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NEW INEQUALITIES FOR MEANS IN TWO VARIABLES

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Abstract. In this paper, some bounds for $I(x,y)$ in terms of $A(x,y)$ and $L(x,y)$, and $L(x,y)$ in terms of $G(x,y)$ and $I(x,y)$ are established.

1. Introduction

Assuming x and y to be two different positive numbers, let $A(x,y)$, $G(x,y)$, $L(x,y)$, and $I(x,y)$ be the arithmetic, geometric, logarithmic, and identric means, respectively. It is well-known that (see [1-6])

$$G(x,y) < L(x,y) < I(x,y) < A(x,y). \tag{1}$$

First, Carlson [7] gives bounds for $L(x,y)$ in terms of $G(x,y)$ and $A(x,y)$ as follows

$$L(x,y) < \frac{1}{3}A(x,y) + \frac{2}{3}G(x,y). \tag{2}$$

The further results are obtained by [8] and [9] (or see [10]) as follows

THEOREM 1. *Inequality*

$$\alpha A(x,y) + (1-\alpha)G(x,y) < L(x,y) < \beta A(x,y) + (1-\beta)G(x,y) \tag{3}$$

holds for all positive x and y such that $x \neq y$ if and only if $\alpha \leqslant 0$ and $\beta \geqslant 1/3$.

Second, Sandor [11] proves that

$$\frac{2}{3}A(x,y) + \frac{1}{3}G(x,y) < I(x,y). \tag{4}$$

Alzer and Qiu [8] obtain the following interesting result:

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THEOREM 2. *Inequality*

$$\lambda A(x,y)+(1-\lambda)G(x,y)<I(x,y)<\mu A(x,y)+(1-\mu)G(x,y) \qquad (5)$$

holds for all positive x *and* y *such that* $x\neq y$ *if and only if* $\lambda\leqslant 2/3$ *and* $\mu\geqslant 2/e$.

On the other hand, Sandor [4,12] gives a lower bound for $I(x,y)$ in terms of $A(x,y)$ and $L(x,y)$, and obtains

$$I(x,y)>\frac{A(x,y)+L(x,y)}{2}. \qquad (6)$$

In fact, we can obtain the following further results

THEOREM 3. *Inequality*

$$pA(x,y)+(1-p)L(x,y)<I(x,y)<qA(x,y)+(1-q)L(x,y) \qquad (7)$$

holds for all positive x *and* y *such that* $x\neq y$ *if and only if* $p\leqslant 1/2$ *and* $q\geqslant 2/e$.

Finally, we give the bounds for $L(x,y)$ in terms of $G(x,y)$ and $I(x,y)$, and obtain the following new results.

THEOREM 4. *Inequality*

$$\eta G(x,y)+(1-\eta)I(x,y)<L(x,y)<\xi G(x,y)+(1-\xi)I(x,y) \qquad (8)$$

holds for all positive x *and* y *such that* $x\neq y$ *if and only if* $\xi\leqslant 1/2$ *and* $\eta\geqslant 1$.

Obviously, the right side of (8) is an extension of the following inequality

$$L(x,y)<\frac{1}{2}(G(x,y)+I(x,y)), \qquad (9)$$

which was given by Alzer [5].

In this paper, we shall present new simple proofs of Theorem 1 and 2, then prove Theorem 3 and 4 by a concise method.

2. Two Lemmas

LEMMA 1 ([13-15]). *Let* $f,g:[a,b]\to\mathbb{R}$ *be two continuous functions which are differentiable on* (a,b). *Further, let* $g'\neq 0$ *on* (a,b). *If* f'/g' *is increasing (or decreasing) on* (a,b), *then the functions*

$$\frac{f(x)-f(b)}{g(x)-g(b)}$$

and

$$\frac{f(x)-f(a)}{g(x)-g(a)}$$

are also increasing (or decreasing) on (a,b).

