inequality (2.5). Similarly we can show (2.5) $\Longrightarrow$ (2.4). $\square$

Suppose that $f_0(t) := f(t^{\frac{1}{2}})^2$ is increasing strictly concave submultiplicative and λ_f is a unique solution of $F(m_1^2, M_1^2, f_0; \lambda) = 0$. If we put A^2 and f_0 instead of A and f in (2.4), respectively, then for each $\lambda \in (0, \lambda_f]$

$$f(\|AB\|)^2 \leqslant \lambda \sup_{t \in [m_2, M_2]} f(t)^2 f\left(\frac{1}{t}\right)^2 \|f(A) f^*(B)\|^2 + F(m_1^2, M_1^2, f_0; \lambda) f(M_2)^2$$

which is a complementary inequality of Theorem A.

Putting $f(t) = t^p$ $(p \geqslant 0)$ in Theorem 2.1, we have the following corollary by using a generalized Kantorovich constant $K(h,p) := \frac{h^p - h}{(p-1)(h-1)} \left(\frac{p-1}{p} \frac{h^p-1}{h^p-h}\right)^p$ for $h > 1$ (see [9]).

COROLLARY 2.2. *Let A and B be positive operators on a Hilbert space H such that $m_1 \leqslant A \leqslant M_1$ and $m_2 \leqslant B \leqslant M_2$ and $h_i = \frac{M_i}{m_i}$ for some scalars $0 < m_i < M_i$ $(i = 1, 2)$. Then the following assertions (i) and (ii) hold and they are equivalent:*

(i) *Suppose that $0 \leqslant p \leqslant 1$. Then for each $\lambda \in (0, K(h,p)^{-1}]$*

$$\|BAB\|^p \leqslant \lambda \|B^p A^p B^p\| + F(m_1, M_1, (\cdot)^p; \lambda) M_2^{2p}. \tag{2.7}$$

(ii) *Suppose that $p \geqslant 1$. Then for each $\lambda \in (0, K(h,p)]$*

$$\|B^p A^p B^p\|^{\frac{1}{p}} \leqslant \lambda \|BAB\| + F\left(m_1^p, M_1^p, (\cdot)^{\frac{1}{p}}, \lambda\right) M_2^2. \tag{2.8}$$

REMARK 2.3. *We have the following ratio inequality by* (2.4):

$$f(\|BAB\|) \leqslant \lambda_f \sup_{t \in [m_2, M_2]} f(t^2) f\left(\frac{1}{t^2}\right) \|f^*(B^2)^{\frac{1}{2}} f(A) f^*(B^2)^{\frac{1}{2}}\|.$$

On the other hand, if $\lambda_f < \lambda$ in Theorem 2.1, then we have similar inequalities. For example we have the following inequality instead of (2.4):

$$f(\|BAB\|) \leqslant \lambda \sup_{t \in [m_2, M_2]} f(t^2) f\left(\frac{1}{t^2}\right) \|f^*(B^2)^{\frac{1}{2}} f(A) f^*(B^2)^{\frac{1}{2}}\| + F(m_1, M_1, f; \lambda) f(m_2^2)$$

because $f(\|Bx\|^2) \geqslant f(m_2^2) > 0$ and $F(m_1, M_1, f; \lambda) < 0$.

By a similar method we have the following:

THEOREM 2.4. *Let A and B be positive operators on a Hilbert space H such that $m_1 \leqslant A \leqslant M_1$ and $m_2 \leqslant B \leqslant M_2$ for some scalars $0 < m_i < M_i$ $(i = 1,2)$. Let f and g be nonnegative real valued differentiable functions on $(0,\infty)$. Then the following assertions (i) and (ii) hold and they are equivalent:*

(i) If f is increasing strictly convex submultiplicative, then for each $\lambda > 0$

$$f(\|BAB\|) \leqslant \lambda \sup_{t\in[m_2,M_2]} f(t^2)f\left(\frac{1}{t^2}\right)\left\|f^*(B^2)^{\frac{1}{2}}f(A)f^*(B^2)^{\frac{1}{2}}\right\| - \lambda F\left(m_1m_2^2, M_1M_2^2, f, \frac{1}{\lambda}\right). \tag{2.9}$$

(ii) If g is increasing strictly concave supermultiplicative, then for each $\lambda > 0$

$$g^{-1}\left(\|g^*(B^2)^{\frac{1}{2}}g(A)g^*(B^2)^{\frac{1}{2}}\|\right) \leqslant \lambda \sup_{t\in[m_2,M_2]} g^{-1}(g^*(t^2))t^{-2}\|BAB\| - \lambda F\left(g(m_1)g^*(m_2^2), g(M_1)g^*(M_2^2), g^{-1}, \frac{1}{\lambda}\right). \tag{2.10}$$