LEMMA 2 ([16-18]). *Let* l_n *and* m_n $(n=0,1,2,\cdots)$ *be real numbers, and let the power series* $L(x)=\sum_{n=0}^{\infty}l_nx^n$ *and* $M(x)=\sum_{n=0}^{\infty}m_nx^n$ *be convergent for* $|x|<R$. *If* $m_n>0$ *for* $n=0,1,2,\cdots$, *and if* l_n/m_n *is strictly increasing (or decreasing) for* $n=0,1,2,\cdots$, *then the function* $L(x)/M(x)$ *is strictly increasing (or decreasing) on* $(0,R)$.

3. A New Proof of Theorem 1

Without loss of generality, we set $0 < x < y$. Let $u = \sqrt{y/x}$. Then $u > 1$ and

$$\frac{L(x,y) - G(x,y)}{A(x,y) - G(x,y)} = \frac{\frac{u^2-1}{2\log u} - u}{\frac{u^2+1}{2} - u}.$$

Let $\log u = t$, or $u = e^t$. Then $t > 0$ and

$$\frac{L(x,y) - G(x,y)}{A(x,y) - G(x,y)} = \frac{\frac{\sinh t}{t} - 1}{\cosh t - 1}.$$

So we complete the proof of Theorem 1 by proving the next result.

THEOREM 5. *Inequality*

$$\alpha \cosh t + (1-\alpha) < \frac{\sinh t}{t} < \beta \cosh t + (1-\beta) \tag{10}$$

holds for all $t > 0$ *if and only if* $\alpha \leqslant 0$ *and* $\beta \geqslant 1/3$.

Proof. Let $f_1(t) = \frac{\sinh t}{t}$, and $g_1(t) = \cosh t$. Then

$$\frac{f_1'(t)}{g_1'(t)} = \frac{f_2(t)}{g_2(t)}, \tag{11}$$

where $f_2(t) = t\cosh t - \sinh t$, and $g_2(t) = t^2 \sinh t$.

It is easy to show that $\frac{\tanh t}{t}$ is decreasing on $(0,+\infty)$, so $\frac{f_2'(t)}{g_2'(t)} = \frac{1}{2+(t/\tanh t)}$ is also decreasing on $(0,+\infty)$. Then $\frac{f_1'(t)}{g_1'(t)} = \frac{f_2(t)}{g_2(t)} = \frac{f_2(t)-f_2(0)}{g_2(t)-g_2(0)}$ is decreasing on $(0,+\infty)$ by Lemma 1. Thus $H(t) = \frac{f_1(t)-f_1(0^+)}{g_1(t)-g_1(0^+)}$ is decreasing on $(0,+\infty)$ by Lemma 1.

Since $\lim_{t\to 0^+} H(t) = \frac{1}{3}$ and $\lim_{t\to+\infty} H(t) = 0$, the proof of Theorem 5 is complete.

4. A New Proof of Theorem 2

Let $t = \frac{1}{2}\log\frac{y}{x}$, we obtain

$$\frac{I(x,y) - G(x,y)}{A(x,y) - G(x,y)} = \frac{e^{t\coth t - 1} - 1}{\cosh t - 1},$$

where $t > 0$.

So we finish the proof of Theorem 2 by proving the following results.

THEOREM 6. *Inequality*

$$\lambda \cosh t + (1-\lambda) < e^{t\coth t - 1} < \mu \cosh t + (1-\mu) \tag{12}$$

holds for all $t > 0$ *if and only if* $\lambda \leqslant 2/3$ *and* $\mu \geqslant 2/e$.

Proof. Let $J(t) \equiv \frac{e^{t\coth t-1}-1}{\cosh t-1} = \frac{q_1(t)-q_1(0^+)}{r_1(t)-r_1(0^+)}$, where $q_1(t) = e^{t\coth t-1}$, and $r_1(t) = \cosh t$. Then

$$\frac{q_1'(t)}{r_1'(t)} = \frac{1}{2}e^{t\coth t-1}\frac{\sinh 2t - 2t}{(\sinh t)^3} \equiv k_1(t). \tag{13}$$

We compute

$$\begin{aligned} k_1'(t) =& e^{t\coth t-1}\frac{(\sinh 2t-2t)^2 + 2(\cosh 2t - 1)^2 - 3\sinh 2t(\sinh 2t - 2t)}{4(\sinh t)^5} \\ =& e^{t\coth t-1}\frac{u_1(t)}{4(\sinh t)^5}, \end{aligned}$$

where

$$u_1(t) = (\sinh 2t-2t)^2 + 2(\cosh 2t - 1)^2 - 3\sinh 2t(\sinh 2t - 2t).$$

Let $w = 2t$. Then $w > 0$ and

$$\begin{aligned} v(w) = u_1(t) &= (\sinh w - w)^2 + 2(\cosh w - 1)^2 - 3\sinh w(\sinh w - w) \\ &= 4 + w^2 + w\sinh w - 4\cosh w \\ &= 4 + w^2 + w\sum_{n=0}^{\infty}\frac{w^{2n+1}}{(2n+1)!} - 4\sum_{n=0}^{\infty}\frac{w^{2n}}{(2n)!} \\ &= \sum_{n=3}^{\infty}\frac{2(n-2)}{(2n)!}w^{2n} > 0, \end{aligned}$$

so $q_1'(t)/r_1'(t) = k_1(t)$ is increasing on $(0,+\infty)$. This computation leads to the conclusion that $J(t)$ is increasing on $(0,+\infty)$ by Lemma 1. At the same time, $\lim\limits_{t\to 0^+} J(t) = 2/3$ and $\lim\limits_{t\to+\infty} J(t) = 2/e$. So the proof of Theorem 6 is complete.

5. A Concise Proof of Theorem 3

Following the same method, we can compute

$$\frac{I(x,y)-L(x,y)}{A(x,y)-L(x,y)} = \frac{\frac{I(x,y)}{G(x,y)} - \frac{L(x,y)}{G(x,y)}}{\frac{A(x,y)}{G(x,y)} - \frac{L(x,y)}{G(x,y)}} = \frac{e^{t\coth t-1} - \frac{\sinh t}{t}}{\cosh t - \frac{\sinh t}{t}} = \frac{\frac{t}{\sinh t}e^{t\coth t-1} - 1}{t\coth t - 1},$$

where $0 < x < y$, and $t = \frac{1}{2}\log\frac{y}{x} > 0$.

So we accomplish the proof of Theorem 3 by proving the following result.

THEOREM 7. *Inequality*

$$p\cosh t+(1-p)\frac{\sinh t}{t}<e^{t\coth t-1}<q\cosh t+(1-q)\frac{\sinh t}{t} \tag{14}$$

holds for all $t>0$ *if and only if* $p\leqslant 1/2$ *and* $q\geqslant 2/e$.

Proof. Let $F(t)\equiv\frac{\frac{t}{\sinh t}e^{t\coth t-1}-1}{t\coth t-1}=\frac{q_2(t)-q_2(0^+)}{r_2(t)-r_2(0^+)}$, where $q_2(t)=\frac{t}{\sinh t}e^{t\coth t-1}$ and $r_2(t)=t\coth t$. Then

$$\frac{q_2'(t)}{r_2'(t)}=e^{t\coth t-1}\left(\frac{\sinh t-t\cosh t}{\sinh t\cosh t-t}+\frac{t}{\sinh t}\right)\equiv k_2(t). \tag{15}$$

we compute

$$k_2'(t)=e^{t\coth t-1}\frac{u_2(t)}{(\sinh t)^3(\sinh t\cosh t-t)^2},$$

where

$$\begin{aligned}
u_2(t)=&2\sinh t(\sinh t\cosh t-t)^2(\sinh t-t\cosh t)+t(\sinh t\cosh t-t)^3\\
&+(\sinh t)^3[t(\sinh t)^2\cosh t+t^2\sinh t-2(\sinh t)^3]\\
=&2(\sinh t)^4-4t(\sinh t)^3\cosh t+2t^2(\sinh t)^2-t(\sinh t\cosh t)^3\\
&+t^2(\sinh t\cosh t)^2+t^3\sinh t\cosh t-t^4+t(\sinh t)^5\cosh t+t^2(\sinh t)^4\\
=&\frac{1}{4}(\cosh 4t-4\cosh 2t+3)-\frac{t}{2}(\sinh 4t-2\sinh 2t)+t^2(\cosh 2t-1)\\
&-\frac{t}{32}(\sinh 6t-3\sinh 2t)+\frac{t^2}{8}(\cosh 4t-1)+t^3\sinh t\cosh t-t^4+\frac{t^3}{2}\sinh 2t-t^4\\
&+\frac{t}{32}(\sinh 6t-4\sinh 4t+5\sinh 2t)+\frac{t^2}{8}(\cosh 4t-4\cosh 2t+3)\\
=&\frac{1}{2}\sum_{n=4}^{\infty}\frac{d_n}{(2n+2)!}t^{2n+2},
\end{aligned}$$