Proof. We only prove (i). If we replace λ and A with $1/\lambda$ and BAB, respectively in (2.3), then it follows from $m_1m_2^2 \leqslant BAB \leqslant M_1M_2^2$ that for each unit vector $x \in H$ and $\lambda > 0$

$$f(\|BAB\|) \leqslant \langle f(BAB)x,x\rangle \leqslant \lambda f(\langle BABx,x\rangle) - \lambda F(m_1m_2^2, M_1M_2^2, f, \frac{1}{\lambda}).$$

Moreover by the increase of f we have for each unit vector $x \in H$

$$\begin{aligned} f(\langle BABx,x\rangle) &\leqslant \left\langle f(A)\frac{Bx}{\|Bx\|}, \frac{Bx}{\|Bx\|}\right\rangle f(\|Bx\|^2) \\ &= \left\langle f(B^{-2})^{-\frac{1}{2}}f(A)f(B^{-2})^{-\frac{1}{2}}\frac{f(B^{-2})^{\frac{1}{2}}Bx}{\|f(B^{-2})^{\frac{1}{2}}Bx\|}, \frac{f(B^{-2})^{\frac{1}{2}}Bx}{\|f(B^{-2})^{\frac{1}{2}}Bx\|}\right\rangle \\ &\quad \times \frac{f(\|Bx\|^2)\|f(B^{-2})^{\frac{1}{2}}Bx\|^2}{\|Bx\|^2} \\ &\leqslant \left\langle f(B^{-2})^{-\frac{1}{2}}f(A)f(B^{-2})^{-\frac{1}{2}}\frac{f(B^{-2})^{\frac{1}{2}}Bx}{\|f(B^{-2})^{\frac{1}{2}}Bx\|}, \frac{f(B^{-2})^{\frac{1}{2}}Bx}{\|f(B^{-2})^{\frac{1}{2}}Bx\|}\right\rangle \\ &\quad \times \sup_{t\in[m_2,M_2]} f(t^2)f\left(\frac{1}{t^2}\right). \end{aligned}$$

So we have the desired inequality (2.9).

3. An application to Bourin's inequality

In [4], Bourin showed the inequality (1.6) which is a reverse inequality of the well-known inequality $r(A) \leqslant \|A\|$ where $r(\cdot)$ is the spectral radius. As a generalization of (1.6), we have the following theorem in our previous note [7]:

THEOREM C. *If A and B are positive operators such that $m_1 \leqslant A \leqslant M_1$ for some scalars $0 < m_1 < M_1$, then for each $\lambda > 0$*

$$\|(BA^pB)^{\frac{1}{p}}\| \leqslant \lambda\, r(AB^{\frac{2}{p}}) + F\left(m_1^p, M_1^p, (\cdot)^{\frac{1}{p}}; \lambda\right) \|B\|^{\frac{2}{p}} \quad \text{for } p > 1. \tag{3.1}$$

In this section, we give a further generalization of Theorem C by Theorems 2.1 and 2.4.

COROLLARY 3.1. *Let A and B be positive operators such that $m_1 \leqslant A \leqslant M_1$ and $m_2 \leqslant B \leqslant M_2$ for some scalars $0 < m_i < M_i$ $(i = 1,2)$. Let f be a nonnegative real valued increasing differentiable function on $(0,\infty)$. Then the following assertions hold:*

(i) *Suppose that f is strictly convex supermultiplicative and λ_f is a unique solution of $F(f(m_1), f(M_1), f^{-1}; \lambda) = 0$. Then for each $\lambda \in (0, \lambda_f]$*

$$\begin{aligned} \|f^{-1}(Bf(A)B)\| \leqslant\ & \lambda \sup_{t\in[m_2,M_2]} f^{-1}(t^2) f^{-1}\left(\frac{1}{t^2}\right) r(A\cdot(f^{-1})^*(B^2)) \\ & + F(f(m_1), f(M_1), f^{-1}; \lambda) f^{-1}(M_2^2). \end{aligned} \tag{3.2}$$

(ii) *Suppose that f is strictly concave supermultiplicative. Then for each $\lambda > 0$*

$$\begin{aligned} \|f^{-1}(Bf(A)B)\| \leqslant\ & \lambda \sup_{t\in[m_2,M_2]} f^{-1}(t^2) f^{-1}\left(\frac{1}{t^2}\right) r(A\cdot(f^{-1})^*(B^2)) \\ & - \lambda F\left(f(m_1)m_2^2, f(M_1)M_2^2, f^{-1}, \frac{1}{\lambda}\right). \end{aligned} \tag{3.3}$$

Proof. If we replace g and B by f and $((f^{-1})^*(B^2))^{\frac{1}{2}}$ in (2.5) and (2.10), respectively, then the desired inequalities (3.2) and (3.3) hold, respectively, by

$$\|XYX\| = r(XYX) = r(YX^2)$$

for positive operators X and Y. $\square$

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