and $d_n=(2n^2-7n-1)16^n+[8n+(2n+2)(2n+1)(n+1)+2(n+1)]4^n>0$ for $n\geqslant 4$. So $k_2'(t)>0$ for $t>0$, and $q_2'(t)/r_2'(t)=k_2(t)$ is increasing on $(0,+\infty)$. Hence $F(t)$ is increasing on $(0,+\infty)$ by Lemma 1. At the same time, $\lim_{t\to 0^+}F(t)=1/2$ and $\lim_{t\to+\infty}F(t)=2/e$. So the proof of Theorem 7 is complete.

6. A Short Proof of Theorem 4

By the transformation $t=\frac{1}{2}\log\frac{y}{x}$, we can compute and obtain

$$\frac{L(x,y)-I(x,y)}{G(x,y)-I(x,y)}=\frac{\frac{L(x,y)}{G(x,y)}-\frac{I(x,y)}{G(x,y)}}{1-\frac{I(x,y)}{G(x,y)}}=\frac{\frac{\sinh t}{t}-e^{t\coth t-1}}{1-e^{t\coth t-1}}=\frac{\frac{\sinh t}{t}e^{1-t\coth t}-1}{e^{1-t\coth t}-1},$$

where $t>0$.

So we complete the proof of Theorem 4 by proving the following result.

THEOREM 8. *Inequality*

$$\eta+(1-\eta)e^{t\coth t-1}<\frac{\sinh t}{t}<\xi+(1-\xi)e^{t\coth t-1} \tag{16}$$

holds for all $t>0$ *if and only if* $\xi\leqslant 1/2$ *and* $\eta\geqslant 1$.

Proof. Let $S(t)\equiv\frac{\frac{\sinh t}{t}e^{1-t\coth t}-1}{e^{1-t\coth t}-1}=\frac{q_3(t)-q_3(0^+)}{r_3(t)-r_3(0^+)}$, where $q_3(t)=\frac{\sinh t}{t}e^{1-t\coth t}$, and $r_3(t)=e^{1-t\coth t}$. Then

$$\frac{q_3'(t)}{r_3'(t)}=\frac{(\sinh t)^3-t^2\sinh t}{t^2(\sinh t\cosh t-t)}\equiv\frac{A(t)}{B(t)}, \tag{17}$$

where $A(t)=(\sinh t)^3-t^2\sinh t$, and $B(t)=t^2(\sinh t\cosh t-t)$. Then

$$A(t)=\sum_{n=2}^{\infty}a_nt^{2n+1}, B(t)=\sum_{n=2}^{\infty}b_nt^{2n+1},$$

where $a_n=\frac{3^{2n+1}-3}{4(2n+1)!}-\frac{1}{(2n-1)!}, b_n=\frac{4^{n-1}}{(2n-1)!}$, $n\geqslant 2$, and $n\in\mathbb{N}^+$.
So

$$c_n\equiv\frac{a_n}{b_n}=\frac{3\cdot 9^n-16n^2-8n-3}{4^n(2n+1)2n},$$

which is increasing for $n=2,3,\cdots$. Thus $\frac{q_3'(t)}{r_3'(t)}=\frac{A(t)}{B(t)}$ is increasing on $(0,+\infty)$ by Lemma 2, and $S(t)$ is increasing on $(0,+\infty)$ by Lemma 1. At the same time, $\lim_{t\to 0^+}S(t)=1/2$ and $\lim_{t\to+\infty}S(t)=1$. So the proof of Theorem 8 is complete.

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